Volume estimation of biomedical objects described by multiple sets of non-trimmed Bézier triangles

Aleksandrs Sisojevs
Ventspils University Colleges
101a Inzhenieru Str.
LV-3601, Ventspils, Latvia
alexiv@inbox.lv

Mihails Kovalovs
Riga Technical University
1 Setas Str.
LV-1048, Riga, Latvia
mihails.kovalovs@rtu.lv

Olga Krutikova
Riga Technical University
1 Setas Str.
LV-1048, Riga, Latvia
olga.krutikova@rtu.lv

ABSTRACT

Estimating the volume of a 3D model of an object is an actual task in many scientific and engineering fields (for example, CAD systems, biomedical engineering tasks etc.). Spline surfaces is one of the most powerful and flexible methods used to describe a 3D model. At the same time, it is rather difficult to estimate the volume of an object described by spline surfaces. A model of a Bézier triangle is a simple type of a spline surface, but it is practically advantageous.

This paper describes a method of estimating the volume for 3D objects that are described by a set of Bézier triangles. The proposed method was tested on 3D models of objects of biomedical origin. A theorem is presented in this paper for volume estimation, based on different properties of researched models, acquired by a projection of a set vertices of a Bézier triangle onto a coordinate system axis. The proposed approach is based on using methods of differential geometry: surface integrals of scalar fields, Euler's integral of the first kind and Beta functions. Experimental results prove the accuracy of presented theorems. The proposed method can be successfully used to calculate the volume of different 3D models, including objects of biomedical origin.

Keywords

Beta function, Bézier triangles, volume estimation.

1. INTRODUCTION

Determining the volume of a three-dimensional surface can be a very complex task, depending on the object's description method. There are different ways a 3D model of an object can be described. Most commonly used approaches are: approximation by primitives, polygonal mesh approximation or use of parametrical surfaces [Sun13a, Wro06a]. While different methods exist for model description, it should be noted that spline surfaces have a significant role in 3D modeling of complex objects, because they can describe curved models with high precision. At the same time, the problem of precise volume estimation can be very critical in many areas. For example, exact volume estimation of biomedical objects can significantly increase the accuracy and reliability of medical diagnosis. Therefore, the use of a precise model and its volume estimation can have a significant impact on biomedical engineering. In work [Sis09a] the author describes a volume estimation method for models of biomedical objects bounded by Bézier surfaces. In [Duer03a] several quantification methods for calculating the volume of soft tissues are discussed that include Geometric Best-Fitting, based on shape assumptions and stereological procedures. In [Matt87a], [Mich88a], [How93a], [McNul00a], the Cavalieri method is discussed which is a quantification method that is commonly used for unbiased estimation of the volume of a variety of biological objects from serial histological sections. In [Mor98a], [Duer00b] a different method is proposed that is used to calculate the volume of a reconstructed 3D model. In these works, two algorithms (Isocontouring and Surface Tiling) for 3D surface reconstruction were tested. These algorithms were used to construct an accurate wireframe surface of a biological object and calculate its volume [Baj96a]. The provided results showed minor quantitative differences between the Isocontouring, Surface Tiling and other standard qualitative approaches [Duer00b].

Calculating the volume of solids bounded by Bézier Surfaces in analytical form, based on integral calculations, was discussed in [Juh00a].

A similar idea was described in [Zha01a], but it was used to calculate the area of a closed 2D mesh and the volume of a bounded 3D mesh.

Approximation of a solid object (that is represented as a triangle mesh) by a bounding set of spheres to minimize the sum of the volume of the sphere and, subsequently, the approximation of the object' volume was discussed in [Wan06a]. A variational approach to computing an optimal segmentation of a 3D shape for estimating a union of tight bounding volumes was described in [Lu07a].

In this paper, an approach for calculating the volume of three-dimensional objects described by a set of Bézier triangles is proposed. 3D models of biomedical origin are considered as the object of study, but the proposed method can be used on any model that is described by Bézier triangles.

2. BÉZIER TRIANGLES AS ELEMENTS OF 3D MODELING

One of the topical problems, when analysing an object, is its volume estimation. In this paper, a method is proposed that can be used to calculate the volume of an object by means of integral calculations. Input data for this method is a 3D model that is described by a set of Bézier triangles. An example of constructing such 3D models, based on biomedical objects can be seen in [Sis12a].

A Bézier triangle is described as follows:

$$S(u,v,w)= \\ =\sum_{\substack{i,j,k=0\\ i+j+k=n}}^{n}\binom{n}{i,j,k}\cdot u^i\cdot v^j\cdot w^k\cdot p_{i,j,k}\,, \qquad (1)$$

where: i,j,k – indexes of sum;

u,v,w – parameters;

$p_{i,j,k}$ – control points

$\binom{n}{i,j,k}=\frac{n!}{i!\cdot j!\cdot k!}$ - trinomial coefficients.

To calculate the volume, the Bézier triangle is transformed as follows:

$$S(u,v)=\sum_{\substack{i,j,k=0\\ i+j+k=n}}^{n}T_{i,j,k}(u,v)\cdot p_{i,j,k}\,, \qquad (2)$$

where:

$$T_{i,j,k}(u,v)=\frac{n!}{i!\cdot j!\cdot k!}\cdot u^i\cdot v^j\cdot(1-u-v)^k. \qquad (3)$$

In this case partial derivatives of base functions could be described as:

$$\frac{\partial T_{i,j,k}(u,v)}{\partial u}=\frac{n!}{i!\cdot j!\cdot k!}\cdot u^{i-1}\cdot v^j\cdot \\ \cdot(1-u-v)^k\cdot(i\cdot(1-u-v)-u\cdot k)\,, \qquad (4)$$

and

$$\frac{\partial T_{i,j,k}(u,v)}{\partial v}=\frac{n!}{i!\cdot j!\cdot k!}\cdot u^{i}\cdot v^{j-1}\cdot \\ \cdot(1-u-v)^k\cdot(j\cdot(1-u-v)-v\cdot k) \qquad (5)$$

3. THE PROPOSED METHOD OF VOLUME ESTIMATION

Let's consider that the volume of an object is a sum of volumes of curvilinear prisms:

$$V_{obj}=\sum V_{prism}\,. \qquad (6)$$

where: V_{obj} – volume of an object;

V_{prism} – volume of a curvilinear prism.

The method of estimating the volume of curvilinear prisms is described in Theorem 1.

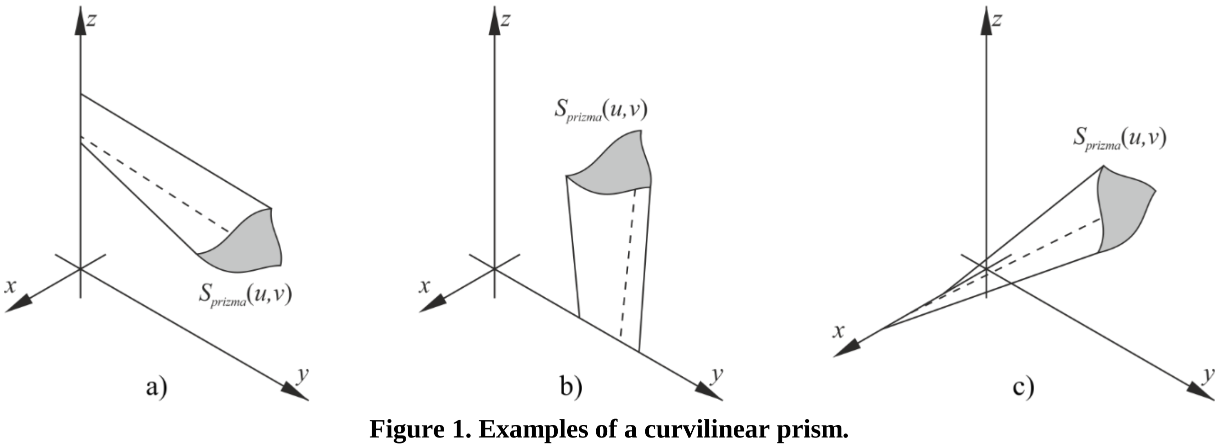

Figure 1. Examples of a curvilinear prism.

Theorem 1

If a curvilinear prism was constructed by projecting the vertices of a Bézier triangle onto coordinate axis, its volume could be calculated as follows:

$$V_{prism} = \frac{(n!)^3}{2} \cdot \sum_{\substack{i1,j1,k1=0 \\ i1+j1+k1=n}}^{n} \sum_{\substack{i2,j2,k2=0 \\ i2+j2+k2=n}}^{n} \sum_{\substack{i3,j3,k3=0 \\ i2+j2+k2=n}}^{n} Q , \quad (7)$$

where:

$$Q = \frac{\det(M_Z) \cdot I}{i1! \cdot j1! \cdot k1! \cdot i2! \cdot j2! \cdot k2! \cdot i3! \cdot j3! \cdot k3!}, \quad (8)$$

When projecting onto the *Oz* axis (Fig.1a.), the matrix takes the following form:

$$M_Z = \begin{bmatrix} x_{i1,j1,k1} & y_{i1,j1,k1} & 0 \\ x_{i2,j2,k2} & y_{i2,j2,k2} & z_{i2,j2,k2} \\ x_{i3,j3,k3} & y_{i3,j3,k3} & z_{i3,j3,k3} \end{bmatrix}. \quad (9)$$

Accordingly, when projecting onto the *Oy* axis (Fig.2b.):

$$M_Y = \begin{bmatrix} x_{i1,j1,k1} & 0 & z_{i1,j1,k1} \\ x_{i2,j2,k2} & y_{i2,j2,k2} & z_{i2,j2,k2} \\ x_{i3,j3,k3} & y_{i3,j3,k3} & z_{i3,j3,k3} \end{bmatrix}, \quad (10)$$

And, accordingly. on the *Ox* axis (see Fig.2c.):

$$M_X = \begin{bmatrix} 0 & y_{i1,j1,k1} & z_{i1,j1,k1} \\ x_{i2,j2,k2} & y_{i2,j2,k2} & z_{i2,j2,k2} \\ x_{i3,j3,k3} & y_{i3,j3,k3} & z_{i3,j3,k3} \end{bmatrix}. \quad (11)$$

A similar method was described in [Sis13a], except it is used for rational Bezier surfaces. The approach that is proposed in Theorem 1 is a logical continuation of ideas described in [Sis09a] and [Sis13a] and is used in this work for the case of Bezier triangle.

The method of calculating the integral *I* in symbol form, is described in Theorem 2.

Theorem 2

Beta function is used to calculate the values of integral I. Historical overview of Beta function with detailed description of its mathematical properties can be found in [Mas09a]. If the argument values of the Beta function are natural numbers ($a, b \in N$), the Beta function could be described as follows:

$$B(a;b) = \frac{(a-1)! \cdot (b-1)!}{(a+b-1)!}. \quad (12)$$

The method of calculating the value of integral *I* depends on the values of coefficients that are calculated as follows:

$$\begin{aligned} i4 &= i1 + i2 + i3 \\ j4 &= j1 + j2 + j3 , \\ k4 &= k1 + k2 + k3 \end{aligned} \quad (13)$$

The integral *I* is calculated using the formulas described below.

1) for case ($i4 > 0$; $j4 > 0$; $k4 > 1$):

$$\begin{aligned} I = {} & C_1 \cdot D_1 \cdot B(i4; j4 + k4 + 1) + \\ & + C_2 \cdot D_2 \cdot B(i4 + 1; j4 + k4) + \\ & + C_3 \cdot D_2 \cdot B(i4; j4 + k4) + \\ & + C_4 \cdot D_3 \cdot B(i4 + 2; j4 + k4 - 1) + \\ & + C_5 \cdot D_3 \cdot B(i4 + 1; j4 + k4 - 1) + \\ & + C_6 \cdot D_3 \cdot B(i4; j4 + k4 - 1) \end{aligned} , \quad (14)$$

where:

$$\begin{aligned} C_1 &= i2 \cdot k3 + i2 \cdot j3 \\ C_2 &= k2 \cdot k3 + i2 \cdot k3 + j3 \cdot k2 + \\ &+ 2 \cdot i2 \cdot j3 \\ C_3 &= -i2 \cdot k3 - 2 \cdot i2 \cdot j3 \\ C_4 &= j3 \cdot k2 + i2 \cdot j3 \\ C_5 &= -j3 \cdot k2 - 2 \cdot i2 \cdot j3 \\ C_6 &= i2 \cdot j3 \end{aligned} , \quad (15)$$

and

$$\begin{aligned} D_1 &= \sum_{q=0}^{k4-2} \binom{k4-2}{q} \cdot \frac{(-1)^q}{j4+q+2}, \\ D_2 &= \sum_{q=0}^{k4-2} \binom{k4-2}{q} \cdot \frac{(-1)^q}{j4+q+1}, \\ D_3 &= \sum_{q=0}^{k4-2} \binom{k4-2}{q} \cdot \frac{(-1)^q}{j4+q}. \end{aligned} \quad (16)$$

2) for case ($i4 = 0$; $j4 > 0$; $k4 > 1$):

$$I = \frac{k2 \cdot k3 \cdot D_2 - j3 \cdot k2 \cdot D_4}{j4 + k4}, \quad (17)$$

where:

$$D_4 = \sum_{q=0}^{k4-1} \binom{k4-1}{q} \cdot \frac{(-1)^q}{j4+q}. \qquad (18)$$

3) for case ($i4 > 0$; $j4 = 0$; $k4 > 1$):

$$I = \frac{k2 \cdot k3}{k4-1} \cdot B(i4+1; k4) - \\ - \frac{i2 \cdot k3}{k4} \cdot B(i4; k4+1). \qquad (19)$$

4) for case ($i4 > 0$; $j4 > 0$; $k4 = 0$):

$$I = \frac{i2 \cdot j3}{j4} \cdot B(i4; j4+1). \qquad (20)$$

5) for case ($i4 > 0$; $j4 > 0$; $k4 = 1$):

$$I = C_7 \cdot B(i4, j4+2) - \\ \frac{j3 \cdot k2}{j4} \cdot B(i4+1, j4+1), \qquad (21)$$

where

$$C_7 = \frac{i2 \cdot (j3 - j4 \cdot k3)}{j4 \cdot (j4+1)}. \qquad (22)$$

6) for case ($i4 = 3n-1$; $j4 = 0$; $k4 = 1$):

$$I = -\frac{i2 \cdot k3}{3n \cdot (3n-1)}. \qquad (23)$$

7) for case ($i4 = 0$; $j4 = 3n-1$; $k4 = 1$):

$$I = -\frac{j3 \cdot k2}{3n \cdot (3n-1)}. \qquad (24)$$

8) for cases ($i4 = 3n$; $j4 = 0$; $k4 = 0$), ($i4 = 0$; $j4 = 3n$; $k4 = 0$) and ($i4 = 0$; $j4 = 0$; $k4 = 3n$):

$$I = 0. \qquad (25)$$

Proof of Theorem 1

To find the differential volume of a prism dV_{prism}, as seen on Fig.2, in the case where vector $\vec{r}$ is perpendicular to $dS*$, the differential volume could be calculated using the following formula:

$$dV_{prism} = \frac{1}{2} \cdot |\vec{r}| \cdot dS*, \qquad (26)$$

where: $\vec{r}$ – perpendicular vector from *Oz* axis to a point on a surface.

Considering that infinitely small values are used in the calculations, it can be assumed that the following expression is valid:

$$dS* = \cos\alpha \cdot dS_{prism}, \qquad (27)$$

where: α – the angle between the vector $\vec{r}$ and the normal vector $\vec{n}$ on dS_{prism}.

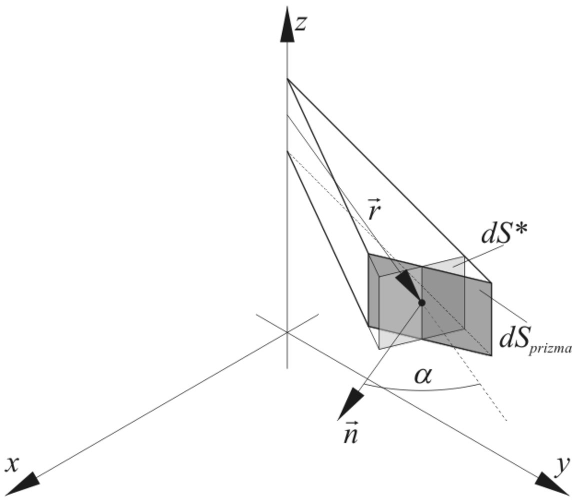

Figure 2. Differential volume calculation.

We calculate $\cos\alpha$ value using the following formula:

$$\cos\alpha = \frac{\vec{r} \cdot \vec{n}}{|\vec{r}| \cdot |\vec{n}|} = \\ = \frac{r_X \cdot n_X + r_Y \cdot n_Y + r_Z \cdot n_Z}{|\vec{r}| \cdot |\vec{n}|}, \qquad (28)$$

differential equation can be obtained from (26), (27) and (28):

$$dV_{prism} = \frac{1}{2} \cdot \frac{\vec{r} \cdot \vec{n}}{|\vec{n}|} \cdot dS_{prism}, \qquad (29)$$

To solve this differential equation, both sides of (29) are integrated with a surface integral:

$$V_{prism} = \frac{1}{2} \cdot \iint_{(S_{prism})} \frac{\vec{r} \cdot \vec{n}}{|\vec{n}|} dS_{prism}, \qquad (30)$$

In order to solve the surface integral of scalar fields (30), it is necessary to convert into a double integral. The following values also need to be considered:

$$A = \begin{vmatrix} \frac{\partial S_Y(u,v)}{\partial u} & \frac{\partial S_Z(u,v)}{\partial u} \\ \frac{\partial S_Y(u,v)}{\partial v} & \frac{\partial S_Z(u,v)}{\partial v} \end{vmatrix}, \qquad (31)$$

and

$$B = \begin{vmatrix} \frac{\partial S_Z(u,v)}{\partial u} & \frac{\partial S_X(u,v)}{\partial u} \\ \frac{\partial S_Z(u,v)}{\partial v} & \frac{\partial S_X(u,v)}{\partial v} \end{vmatrix}, \quad (32)$$

and

$$C = \begin{vmatrix} \frac{\partial S_X(u,v)}{\partial u} & \frac{\partial S_Y(u,v)}{\partial u} \\ \frac{\partial S_X(u,v)}{\partial v} & \frac{\partial S_Y(u,v)}{\partial v} \end{vmatrix}, \quad (33)$$

where: $S_X(u,v), S_Y(u,v), S_Z(u,v)$ are coordinates of function (1).

Considering the *A*, *B* and *C* values it can be concluded that the normal vector could be described as follows:

$$\vec{n} = \begin{pmatrix} A & B & C \end{pmatrix}, \quad (34)$$

In this case the vector $\vec{n}$ module can be calculated as follows:

$$|\vec{n}| = \sqrt{A^2 + B^2 + C^2}, \quad (35)$$

By placing (35) in (30) and transforming (30) to a double integral (using the transition properties from surface integral of the first kind to double integral), we get:

$$V_{prism} = \frac{1}{2} \cdot \int_{v\_\min}^{v\_\max} \int_{u\_\min}^{u\_\max} (r_X \cdot n_X + + r_Y \cdot n_Y + r_Z \cdot n_Z) du dv, \quad (36)$$

Considering the nature of vector $\vec{r}$ it could be described as follows:

$$\vec{r} = \begin{pmatrix} S_X(u,v) & S_Y(u,v) & 0 \end{pmatrix}, \quad (37)$$

Considering the range of parameters *u* and *v*, the final integral of volume estimation of a curvilinear prism could be described as follows:

$$V_{prism} = \frac{1}{2} \cdot \int_0^1 \left\{ \int_0^{1-u} f_1(u,v) dv \right\} du, \quad (38)$$

where:

$$f_1(u,v) = A \cdot S_X(u,v) + B \cdot S_Y(u,v), \quad (392)$$

By substituting the values (2) into (39) and after transforming we get:

$$f_1(u,v) = = \sum_{\substack{i1,j1,k1=0 \\ i1+j1+k1=n}}^{n} \sum_{\substack{i2,j2,k2=0 \\ i2+j2+k2=n}}^{n} \sum_{\substack{i3,j3,k3=0 \\ i2+j2+k2=n}}^{n} \Big\{ \det(M_Z) \cdot \cdot T_{i1,j1,k1}(u,v) \cdot \frac{\partial T_{i2,j2,k2}(u,v)}{\partial u} \cdot \cdot \frac{\partial T_{i3,j3,k3}(u,v)}{\partial v} \Big\}, \quad (40)$$

where M_Z is described in (9)

It is easy to see that if the curvilinear prism is constructed by projecting the points of Bézier triangle onto the coordinate axis *Oy* or *Ox* (Fig.6b and Fig.6c.), the matrices in formula (9) take the form (10) and (11).

By placing (3)-(5) into (39) and after transforming the volume estimation integral could be described as follows:

$$V_{prism} = \frac{(n!)^3}{2} \cdot \cdot \sum_{\substack{i1,j1,k1=0 \\ i1+j1+k1=n}}^{n} \sum_{\substack{i2,j2,k2=0 \\ i2+j2+k2=n}}^{n} \sum_{\substack{i3,j3,k3=0 \\ i2+j2+k2=n}}^{n} G_{prism}, \quad (41)$$

where:

$$G_{prism} = \frac{\det(M_Z) \cdot I}{i1!\,j1!\,k1!\,i2!\,j2!\,k2!\,i3!\,j3!\,k3!}, \quad (42)$$

where:

$$I = \int_0^1 u^{i4-1} \cdot I_u du, \quad (43)$$

where:

$$I_u = \int_0^{1-u} v^{j4-1} \cdot (1-u-v)^{k4-2} \cdot \cdot (i2 \cdot (1-u-v) - u \cdot k2) \cdot \cdot (j3 \cdot (1-u-v) - v \cdot k3) dv. \quad (44)$$

Solving the integral (44) is described in the Proof of Theorem 2.

Proof of Theorem 2.

1) for case ($i4 > 0$; $j4 > 0$; $k4 > 1$) the integral (44) after simplifying and transforming takes the following form:

$$I_u = C_1 \cdot \int_0^{1-u} v^{j4+1} \cdot (1-u-v)^{k4-2} dv + \\ + C_2 \cdot u \cdot \int_0^{1-u} v^{j4} \cdot (1-u-v)^{k4-2} dv + \\ + C_3 \cdot \int_0^{1-u} v^{j4} \cdot (1-u-v)^{k4-2} dv + \\ \{C_4 \cdot u^2 + C_5 \cdot u + C_6\} \cdot \\ \int_0^{1-u} v^{j4-1} \cdot (1-u-v)^{k4-2} dv \quad , \quad (45)$$

where the $C_1 - C_6$ values are described in (15). The separate integral (45) is solved using binomial theorem:

$$\int_0^{1-u} v^{j4+1} \cdot (1-u-v)^{k4-2} dv = \\ = \sum_{q=0}^{k4-2} (-1)^q \cdot \binom{k4-2}{q} \cdot (1-u)^{k4-2-q} \cdot \\ \cdot \int_0^{1-u} v^{j4+q+1} dv = \\ = \sum_{q=0}^{k4-2} (-1)^q \cdot \binom{k4-2}{q} \cdot \frac{(1-u)^{k4+j4}}{j4+q+2} \quad . \quad (46)$$

Similarly, the other integrals in (45) can be solved. Substituting the results in (43), after transformations, we get:

$$I_b = C_1 \cdot D_1 \cdot \int_0^1 u^{i4-1} \cdot (1-u)^{k4+j4} du + \\ + C_2 \cdot D_2 \cdot \int_0^1 u^{i4} \cdot (1-u)^{j4+k4-1} du + \\ + C_3 \cdot D_2 \cdot \int_0^1 u^{i4-1} \cdot (1-u)^{j4+k4-1} du + \\ + C_4 \cdot D_3 \cdot \int_0^1 u^{i4+1} \cdot (1-u)^{j4+k4-2} du + \\ + C_5 \cdot D_3 \cdot \int_0^1 u^{i4} \cdot (1-u)^{j4+k4-2} du + \\ C_6 \cdot D_3 \cdot \int_0^1 u^{i4-1} \cdot (1-u)^{j4+k4-2} du \quad (47)$$

where D_1 - D_3 are described in (16). Some of the integrals in (47) are Euler's integrals of the first kind. They can be solved with a Beta function, described in (12). The solution of (47) is (14).

2) for case ($i4 = 0$; $j4 > 0$; $k4 > 1$) integral (44) takes the following form:

$$I_u = (-u \cdot k2) \cdot \\ \cdot \left[j3 \int_0^{1-u} v^{j4-1} \cdot (1-u-v)^{k4-1} dv - \right. \\ \left. - k3 \int_0^{1-u} v^{j4} \cdot (1-u-v)^{k4-2} dv \right] \quad (48)$$

Some of the integrals in (48) can be solved similarly as in (46). The result of integrating (48) is as follows:

$$I_u = u \cdot (1-u)^{j4+k4-1} \cdot \\ \cdot [k2 \cdot k3 \cdot D_2 - j3 \cdot k2 \cdot D_4], \quad (49)$$

Where D_4 is described in (18). By placing (49) into (43) after integration we get (17).

3) for case ($i4 > 0$; $j4 = 0$; $k4 > 1$) integral (44) takes the following form:

$$I_u = u \cdot k2 \cdot k3 \cdot \int_0^{1-u} (1-u-v)^{k4-2} dv - \\ - i2 \cdot k3 \cdot \int_0^{1-u} (1-u-v)^{k4-1} dv = \\ = \frac{k2 \cdot k3}{k4-1} \cdot \{u \cdot (1-u)^{k4-1}\} - \\ - \frac{i2 \cdot k3}{k4} \cdot \{(1-u)^{k4}\} \quad (50)$$

By placing (50) into (43) after integration we get (19).

4) for case ($i4 > 0$; $j4 > 0$; $k4 = 0$) integral (44) takes the following form:

$$I_u = i2 \cdot j3 \int_0^{1-u} v^{j4-1} dv = \frac{i2 \cdot j3}{j4} (1-u)^{j4}. \quad (51)$$

By placing (51) into (43) after integration we get (20).

5) for case ($i4 > 0$; $j4 > 0$; $k4 = 1$) the value of (44) is calculated for three sub cases:

a) for case $k1 = 1$ and $k2 = k3 = 0$:

$$I_u = i2 \cdot j3 \cdot \int_0^{1-u} v^{j4-1} \cdot (1-u-v) dv \quad (52)$$

b) for case $k1 = 0$, $k2 = 1$ and $k3 = 0$:

$$I_u = i2 \cdot j3 \cdot \int_0^{1-u} v^{j4-1} \cdot (1-u-v) dv - \\ - j3 \cdot u \cdot \int_0^{1-u} v^{j4-1} dv \quad (53)$$

c) for case $k1 = 0$, $k2 = 0$ and $k3 = 1$:

$$I_u = i2 \cdot j3 \cdot \int_0^{1-u} v^{j4-1} \cdot (1-u-v) dv - \\ - i2 \cdot \int_0^{1-u} v^{j4} dv \quad (54)$$

Combining (52)-(54) after transformations we get:

$$I_u = i2 \cdot j3 \cdot (1-u) \cdot \int_0^{1-u} v^{j4-1} dv - \\ - j3 \cdot k2 \cdot u \cdot \int_0^{1-u} v^{j4-1} dv - \\ (i2 \cdot j3 + i2 \cdot k3) \cdot \int_0^{1-u} v^{j4} dv , \quad (55)$$

After substituting the integration results (55) into (43) we get (21).

6) for case ($i4 = 3n-1$; $j4 = 0$; $k4 = 1$) the value of (44) is calculated for three sub cases:

a) for cases ($k1 = 1$ and $k2 = k3 = 0$) and ($k1 = 0$, $k2 = 1$ and $k3 = 0$):

$$I_u = 0; \quad (56)$$

b) for case $k1 = 0$, $k2 = 0$ and $k3 = 1$:

$$I_c = -i2 \cdot \int_0^{1-u} dv = -i2 \cdot (1-u). \quad (57)$$

Combining (56) and (57) after transformations we get:

$$I_u = -i2 \cdot k3 \cdot (1-u). \quad (58)$$

After substituting the integration results (58) into (43) we get (23).

7) for case ($i4 = 0$; $j4 = 3n-1$; $k4 = 1$) the value of (44) is calculated for three sub cases:

a) for cases ($k1 = 1$ and $k2 = k3 = 0$) and ($k1 = 0$, $k2 = 0$ and $k3 = 1$):

$$I_u = 0; \quad (59)$$

b) for case $k1 = 0$, $k2 = 1$ and $k3 = 0$

$$I_u = -j3 \cdot u \cdot \int_0^{1-u} v^{3n-2} dv; \quad (60)$$

Combining (59) and (60) after transformations we get:

$$I_c = -j3 \cdot k2 \cdot u \cdot \int_0^{1-u} v^{3n-2} dv = \\ = u \cdot (1-u)^{3n-1} \cdot \left[\frac{-j3 \cdot k2}{3n-1} \right] . \quad (61)$$

After substituting the integration results (61) into (43) we get (24).

8) for cases ($i4 = 3n$; $j4 = 0$; $k4 = 0$), ($i4 = 0$; $j4 = 3n$; $k4 = 0$) and ($i4 = 0$; $j4 = 0$; $k4 = 3n$):

It is necessary to consider that in the described cases the value of the determinant from the formula (8) takes the values:

$$\det(M_Z) = \det(M_Y) = \det(M_X) = 0. \quad (62)$$

In this case the value of integral *I* does not affect the value of *Q* (from (8)):

$$\det(M_Z) \cdot I = 0 \Rightarrow \forall I \in R. \quad (63)$$

For the convenience of software implementation of the method, the integral *I* can take the zero value as in (25).

An illustration of cases with different values of *i*4, *j*4 and *k*4 can be seen in Fig.4 in Appendix.

Numerical integration

To check the accuracy of symbolic integration, the numerical integration was also used to calculate the approximate value of integral *I*.

The range of parameters *u* and *v* is divided into *m* parts. In this case, the volume of the curvilinear prism could be approximated as follows:

$$I_b \approx \frac{1}{m^2} \cdot \left\{ \left(\sum_{q1=0}^{(m-1)} \sum_{q2=0}^{(m-2-q1)} f_2(u_{q1}, v_{q2}) \right) + \right. \\ \left. + \frac{1}{2} \left(\sum_{q1=0}^{(m-1)} f_b(u_{q1}, v_{m-2-q1}) \right) \right\} , \quad (64)$$

where:

$$u_{q1} = \frac{2 \cdot q1 + 1}{2 \cdot m}, \quad (65)$$

and

$$v_{q2} = \frac{2 \cdot q2 + 1}{2 \cdot m}, \tag{66}$$

and

$$f_2(u,v) = u^{i4-1} \cdot v^{j4-1} \cdot (1-u-v)^{k4-2} \cdot \\ \cdot (i2 \cdot (1-u-v) - u \cdot k2) \cdot \\ \cdot (j3 \cdot (1-u-v) - v \cdot k3). \tag{67}$$

4. EXPERIMENTS

To test the accuracy of the proposed method, the volume was estimated for a 3D model of an antiprism (Fig.3a.), with an approximately known volume in a range from 9351487,18 to 9732392,24.

The proposed volume estimation method was also used to calculate the volume of 3D models of real biological objects. Three models were acquired through 3D scanning (foot, hand, human torso), and two models of the head, that were automatically generated through processing Computed Tomography images (Fig 3e.) and regular photographs (Fig. 3f.), these objects can be seen on Figure 3. A method, described in [Sis12a], was used to create spline models of biomedical objects.

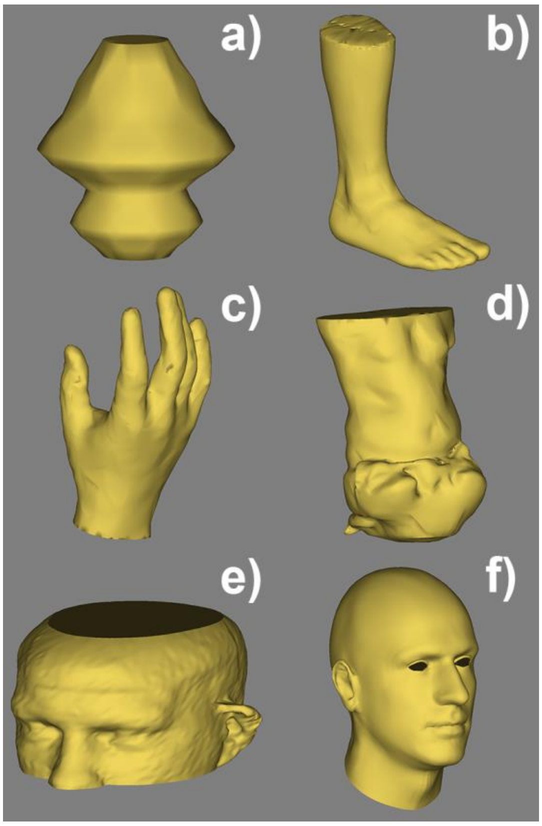

Figure 3. 3D models of scanned biological objects, a) Antiprism, b) scanned foot, c) scanned hand, d) scanned torso, e) head (from CT), f) head (from photographs)

The volume was estimated using both the symbolic and numerical integration (m=10). The error of numerical integration was also calculated. The results can be seen in Table 1.

Object	Numerical	Symbolic	Error, %
Antiprism	9632649,96	9614998,57	0,1832
foot	3391616,27	3391342,52	0,0081
hand	374145,34	374030,57	0,0307
body	54876035,86	54870496,04	0,0101
head (CT)	21059419,31	21055516,85	0,0185
Head (photo)	7951163,16	7949147,47	0,0254

Table 1. Volume estimation of biological objects

The results of the experiments show that the estimated volume of the antiprism falls within the known volume range (9351487,18 to 9732392,24). The proposed algorithm has also managed to estimate volume of all the other 3D models.

The error of the numerical integration was very low (below 0,2%). The highest error appeared when estimating the volume of the antiprism, due to the size of the triangles relative to full 3D model size. The size of the antiprism triangles was considerably greater compared to the other models.

To evaluate the proposed method, it is also necessary to consider the time of calculation of volumes, as well as the models. The calculation times were measured on a computer with the following specifications: Processor - Intel Core i7-3770K 3.50GHz, RAM - 8GB, OS - Windows 7 64bit. The acquired calculation times can be seen in Table2.

Object	Numerical, s.	Symbolic, s.	Number of Bezier triangles
Antiprism	0,2646	0,0204	92
Foot	5,7684	0,4435	1996
Hand	5,7181	0,4395	1979
body	5,7910	0,4455	1996
Head (CT)	43,3005	3,3389	14695
Head (photo)	8,0820	0,6228	2794

Table 2. Calculation times of volume estimation

As can be seen in Table 2 the calculation time of volume estimation using symbolic integration is 13 times faster than using the numerical integration. In addition, when using symbolic integration, an average of 4481 Bezier triangles are processed in 1 second. However, when numerical integration is used, an average of 345 Bezier triangles are processed in 1 second.

5. CONCLUSIONS

In this paper, a volume estimation method was proposed, for objects described by a set of Bézier triangles. This proposed approach is based on integral calculations methods: Surface integrals of scalar fields, Euler's integral of the first kind and Beta function for natural arguments.

The proposed method was tested on several models: an etalon object and five biomedical objects that were generated in different ways. Two volume estimation approaches were used in the experiments: symbolic integration and numerical integration where the value of the parameter m was 10. The results of the experiments can be seen in Table 1. The experiments have proved the accuracy of the proposed volume estimation method. Based on the results of the experiments it is possible to conclude that using the numerical integration the error was bellow 0,2%.

The proposed method could be used to estimate volume of objects that are described by spline surfaces (in our case - Bezier triangles). If the model of the object is described using different kinds of patches (for example, triangular + rectangular), then the proposed method will only estimate a part of the object's volume. If the object is more precisely described with a polygonal mesh, then the volume can be estimated using the proposed method (in particular cases) or using the approach from [Zha01a].

Further research on this subject would be related to optimizing the volume estimation functions to decrease the number of necessary calculations and decrease the calculation time.

6. ACKNOWLEDGEMENT

The research was supported by the National Research Programme "The next generation of information and communication technologies" (NexIT, Project 2).

7. REFERENCES

[Baj96a] Bajaj, C., Cole, E. & Lin, K. Boundary and 3D triangular meshes from planar cross sections. Proceedings of the 5th International Imaging Roundtable, Sandia National Laboratories, Sandia Report SAND 96–2301, VC-405, Pittsburgh, PA, USA. pp. 169–178, 1996.

[Bres91a] Bresnahan, J.C., Beattie, M.S., Stokes, B.T. & Conway, K.M. (1991) Three dimensional computer-assisted analysis of graded contusion lesions in the spinal cord of the rat. J. Neurotrauma, 8, 91–97, 1991.

[Col11a] Colley, S.J. Vector Calculus (4th Edition), 2011, 624 p.

[Duer00b] Duerstock, B.S., Bajaj, C.L., Pascucci, V., Shikore, D., Lin, K. & Borgens, R.B. (2000) Advances in three-dimensional reconstruction of the experimental spinal cord injury. Comput. Med. Imag. Grap., 24, 389–406, 2000.

[Duer03a] Duerstock, B.S., Bajaj C. L., A comparative study of the quantitative accuracy of three-dimensional reconstructions of spinal cord from serial histological sections, Journal of Microscopy, Vol. 2010, Pt 2, pp. 138-148, 2003.

[How93a] Howard, C.V., et al. Measurement of total neuronal volume, surface area and dendrite length following intracellular physiological recording. Neuroprotocols. 2, 113–120, 1993.

[Juh00a] Juhász, I. Computing Volume of Solids Bounded by Bézier Surfaces. Mathematical Notes, Miskolc, Vol. 1., No. 2., 2000, pp. 127—133.

[Lu07a] Lu, L., Choi, Y.-K., Wang, W., Kim, M.-S. Variational 3D Shape Segmentation for Bounding Volume Computation. Journal Computer Graphics Forum, Volume 26, Issue 3, 2007, pp. 329–338.

[Mas09a] Massa Esteve, M. R., and Delshams, A. Euler's beta integral in Pietro mengoli's works. Archive for History of Exact Sciences, 2009, Volume 63, Issue 3, pp 325–356.

[Matt87a] Mattfeldt, T. Volume estimation of biological objects by systemic sampling. J. Math. Biol. 25, 685–695, 1987.

[McNul00a] McNulty, V., Cruz-Orive, L.M., Roberts, N., Holmes, C.J. & Gual-Arnau, X. Estimation of brain compartment Volume from MR Cavalieri slices. J. Comput. Assist. Tomogr. 24, 466–477, 2000.

[Mich88a] Michel, R.P. & Cruz-Orive, L.M. Application of the Cavalieri principle and vertical sections method to lung: estimation of Volume and pleural surface area. J. Microsc. 150, 117–136, 1988.

[Mor98a] Moriarty, L.J., Duerstock, B.S., Bajaj, C.L., Lin. K. & Borgens, R.B. Two and three dimensional computer graphic evaluation of the subacute spinal cord injury. J. Neurol. Sci. 155, 121–137.

[Ras05a] Rasmussen, A.F. and Floater M.S A point-based method for estimating surface area. Electronic Proceedings of the SIAM conference on Geometric Design, Phoenix, 2005.

[Sis09a] Sisojevs, A., Boločko, K. and Glazs, A. 3D Modeling of Free-Form Object (Interpolation, Visualization and Volume Estimation). The 17th International Conference WSCG'2009: Communication Papers Proceedings, 2009, pp. 125.-128.

[Sis12a] Sisojevs A., Kovalovs M. and Glazs A. Medical Object 3D Visualization Method Based on the Bézier Triangles. IADIS International Conference CGVCVIP'2012 proceedings. Lisbon, 2012, pp. 185 – 187.

[Sis13a] Sisojevs, A. An Approach of Semi-numerical Computing Volume of Solids Bounded by Rational Bézier Surfaces. Scientific Journal of RTU: Technologies of Computer Control, Vol.14, 2013, pp. 25.-31.

[Sun13a] Sun, F., Choi, Y.-K., Yu, Y., Wang, W. Medial meshes for volume approximation. http://dblp.uni-trier.de/db/journals/corr/corr1308.html#SunCYW13

[Wan06a] Wang, R., Zhou, K., Snyder, J., Liu, X., Bao, H., Peng, Q., Guo, B. Variational Sphere Set Approximation for Solid Objects. Journal Visual Computer, 2006, Vol. 22, Issue 9, pp. 612-621.

[Wro06a] Wronecki, J., Concept Modeling with NURBS, Polygon and Subdivision Surfaces. Proceedings of the 2006 American Society for Engineering Education Annual Conference & Exposition

[Zha01a] Zhang, C., and Chen, T. Efficient Feature Extraction for 2D/3D Objects in Mesh Representation. Proceedings of IEEE International Conference on Image Processing ICIP, 2001, pp.935-938.

8. APPENDIX

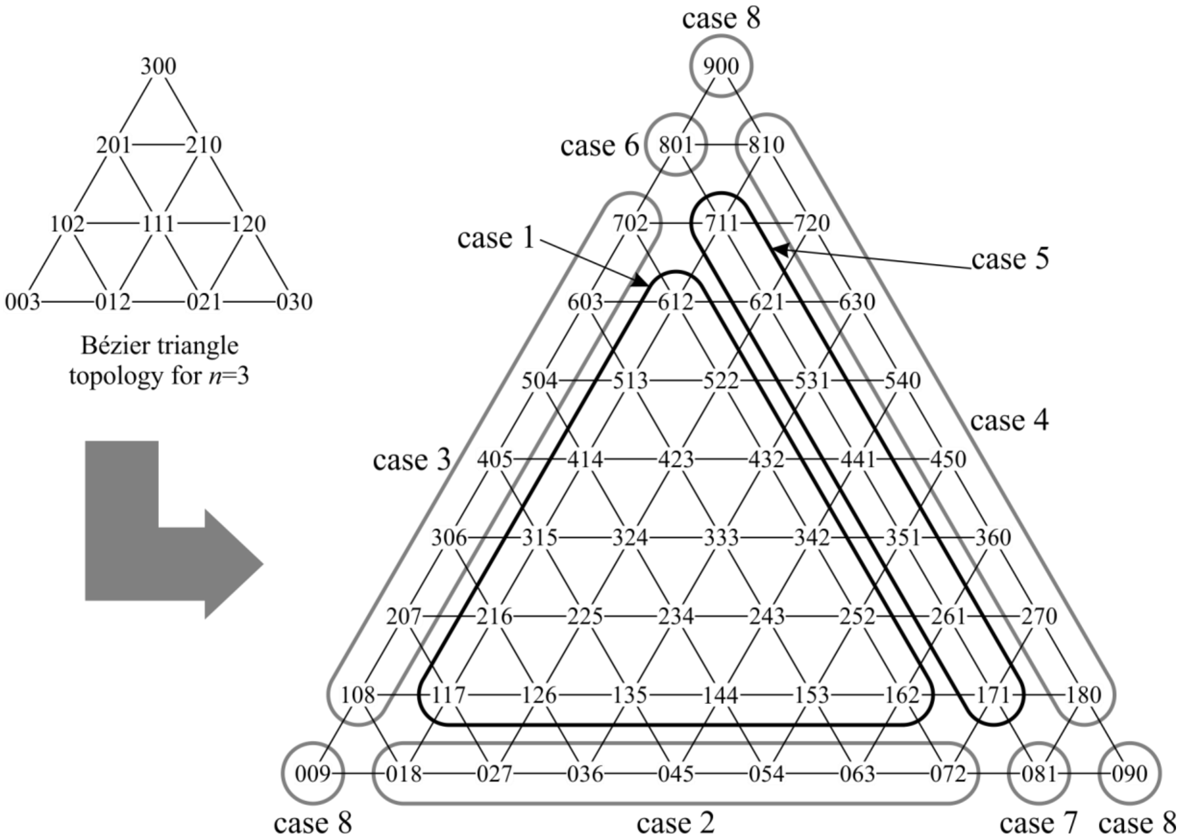

Figure 4. Possible values of *i*4, *j*4, *k*4 when *n*=3