Parametric Discretization of Supercover Simplex

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ABSTRACT

Within the framework of the development of a software which aims at unifying geometrical modelisation and image processing, based on both analytical and voxels representation of objects, we wish to extend this to parametrical objects. We focus, in this paper, on the study of parametrical simplexes and their relation with the discrete supercover model. Besides, we obtain a simple and incremental algorithm to run over n-D discrete objects. This is also the basis of parametrical curves and surfaces studies.

Keywords

Discrete geometry, geometrical and topological modelling, parametric rasterization.

1. INTRODUCTION

2D and 3D geometrical modelling and image processing do not use similar tools. Geometrical modelling needs high level informations such as topological and geometrical models, although image processing uses low level informations like pixels and voxels. In many applications, we would like to use both type of representations at the same time, for example virtual scalpels in a discrete image resulting of a scanner to simulate a medical operation. For a few years, a discrete modeler called SPAMOD (for *Spatial Modeler*) is being developed [DDLA05]. The purpose of this modeler is to permit, in the same software, to have a representation of an object in an Euclidean form together with a discrete form. The various forms (embeddings) of the object are hold in a coherent way using a hierarchical structure based on topology.

The hierarchical structure of SPAMOD, on one hand, a geometrical modeler based on topology. On the other hand, on a discrete analytical model, the supercover model [ANF97], sets up the pixels and voxels. In a discrete analytical model, we can describe a discrete object with a number of inequations which do not depend on the total number of pixels that compose the object. So, for the supercover model, a 3D triangle of whatever size is described by 17 inequations at the most. In fact, these inequations represent the convex hull of the voxels, and this convex hull is connected to the boundary representation of the Euclidean geometrical models.

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The purpose of our study is to provide an appropriate parametrical description of supercover objects unlike [Kau87]. The first step, which is developed in this paper, aims at proposing a parametrical description of the supercover simplexes thanks to a parametrical method and also its discretisation algorithms.

First, in Section 2, with the example of a 2D segment, we describe the principle on which the parametrical description of a simplex works. We extend our analysis to simplexes of higher dimensions in Section 3, and conclude in Section 4.

2. A PARAMETRICAL APPROACH OF THE SEGMENTS

We focus, in a first step, on the 1-dimensional simplex (straight line segment). Let be a segment defined by two points P and Q which have these coordinates (P_x, P_y) and (Q_x, Q_y) respectively. We will study this problem with a parametrical approach. The segment PQ can also be represented by a piece of parametrical line, such as $\vec{PX} = t\ \vec{PQ}$ with t between 0 and 1 :

$$D(x,y) = \begin{cases} bt \\ at \end{cases} \text{, with } t \in [0,1]$$

We focus on the supercover discretization of points which have coordinates (bt, at) with t between 0 and 1. This parametrical formula has the advantage to show that the incremental step on X (respectively on Y) is only linked with the parameter t and the constant value b (respectively a).

Our goal is to characterise the values of t for which an incremental step on X or on Y is defined. To do this, we can compute the value of t where the segment cuts a pixel border, that is a point with one half-integer coordinate. We deduce a series S_x (respectively S_y) of parameters t where we have an incrementation on X (respectively Y) : $S_x(k)$ series of t such as

$$bt = k + \frac{1}{2} \Rightarrow t = \frac{2k+1}{2b} \text{ for } k \geq 0$$

and $S_y(k)$ series of t such as

$$at = k + \frac{1}{2} \Rightarrow t = \frac{2k+1}{2a} \text{ for } k \geq 0$$

Now, we only have to sort the two series S_x and S_y to know what order of incrementations on X or on Y we have to make so as to draw the line.

For limiting the line to the segment PQ, we only need to restrict t between 0 and 1, that is we only need to take values of the two series S_x and S_y with k integer from 0 to $b-1$ (respectively $a-1$) for S_x (respectively S_y).

This construction is directly extendable in 3D. We only have to sort not two series but three S_x, S_y and S_z.

3. SIMPLEXES OF HIGHER DIMENSIONS

Now, we extend this principle to the 2D simplex: the triangle. Let P_1, P_2 and P_3 be three points with coordinates (X_1, Y_1), (X_2, Y_2) et (X_3, Y_3) respectively. A point U belongs to the triangle if and only if we can write: $\overrightarrow{P_1U} = t\ \overrightarrow{P_1P_2} + t'\ \overrightarrow{P_1P_3}$ with $t, t', t+t' \in [0, 1]$.

We use a parametrical representation to obtain a new supercover discretisation algorithm. With A_1 and B_1 (resp. A_2 and B_2) are the directrix coefficients of the line which contains the segment P_1P_2 (resp. P_1P_3) we have :

$$T(x, y) = \begin{cases} A_1t + A_2t' \\ B_1t + B_2t' \end{cases} \text{, with } t, t' \in [0, 1]$$

From these equations, we can deduce an incremental step on X (respectively on Y) when $A_1t + A_2t' = \frac{2k+1}{2}$ (respectively $B_1t + B_2t' = \frac{2k+1}{2}$) for $k \geq 0$. We deduce the two following series S_x and S_y :

$$S_x(k) = \{t, t'\} \text{ such as } A_1t + A_2t' = \frac{2k+1}{2} \text{ for } k \geq 0$$

and

$$S_y(k) = \{t, t'\} \text{ such as } B_1t + B_2t' = \frac{2k+1}{2} \text{ for } k \geq 0$$

With a geometrical interpretation of these two series, in a space built by the parameters t and t', we obtain that S_x (respectively S_y) is nothing else but the series of parallel lines such as $A_1t + A_2t' = \frac{2k+1}{2}$ (respectively $B_1t + B_2t' = \frac{2k+1}{2}$) for $k \geq 0$.

If we study the relative position of these two sets of lines. We can find what incremental step we must do without explicitly computing the values of t et t', using only a comparison.

As for segments, we do not need to work with two series but three to obtain a 3D incremental algorithm of supercover discretisation for a discrete triangle.

In practice, the algorithm is based on a notion of order built on parameters t and t', providing a traversal order on the pixels or voxels of the triangle. This order provides a direct generation of all voxels instead of an algorithm based on 2D-filling and then retro-projection in 3D space.

It is also possible to apply the same process for a 3D simplex (tetrahedron). With the same principle, we have to study the relative position of three pieces of planes, which can be computed by many differents methods [EOS86]. For higher dimensions, we only need to check if the geometrical objects we need to handle, always allow us to extend our algorithm.

4. CONCLUSION

In this paper, we show, for the supercover model, a parametrical representation of the supercover of various discrete simplexes. It provides a natural order for the pixels, voxels of simplexes. In discrete modelling, we often have to run over a discrete object so as to realise various operation. As soon as we do not handle a line segment, we loose the notion of order. In the higher dimensions, coverage algorithms are often complicated. In this particular application we hope that parametrical description could help.

5. REFERENCES

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