University of West Bohemia
Faculty of Applied Sciences
Department of Mathematics

BACHELOR'S THESIS

The Fučík spectrum for problems with nonlocal boundary conditions

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Prohlášení

Prohlašuji, že jsem bakalářskou práci vypracoval pod vedením vedoucího bakalářské práce samostatně za použití v práci uvedených pramenů a literatury.

V Plzni, dne 20. července 2020

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podpis

Poděkování

Děkuji mému vedoucímu, panu Ing. Petru Nečesalovi, Ph.D., za dohled nad prací a cenné rady.

Abstrakt

V této práci vyšetřujeme Fučíkovo spektrum okrajové úlohy druhého řádu s jednou Robinovou a jednou nelokální okrajovou podmínkou. Dokážeme, že příslušná lineární úloha má nekonečně mnoho vlastních čísel a poskytneme jejich popis. Představíme implicitní popis Fučíkova spektra v prvním kvadrantu. Pro speciální nastavení parametrů také dokážeme, že se Fučíkovo spektrum skládá ze dvou spojitých křivek a najdeme parametrizaci těchto křivek.

Klíčová slova: Fučíkovo spektrum, nelokální okrajové podmínky, Robinova podmínka, vlastní čísla

Abstract

In this thesis, we investigate the Fučík spectrum for the second order boundary value problem with one Robin and one non-local boundary conditions. We prove that the corresponding linear boundary value problem has infinitely many eigenvalues and we provide the description of these eigenvalues. We present a compact form of the implicit description of the Fučík spectrum in the first quadrant. We also prove for a specific setting of the parameters, that the Fučík spectrum consists of two continuous curves and the parametrization of these curves is provided.

Keywords: Fučík spectrum, non-local boudary condition, Robin condition, eigenvalues

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Chapter 1

Introduction

The goal of this thesis is to investigate the Fučík spectrum for the following boundary value problem with one Robin and one non-local boundary conditions

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, & x \in (0,1), \\ u(0) \cdot \sin c = u'(0) \cdot \cos c, & \int_0^1 u(x)\, \mathrm{d}x = 0, \end{cases} \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}$ and $\frac{\pi}{2} < c \le \frac{\pi}{2}$. By the Fučík spectrum for the problem (1.1), we mean the set (see Figure 1.1 and 1.2)

$$\Sigma_c := \Big\{ (\alpha, \beta) \in \mathbb{R}^2 : \text{the problem (1.1) has a non-trivial solution } u \Big\}.$$

In the special case of $c = \frac{\pi}{2}$, the problem (1.1) reads

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, & x \in (0,1), \\ u(0) = 0, & \int_0^1 u(x)\, \mathrm{d}x = 0, \end{cases}$$

and its corresponding Fučík spectrum $\Sigma_{\frac{\pi}{2}}$ is investigated in details in [3]. Thus, in this thesis, we focus mainly to values of the parameter $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we use the paper [3] as a starting point for our study and we also follow the notation used in [3].

For $\alpha = \beta = \lambda$, the problem (1.1) reduces to the following linear problem

$$\begin{cases} u''(x) + \lambda u(x) = 0, & x \in (0,1), \\ u(0) \cdot \sin c = u'(0) \cdot \cos c, & \int_0^1 u(x)\, \mathrm{d}x = 0, \end{cases} \tag{1.2}$$

and its eigenvalues λ (the values $\lambda \in \mathbb{R}$ such that the problem (1.2) has a non-trivial solution u) determine points $(\lambda, \lambda) \in \Sigma_c$ on the diagonal $\alpha = \beta$. For $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we show that $\lambda > 0$ is an eigenvalue for the problem (1.2) if and only if

$$\cos\left(\sqrt{\lambda} - \sqrt{\lambda} \cdot p\left(\sqrt{\lambda}, c\right)\right) = \cos\left(\sqrt{\lambda} \cdot p\left(\sqrt{\lambda}, c\right)\right), \tag{1.3}$$

where $p\left(\sqrt{\lambda}, c\right) = -\frac{1}{\sqrt{\lambda}} \operatorname{arccot}\left(\frac{1}{\sqrt{\lambda}} \tan c\right)$. Moreover, let us note that in the special case of $c = 0$, the equation (1.3) simplifies to

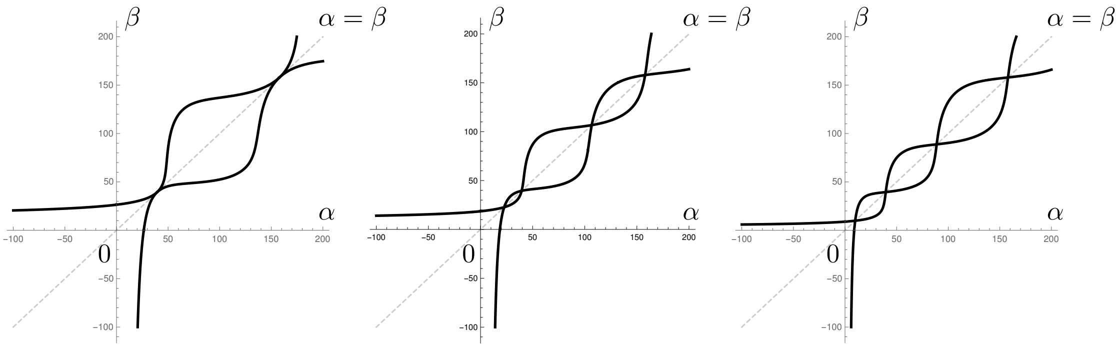
$$\cos\left(\sqrt{\lambda} + \frac{\pi}{2}\right) = 0,$$

the linear problem (1.2) reduces to

$$\begin{cases} u''(x) + \lambda u(x) = 0, & x \in (0,1), \\ u'(0) = 0, & \int_0^1 u(x)\, \mathrm{d}x = 0, \end{cases}$$

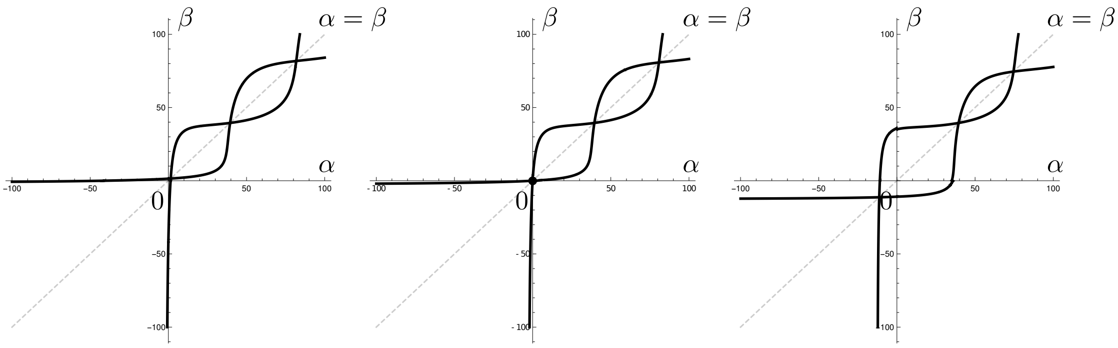
and all its positive eigenvalues are of the form $\lambda_k = k^2 \pi^2$, $k \in \mathbb{N}$.

For the original problem (1.1), we provide the following description for its Fučík spectrum Σ_c in the first quadrant of the $\alpha\beta$-plane. For $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\alpha, \beta > 0$, we prove that $(\alpha, \beta) \in \Sigma_c$ if and only if



(a) $c = \frac{\pi}{2}$ (b) $c = \frac{7\pi}{16}$ (c) $c = 0$

Fig. 1.1: The Fučík spectrum Σ_c for different non-negative values of the parameter c.



(a) $c = -1.05$ (b) $c = \arctan(-2)$ (c) $c = -1.3$

Fig. 1.2: The Fučík spectrum Σ_c for different negative values of the parameter c.

$$\mathcal{G}\left(\sqrt{\alpha}, \sqrt{\beta}, \frac{2\sqrt{\alpha\beta}}{\sqrt{\alpha}+\sqrt{\beta}}\right) = 1 + \frac{2}{\pi}\left(\sqrt{\beta} - \sqrt{\alpha}\right) \tag{1.4}$$

or

$$\mathcal{G}\left(\sqrt{\beta}, \sqrt{\alpha}, \frac{2\sqrt{\alpha\beta}}{\sqrt{\alpha}+\sqrt{\beta}}\right) = 1 + \frac{2}{\pi}\left(\sqrt{\alpha} - \sqrt{\beta}\right), \tag{1.5}$$

where the function $\mathcal{G} = \mathcal{G}(a, b, t)$ is 2π-periodic in the third variable t and is defined by parts in Definition 3. Let us point out that for $\alpha = \beta = \lambda$, we have

$$\mathcal{G}\left(\sqrt{\lambda}, \sqrt{\lambda}, t\right) = \cos\left(t - \sqrt{\lambda} \cdot p\left(\sqrt{\lambda}, c\right)\right) - \cos\left(\sqrt{\lambda} \cdot p\left(\sqrt{\lambda}, c\right)\right) + 1$$

and thus, both equations (1.4) and (1.5) can be simplified to (1.3) in this case.

Finally, the next chapters are organized in the following way. The second chapter is devoted to the linear problem (1.2) and we provide the description of all its eigenvalues. In the third chapter, we return back to the original problem (1.1) and provide the implicit description of the Fučík spectrum Σ_c in the first quadrant and partially also in the second and fourth quadrants of the $\alpha\beta$-plane. The fourth chapter is devoted to the problem (1.1) for the special case of $c = 0$ and we show how to get the parametrization of the Fučík spectrum Σ_0 by two continuous curves.

Chapter 2

Solvability of linear problems

In this chapter, we study the following linear boundary value problem

$$\begin{cases} u''(x) + \lambda u(x) = 0, & x \in (0,1)\,, \\ u\,(0) \cdot \sin c = u'\,(0) \cdot \cos c, & \int_0^1 u(x)\,\mathrm{d}x = 0, \end{cases} \tag{2.1}$$

where $\lambda \in \mathbb{R}$ and $-\frac{\pi}{2} < c \leq \frac{\pi}{2}$. More precisely, for fixed $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we describe all values $\lambda \in \mathbb{R}$ such that the problem (2.1) has a non-trivial solution u, i.e. we describe all eigenvalues for the problem (2.1). By a non-trivial solution of the problem (2.1) we mean a function $u \in C^2(0,1) \cap C^1\,[0,1]$ which is not identically zero and satisfies the differential equation in (2.1) pointwise on the interval $(0,1)$, satisfies the Robin boundary condition at the point $x = 0$ (containing the parameter c) and the non-local boundary condition of integral form.
The general solution of the linear differential equation $u''(x) + \lambda u(x) = 0$, $x \in \mathbb{R}$, can be expressed as

$$u(x) = \begin{cases} \frac{C_1}{\sqrt{\lambda}} \sin \sqrt{\lambda} x + C_0 \cos \sqrt{\lambda} x & \text{for } \lambda > 0, \\ A \cdot (x - x_0) & \text{for } \lambda = 0, \\ \frac{C_1}{\sqrt{-\lambda}} \sinh \sqrt{-\lambda} x + C_0 \cosh \sqrt{-\lambda} x & \text{for } \lambda < 0, \end{cases} \tag{2.2}$$

where $C_0, C_1, A, x_0 \in \mathbb{R}$.

In the following lemma, we investigate eigenvalues of the problem (2.1) in the special case of $c = \frac{\pi}{2}$.

Lemma 1. *For $c = \frac{\pi}{2}$, there are infinitely many eigenvalues for the boundary value problem* (2.1). *All these eigenvalues are positive and form the sequence $(\lambda_k^{\mathrm{I}})_{k=1}^{+\infty}$, where $\lambda_k^{\mathrm{I}} = 4k^2\pi^2$, $k \in \mathbb{N}$.*

Proof. For $c = \frac{\pi}{2}$, the first boundary condition in (2.1), i.e. $u(0) \cdot \sin c = u'(0) \cdot \cos c$, reads

$$u(0) = 0, \tag{2.3}$$

and the integral condition in (2.1) remains the same as

$$\int_0^1 u(x)\,\mathrm{d}x = 0. \tag{2.4}$$

Now we split the proof according to the value of λ.

1. For $\lambda > 0$, we obtain using the general solution in (2.2) that

$$u(0) = C_0.$$

Therefore $C_0 = 0$ due to (2.3) and thus, all solutions u of the differential equation in (2.1) such that the first boundary condition is satisfied are of the following form

$$u(x) = \frac{C_1}{\sqrt{\lambda}} \sin \sqrt{\lambda} x.$$

Let us assume $C_1 \neq 0$ to obtain a non-trivial solution u and let us evaluate the following integral

$$\begin{aligned}\int_0^1 u(x)\,\mathrm{d}x &= \int_0^1 \frac{C_1}{\sqrt{\lambda}} \sin\sqrt{\lambda}x\,\mathrm{d}x \\ &= -\frac{C_1}{\sqrt{\lambda}}\left[\frac{1}{\sqrt{\lambda}}\cos\sqrt{\lambda}x\right]_0^1 \\ &= -\frac{C_1}{\lambda}\left(\cos\sqrt{\lambda}-1\right).\end{aligned}$$

The integral condition (2.4) is thus satisfied if and only if $\cos\sqrt{\lambda} = 1$ or if and only if (recall $\lambda > 0$)

$$\lambda = 4k^2\pi^2, k \in \mathbb{N}.$$

2. For $\lambda = 0$, the linear differential equation in (2.1) simplifies to $u''(x) = 0$ and thus the general solution u can be expressed for $A, x_0 \in \mathbb{R}$ as

$$u(x) = A \cdot (x - x_0).$$

This solution is trivial for $A = 0$ and thus let us assume that $A \neq 0$ in the following text.

Now we will evaluate the integral

$$\begin{aligned}\int_0^1 u(x)\,\mathrm{d}x &= \int_0^1 A \cdot (x - x_0)\,\mathrm{d}x \\ &= A\left[\frac{x^2}{2} - x_0 x\right]_0^1 \\ &= A\left(\frac{1}{2} - x_0\right).\end{aligned}$$

Using (2.4), we get $x_0 = \frac{1}{2}$, since $A \neq 0$. Thus we have $u(0) = -\frac{A}{2} \neq 0$, which is a contradiction with (2.3). For $c = \frac{\pi}{2}$, $\lambda = 0$ is not an eigenvalue for the problem (2.1).

3. For $\lambda < 0$, the linear differential equation in (2.1) has the general solution

$$u(x) = \frac{C_1}{\sqrt{-\lambda}}\sinh\sqrt{-\lambda}x + C_0\cosh\sqrt{-\lambda}x$$

for $C_0, C_1 \in \mathbb{R}$ and we get the value $u(0) = C_0$. Thus $C_0 = 0$ according to the condition (2.3). Now we can evaluate the integral

$$\begin{aligned}\int_0^1 u(x)\,\mathrm{d}x &= \int_0^1 \frac{C_1}{\sqrt{-\lambda}}\sinh\sqrt{-\lambda}x\,\mathrm{d}x \\ &= \frac{C_1}{\sqrt{-\lambda}}\left[\frac{1}{\sqrt{-\lambda}}\cosh\sqrt{-\lambda}x\right]_0^1 \\ &= \frac{C_1}{-\lambda}\left(\cosh\sqrt{-\lambda}-1\right).\end{aligned}$$

We assume $C_1 \neq 0$ to achieve a non-trivial solution. Therefore the integral condition (2.4) reads $\cosh\sqrt{-\lambda} = 1$, which cannot be satisfied since $\lambda < 0$. Thus, there are no negative eigenvalues for $c = \frac{\pi}{2}$.

□

Now, let us discuss the case $\lambda > 0$ for $-\frac{\pi}{2} < c \leq \frac{\pi}{2}$. For fixed $C_0, C_1 \in \mathbb{R}$, the function u in (2.2) can be also viewed as the solution of the following initial value problem (see Figure 2.1)

$$\begin{cases} u''(x) + \lambda u(x) = 0, & x \in \mathbb{R}, \\ u(0) = C_0, \quad u'(0) = C_1. \end{cases} \tag{2.5}$$

The following lemma gives us another form of the general solution (2.2) for $\lambda > 0$ (see Figure 2.2).

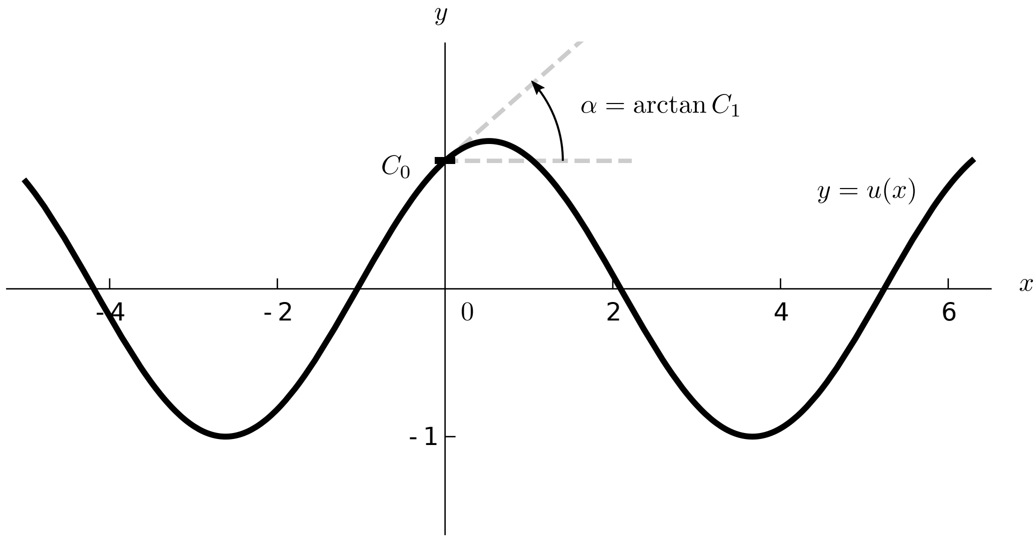


Fig. 2.1: The solution u of the initial value problem (2.5) for $\lambda = 1$, $C_0 = \frac{\sqrt{3}}{2}$, $C_1 = \frac{1}{2}$.

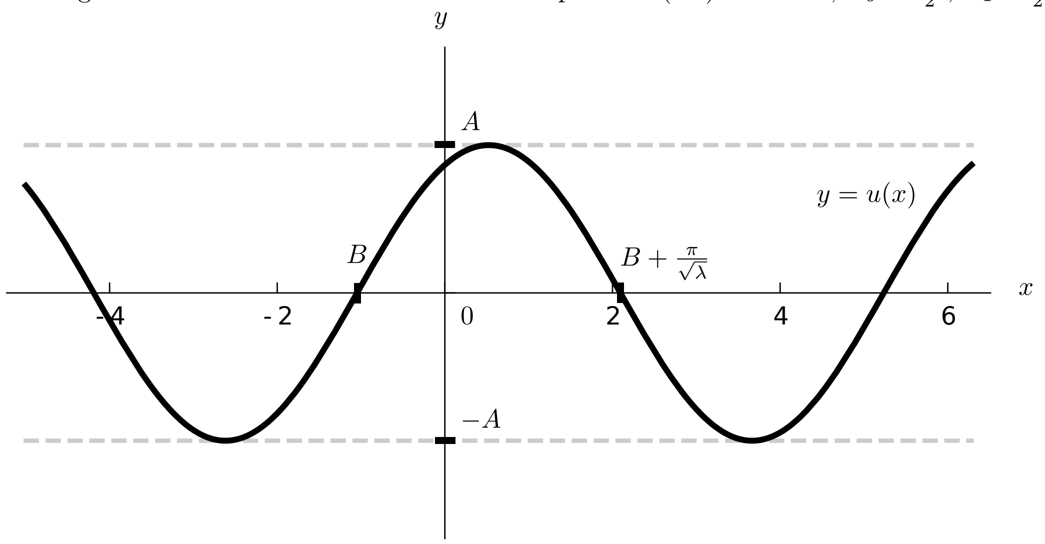


Fig. 2.2: The solution u of the initial value problem (2.5) in the form of (2.6) for $\lambda = 1$, $A = 1$, $B = -\frac{\pi}{3}$.

Lemma 2. *For $\lambda > 0$, the general solution of the linear equation*

$$u''(x) + \lambda u(x) = 0, \qquad x \in \mathbb{R},$$

is given by

$$u(x) = A \sin\left(\sqrt{\lambda}\,(x - B)\right), \qquad A \in \mathbb{R},\ B \in \left(-\tfrac{\pi}{\sqrt{\lambda}}, 0\right]. \tag{2.6}$$

Proof. We prove that for $\lambda > 0$, the general solution in the form of (2.2) can be equivalently written as (2.6).

1. Let's start with (2.6), i.e. with $u(x) = A \sin\left(\sqrt{\lambda}\,(x - B)\right),\ A \in \mathbb{R},\ B \in \left(-\frac{\pi}{\sqrt{\lambda}}, 0\right]$. We can manipulate the expression

$$\begin{aligned} u(x) &= A \sin\left(\sqrt{\lambda}\,(x - B)\right) \\ &= A \sin\left(\sqrt{\lambda}x - \sqrt{\lambda}B\right) \\ &= A \sin\left(\sqrt{\lambda}x\right) \cos\left(\sqrt{\lambda}B\right) + A \cos\left(\sqrt{\lambda}x\right) \sin\left(-\sqrt{\lambda}B\right). \end{aligned}$$

Now we can denote

$$C_1 = \sqrt{\lambda} A \cos\left(-\sqrt{\lambda}B\right), \tag{2.7}$$

and

$$C_0 = A \sin\left(-\sqrt{\lambda}B\right). \tag{2.8}$$

Therefore we have $C_0, C_1 \in \mathbb{R}$ and we achieved the desired outcome in (2.2), where

$$u(x) = \frac{C_1}{\sqrt{\lambda}} \sin\sqrt{\lambda}x + C_0 \cos\sqrt{\lambda}x.$$

2. Secondly, we find constants A and B to the fixed C_0 and C_1 in order to express (2.6). Thus we will manipulate the expression (2.2).

   (a) In the case of $C_0 = 0$, we have

   $$u(x) = \frac{C_1}{\sqrt{\lambda}} \sin\sqrt{\lambda}x = \frac{C_1}{\sqrt{\lambda}} \sin\sqrt{\lambda}(x-0),$$

   and we can define

   $$A = \frac{C_1}{\sqrt{\lambda}} \tag{2.9}$$

   and

   $$B = 0. \tag{2.10}$$

   We have $C_1 \in \mathbb{R}$ and therefore $A \in \mathbb{R}$ as well. With new variables, we have achieved (2.6), where

   $$u(x) = A \sin\left(\sqrt{\lambda}\,(x-B)\right).$$

   (b) Finally, let's take a look at the case $C_0 \neq 0$. We can define

   $$B = -\frac{1}{\sqrt{\lambda}} \operatorname{arccot}\left(\frac{1}{\sqrt{\lambda}} \cdot \frac{C_1}{C_0}\right), \tag{2.11}$$

   with values $-\frac{\pi}{\sqrt{\lambda}} < B < 0$, due to the range of the function arccot. In the case of $C_1 = 0$, we have

   $$u(x) = C_0 \cos\left(\sqrt{\lambda}x\right) = C_0 \sin\left(\sqrt{\lambda}\left(x + \frac{\pi}{2\sqrt{\lambda}}\right)\right),$$

   which is the form of $u(x)$ in (2.6) for $A = C_0$ and $B = -\frac{\pi}{2\sqrt{\lambda}}$. It remains to deal with $C_1 \neq 0$. Thus, let us assume that $C_1 \neq 0$ and rewrite (2.2) into the following form

   $$\begin{aligned} u(x) &= \frac{C_1}{\sqrt{\lambda}} \sin\sqrt{\lambda}x + C_0 \cos\sqrt{\lambda}x \\ &= \frac{C_1}{\sqrt{\lambda}\cos(-\sqrt{\lambda}B)} \cdot \sin\sqrt{\lambda}x \cdot \cos(-\sqrt{\lambda}B) + \\ &\quad \frac{C_0}{\sin(-\sqrt{\lambda}B)} \cdot \cos\sqrt{\lambda}x \cdot \sin(-\sqrt{\lambda}B). \end{aligned}$$

   Using (2.11) we obtain

   $$\frac{C_1}{\sqrt{\lambda}\cos(-\sqrt{\lambda}B)} = \frac{C_0}{\sin(-\sqrt{\lambda}B)},$$

   and therefore

   $$\begin{aligned} u(x) &= \frac{C_0}{\sin(-\sqrt{\lambda}B)} \cdot \left(\sin\sqrt{\lambda}x \cdot \cos(-\sqrt{\lambda}B) + \cos\sqrt{\lambda}x \cdot \sin(-\sqrt{\lambda}B)\right) \\ &= \frac{C_0}{\sin(-\sqrt{\lambda}B)} \cdot \sin\sqrt{\lambda}(x-B) \\ &= \frac{C_0}{\sin\operatorname{arccot}\left(\frac{C_1}{\sqrt{\lambda}C_0}\right)} \cdot \sin\sqrt{\lambda}(x-B). \end{aligned}$$

Since for all $x \in \mathbb{R}$, we have $\sin \operatorname{arccot}(x) = \frac{1}{\sqrt{1+x^2}}$ (see Chapter 35 in [4]), we obtain

$$u(x) = C_0 \sqrt{1 + \frac{1}{\lambda} \cdot \frac{C_1^2}{C_0^2}} \cdot \sin \sqrt{\lambda}(x - B).$$

Now let's define

$$A = C_0 \sqrt{1 + \frac{1}{\lambda} \cdot \frac{C_1^2}{C_0^2}}, \tag{2.12}$$

where we have $A \in \mathbb{R}$, since $C_0, C_1 \in \mathbb{R}$, and we have

$$u(x) = A \sin \sqrt{\lambda}(x - B),$$

which finishes the proof.

□

Remark 1. *For $\lambda > 0$, we have obtained two possible expressions of the general solution for the linear differential equation $u''(x) + \lambda x = 0$, $x \in (0, 1)$ of the problem (2.1).*

1. *The first expression (2.2), i.e. $u(x) = \frac{C_1}{\sqrt{\lambda}} \sin \sqrt{\lambda}x + C_0 \cos \sqrt{\lambda}x$, for $C_0, C_1 \in \mathbb{R}$, yields a trivial solution if and only if $C_0 = C_1 = 0$.*
2. *The second expression (2.6), i.e. $u(x) = A \sin\left(\sqrt{\lambda}\,(x - B)\right)$, for $A \in \mathbb{R}$, $B \in \left(-\frac{\pi}{\sqrt{\lambda}}, 0\right]$, yields a trivial solution if and only if $A = 0$.*

The proof of Lemma 2 showed us a way of converting between two different expressions (2.2) and (2.6) of the general solution of (2.1), i.e. in between $u(x) = \frac{C_1}{\sqrt{\lambda}} \sin \sqrt{\lambda}x + C_0 \cos \sqrt{\lambda}x,\ C_0, C_1 \in \mathbb{R}$ and $u(x) = A \sin\left(\sqrt{\lambda}\,(x - B)\right),\ A \in \mathbb{R},\ -\frac{\pi}{\sqrt{\lambda}} < B \leq 0$. The conversion from (2.6) to (2.2) can be achieved by (2.7) and (2.8), i.e. by placing

$$\begin{aligned} C_0 &= A \sin\left(-\sqrt{\lambda}B\right), \\ C_1 &= \sqrt{\lambda} A \cos\left(-\sqrt{\lambda}B\right). \end{aligned}$$

And the conversion from (2.2) to (2.6) can be achieved by (2.9), (2.10), (2.11), and (2.12), i.e. by placing

$$A = \begin{cases} C_0 \cdot \sqrt{1 + \frac{1}{\lambda} \cdot \frac{C_1^2}{C_0^2}} & \text{for } C_0 \neq 0, \\ \sqrt{\lambda} C_1 & \text{for } C_0 = 0, \end{cases}$$

$$B = \begin{cases} -\frac{1}{\sqrt{\lambda}} \operatorname{arccot}\left(\frac{1}{\sqrt{\lambda}} \cdot \frac{C_1}{C_0}\right) & \text{for } C_0 \neq 0, \\ 0 & \text{for } C_0 = 0. \end{cases}$$

Using the general solution in the form of (2.6), we examine eigenvalues of the boundary value problem (2.1) for $\lambda > 0$.

Lemma 3. *For $\lambda > 0$, the boundary value problem (2.1) has a non-trivial solution if and only if*

$$\cos\left(\sqrt{\lambda} - \sqrt{\lambda} \cdot p(\sqrt{\lambda}, c)\right) = \cos\left(\sqrt{\lambda} \cdot p(\sqrt{\lambda}, c)\right), \tag{2.13}$$

where the function $p : \mathbb{R}^+ \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ is defined as

$$p(l, c) := \begin{cases} -\frac{1}{l} \operatorname{arccot}\left(\frac{1}{l} \tan c\right) & \text{for } c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 0 & \text{for } c = \frac{\pi}{2}. \end{cases} \tag{2.14}$$

Proof. Using Lemma 2, the general solution of the differential equation in the boundary value problem (2.1) is given by $u(x) = A\sin\left(\sqrt{\lambda}(x-B)\right), \quad A \in \mathbb{R},\, B \in \left(-\frac{\pi}{\sqrt{\lambda}}, 0\right]$. Thus, we get

$$u'(x) = A\sqrt{\lambda}\cos\left(\sqrt{\lambda}(x-B)\right).$$

Using the first boundary condition in (2.1), i.e.

$$u(0)\cdot\sin c = u'(0)\cdot\cos c, \tag{2.15}$$

we can calculate B with respect to the value of c.

1. In the case of $c = \frac{\pi}{2}$, the boundary condition (2.15) reads

   $$u(0) = 0$$

   and thus we have

   $$A\sin\left(-\sqrt{\lambda}B\right) = 0.$$

   To achieve a non-trivial solution, we have $A \neq 0$ (see Remark 1) and we obtain

   $$B = 0.$$

2. Now, let us fix $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Since $\cos c \neq 0$, we can manipulate the condition (2.15) in the following way

   $$\begin{aligned}\frac{u'(0)}{u(0)} &= \frac{\sin c}{\cos c},\\ \frac{u'(0)}{u(0)} &= \tan c,\end{aligned}$$

   and therefore using (2.11)

   $$\begin{aligned}B &= -\frac{1}{\sqrt{\lambda}}\operatorname{arccot}\left(\frac{1}{\sqrt{\lambda}}\cdot\frac{C_1}{C_0}\right)\\ &= -\frac{1}{\sqrt{\lambda}}\operatorname{arccot}\left(\frac{1}{\sqrt{\lambda}}\cdot\frac{u'(0)}{u(0)}\right)\\ &= -\frac{1}{\sqrt{\lambda}}\operatorname{arccot}\left(\frac{1}{\sqrt{\lambda}}\tan c\right).\end{aligned}$$

To conclude, we get that $B = p(\sqrt{\lambda}, c)$, where the function p is defined in (2.14). Moreover, we have another expression for solutions of the differential equation in (2.1), such that the first condition of (2.1) holds

$$u(x) = A\sin\left(\sqrt{\lambda}\left(x - p\left(\sqrt{\lambda},\, c\right)\right)\right), \quad A \in \mathbb{R},\, c \in \left(-\tfrac{\pi}{2},\, \tfrac{\pi}{2}\right].$$

The integral condition in (2.1) reads

$$\int_0^1 u(x)\,\mathrm{d}x = 0, \tag{2.16}$$

where u is given by $u(x) = A\sin\left(\sqrt{\lambda}\left(x - p(\sqrt{\lambda}, c)\right)\right)$. Therefore we get

$$\begin{aligned}\int_0^1 A\sin\left(\sqrt{\lambda}\left(x - p\left(\sqrt{\lambda},\, c\right)\right)\right)\mathrm{d}x &= -A\frac{1}{\sqrt{\lambda}}\left[\cos\left(\sqrt{\lambda}(x - p(\sqrt{\lambda}, c))\right)\right]_0^1\\ &= -A\frac{1}{\sqrt{\lambda}}\left(\cos\left(\sqrt{\lambda}(1 - p(\sqrt{\lambda}, c))\right) - \cos\left(\sqrt{\lambda}\cdot p(\sqrt{\lambda}, c)\right)\right),\end{aligned}$$

and thus the integral condition (2.16) is justified if and only if

$$\cos\left(\sqrt{\lambda}(1 - p(\sqrt{\lambda}, c))\right) = \cos\left(\sqrt{\lambda}\cdot p(\sqrt{\lambda}, c)\right), \quad c \in \left(-\tfrac{\pi}{2},\, \tfrac{\pi}{2}\right].$$

□

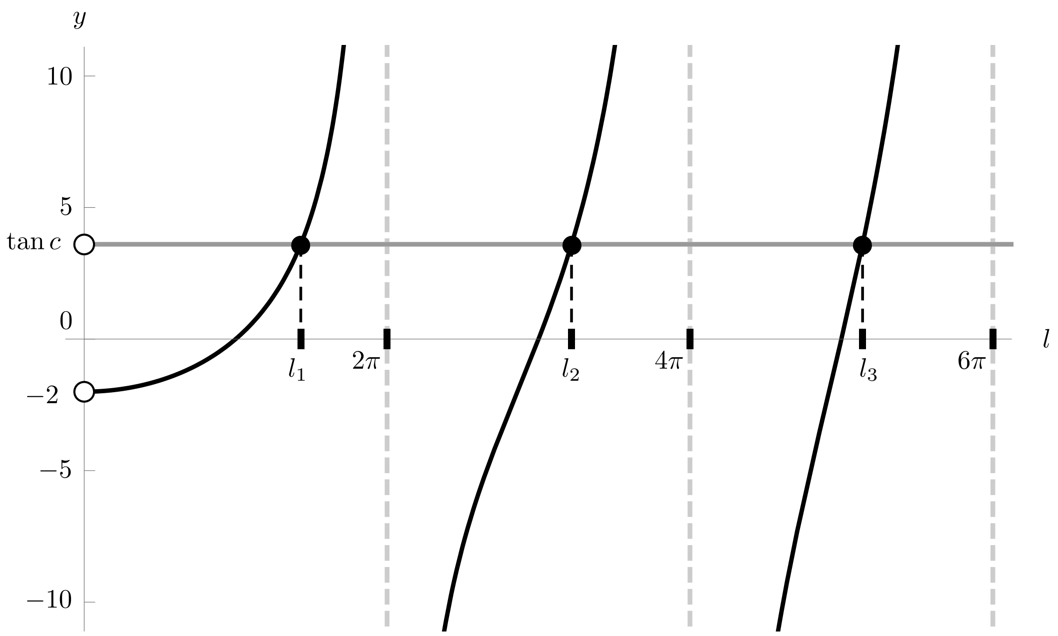


Fig. 2.3: The graph of the non-linear function $y = -l \cdot \cot\left(\frac{l}{2}\right)$ (black curves) and the graph of the constant function $y = \tan c$ for $c = 1.3$ (the grey line).

In the following two lemmas, we show that the solutions λ of the equation (2.13) (and therefore the eigenvalues of the boundary value problem (2.1)) can be figured out using a simpler non-linear equation (see Figure 2.3).

Lemma 4. *For* $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, *the equation*

$$-\,l \cdot \cot\left(\frac{l}{2}\right) = \tan c \tag{2.17}$$

has infinitely many solutions $l > 0$. *All these positive solutions form the sequence* $(l_k)_{k=1}^{+\infty}$ *for* $\tan c > -2$ *and the sequence* $(l_k)_{k=2}^{+\infty}$ *for* $\tan c \leq -2$. *Moreover, we have that*

$$2\,(k-1)\,\pi < l_k < 2k\pi, \quad k \in \mathbb{N}. \tag{2.18}$$

Proof. Let us define the function g as the left hand side of the equation (2.17)

$$g(l) := -l \cdot \cot\left(\frac{l}{2}\right), \quad l > 0,\ l \neq 2k\pi,\ k \in \mathbb{N},$$

the interval

$$I_k := (2(k-1)\pi,\, 2k\pi)\,,$$

and finally let us denote by g_k the restriction of g on the interval I_k.

We prove, that for all $k \in \mathbb{N}$, the function g_k is strictly increasing and therefore injective. The right hand side of the equation (2.17) is $\tan c$, which is a constant for fixed $c \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and there exists at most one $l \in I_k$ such that $g_k(l) = \tan c$. For all $k \geq 2$, g_k has the range of $\mathbb{R}$ and therefore there is exactly one solution of (2.17) on I_k. For the case of $k = 1$, we proceed separately and we show that the range of the function g_1 is $(-2, +\infty)$.
For every interval $I_k, k \in \mathbb{N}$, we get

$$\frac{\mathrm{d}g_k(l)}{\mathrm{d}l} = -\cot\frac{l}{2} + \frac{l}{2 \cdot \sin^2\frac{l}{2}} = -\frac{2 \cdot \cos\frac{l}{2}\sin\frac{l}{2}}{2 \cdot \sin^2\frac{l}{2}} + \frac{l}{2 \cdot \sin^2\frac{l}{2}} = \frac{l - \sin l}{2 \cdot \sin^2\frac{l}{2}} > 0,$$

since $l > \sin l$ for all $l > 0$.
Therefore, the function g_k is strictly increasing and injective on the interval I_k, $k \in \mathbb{N}$.

For $k \geq 2$, the range of the function g_k is $\mathbb{R}$ since

$$\lim_{l \to 2(k-1)\pi+} g_k(l) = \lim_{l \to 2(k-1)\pi+} -l \cot \frac{l}{2} = -\infty,$$

and

$$\lim_{l \to 2k\pi-} g_k(l) = \lim_{l \to 2k\pi-} -l \cot \frac{l}{2} = +\infty.$$

And therefore there is exactly one solution l_k of (2.17) for every interval $I_k, k \geq 2$.
Now, let us consider the case of $k = 1$. The range of the function g_1 is $(-2, +\infty)$ due to

$$\lim_{l \to 0+} g_1(l) = \lim_{l \to 0+} -l \cdot \cot \frac{l}{2} = \lim_{l \to 0+} -l \cdot \frac{\cos \frac{l}{2}}{\sin \frac{l}{2}} = \lim_{l \to 0+} -2 \cdot \frac{\frac{l}{2}}{\sin \frac{l}{2}} \cos \frac{l}{2} = -2,$$

and

$$\lim_{l \to 2\pi-} g_1(l) = \lim_{l \to 2\pi-} -l \cdot \cot \frac{l}{2} = +\infty.$$

Given this range, there is exactly one solution l_1 on the interval $I_1 = (0, 2\pi)$ for $\tan c > -2$ and there is no solution for $\tan c \leq -2$.

□

Lemma 5. *For $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, there are infinitely many positive eigenvalues for the boundary value problem* (2.1)*. For $\tan c > -2$, all these positive eigenvalues form two sequences $(\lambda_k^{\mathrm{I}})_{k=1}^{+\infty}$ and $(\lambda_k^{\mathrm{II}})_{k=1}^{+\infty}$ such that*

$$0 < \lambda_1^{\mathrm{II}} < \lambda_1^{\mathrm{I}} < \lambda_2^{\mathrm{II}} < \lambda_2^{\mathrm{I}} < \ldots < \lambda_k^{\mathrm{II}} < \lambda_k^{\mathrm{I}} < \ldots$$

For $\tan c \leq -2$, all positive eigenvalues form two sequences $(\lambda_k^{\mathrm{I}})_{k=1}^{+\infty}$ and $(\lambda_k^{\mathrm{II}})_{k=2}^{+\infty}$ such that

$$0 < \lambda_1^{\mathrm{I}} < \lambda_2^{\mathrm{II}} < \lambda_2^{\mathrm{I}} < \ldots < \lambda_k^{\mathrm{II}} < \lambda_k^{\mathrm{I}} < \ldots$$

Moreover, in both cases, we have that $\lambda_k^{\mathrm{I}} = 4k^2\pi^2$ and $\lambda_k^{\mathrm{II}} = l_k^2$, where l_k is the unique solution of the nonlinear equation (2.17) *such that* (2.18) *holds.*

Proof. For $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, there are two possible types of solutions of the equation (2.13). Two cosines are equal if and only if their arguments are equal except for the $2k\pi$-shift, $k \in \mathbb{Z}$, or one argument is equal to the negative of the second argument, again except for the $2k\pi$-shift, $k \in \mathbb{Z}$.

1. In the first case, we have for $k \in \mathbb{Z}$ that

$$\begin{aligned} \sqrt{\lambda}\left(1 - p(\sqrt{\lambda}, c)\right) &= \sqrt{\lambda} p(\sqrt{\lambda}, c) + 2k\pi, \\ \sqrt{\lambda}\left(1 - 2p(\sqrt{\lambda}, c)\right) &= 2k\pi, \\ p(\sqrt{\lambda}, c) &= \frac{1}{2} - \frac{k\pi}{\sqrt{\lambda}}. \end{aligned} \tag{2.19}$$

Since the value $p(\sqrt{\lambda}, c)$ is negative for $\lambda > 0$ and $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the equation (2.19) can be satisfied only for $k \in \mathbb{N}$. Now, using the definition (2.14) of p, we get for $k \in \mathbb{N}$ that

$$\begin{aligned} -\frac{1}{\sqrt{\lambda}} \operatorname{arccot}\left(\frac{1}{\sqrt{\lambda}} \tan c\right) &= \frac{1}{2} - \frac{k\pi}{\sqrt{\lambda}}, \\ \operatorname{arccot}\left(\frac{1}{\sqrt{\lambda}} \tan c\right) &= k\pi - \frac{\sqrt{\lambda}}{2}. \end{aligned} \tag{2.20}$$

The range of arccot is $(0, \pi)$ and thus, using (2.20), we obtain for $k \in \mathbb{N}$ that

$$\begin{aligned} 0 < k\pi - &\frac{\sqrt{\lambda}}{2} < \pi, \\ 2(k-1)\pi < &\sqrt{\lambda} < 2k\pi. \end{aligned}$$

Moreover, (2.20) implies that

$$\tan c = -\sqrt{\lambda}\cot\left(\frac{\sqrt{\lambda}}{2}\right), \tag{2.21}$$

which is exactly the nonlinear equation (2.17) for $l = \sqrt{\lambda}$.

Thus, using Lemma 4, we obtain solutions λ of (2.21) as $\lambda_k^{\mathrm{II}} = l_k^2$, $k \in \mathbb{N}$.

2. In the second case, we have for $k \in \mathbb{Z}$ that

$$\begin{aligned}\sqrt{\lambda}\left(1 - p(\sqrt{\lambda}, c)\right) &= -\sqrt{\lambda}p(\sqrt{\lambda}, c) + 2k\pi,\\ \sqrt{\lambda} &= 2k\pi.\end{aligned} \tag{2.22}$$

The equation (2.22) can be satisfied only for $k \in \mathbb{N}$ due to $\lambda > 0$. Thus, we obtain solutions of (2.13) of the second type $\lambda_k^I = 4k^2\pi^2$, $k \in \mathbb{N}$.

□

Now, let us continue to study the problem (2.1) for $\lambda \leq 0$.

Lemma 6. *For $\lambda = 0$, the boundary value problem* (2.1) *has a non-trivial solution if and only if*

$$\tan c = -2.$$

Proof. Let us split the proof according to the value of c. Firstly, in the case of $c = \frac{\pi}{2}$, zero is not the eigenvalue for (2.1) due to Lemma 1. Secondly, for $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\lambda = 0$, the differential equation of the problem (2.1) simplifies to $u''(x) = 0,\ x \in \mathbb{R}$. Its general solution is therefore $u(x) = A\,(x - x_0)\,,\ A \in \mathbb{R},\ x_0 \in \mathbb{R}$. Let us assume $A \neq 0$ since $A = 0$ leads to the trivial solution. Using the integral condition of the boundary value problem (2.1), the value of x_0 can be determined as $\frac{1}{2}$ since

$$\int_0^1 u(x)\,\mathrm{d}x = \int_0^1 A\,(x - x_0)\ \mathrm{d}x = A\left(\frac{1}{2} - x_0\right).$$

Indeed, $x_0 = \frac{1}{2}$ due to $\int_0^1 u(x)\ \mathrm{d}x = 0$ and $A \neq 0$. Moreover, we have $u(0) = -Ax_0 = -\frac{1}{2}A \neq 0$. Now, the first boundary condition in (2.1) can be manipulated in the following way ($\cos c \neq 0$)

$$\begin{aligned}\frac{\sin c}{\cos c} &= \frac{u'(0)}{u(0)},\\ \tan c &= \frac{A}{-\frac{A}{2}},\\ \tan c &= -2.\end{aligned}$$

□

Lemma 7. *For $\lambda < 0$, the boundary value problem* (2.1) *has a non-trivial solution if and only if*

$$\tan c = \sqrt{-\lambda}\coth\left(\frac{-\sqrt{-\lambda}}{2}\right). \tag{2.23}$$

Moreover, for $\tan c < -2$, there exists exactly one $\lambda < 0$ such that (2.23) *holds, and for $\tan c \geq -2$, there is no $\lambda < 0$ such that* (2.23) *is satisfied.*

Proof. Let us split the proof according to the value of c.

1. For $c = \frac{\pi}{2}$, there are no negative eigenvalues according to Lemma 1.

2. For $c \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\lambda < 0$, the differential equation of the boundary value problem (2.1) has the general solution in the form of

$$u(x) = \frac{C_1}{\sqrt{-\lambda}} \sinh(\sqrt{-\lambda}x) + C_0 \cosh(\sqrt{-\lambda}x), \quad C_0, C_1 \in \mathbb{R}.$$

This general solution can be equivalently written as

$$u(x) = A \sinh(\sqrt{-\lambda}(x - x_0)), \quad A, x_0 \in \mathbb{R},$$

which can be proven in an analogous way as Lemma 2.

Let us assume $A \neq 0$ since $A = 0$ yields a trivial solution.

Using the integral condition of the boundary value problem (2.1), the value of x_0 can be determined . First we evaluate the integral as

$$\begin{aligned}\int_0^1 u(x)\,\mathrm{d}x &= A \int_0^1 \sinh\left(\sqrt{-\lambda}\,(x - x_0)\right)\,\mathrm{d}x \\ &= A\frac{1}{\sqrt{-\lambda}}\left(\cosh\sqrt{-\lambda}(1 - x_0) - \cosh\left(-\sqrt{-\lambda}x_0\right)\right).\end{aligned}$$

The integral condition $\int_0^1 u(x)\,\mathrm{d}x = 0$ can therefore be satisfied if and only if

$$\cosh\sqrt{-\lambda}(1 - x_0) = \cosh\left(-\sqrt{-\lambda}x_0\right).$$

Since cosh is an even function, the equality holds true if arguments on both sides are equal or if one of them is equal to the negative of the second one. In the first case we obtain ($\sqrt{-\lambda} \neq 0$)

$$\begin{aligned}\sqrt{-\lambda}(1 - x_0) &= -\sqrt{-\lambda}x_0, \\ 1 - x_0 &= -x_0,\end{aligned}$$

and therefore the condition cannot be satisfied.

In the second case, we have

$$\begin{aligned}\sqrt{-\lambda}(1 - x_0) &= \sqrt{-\lambda}x_0, \\ 1 - x_0 &= x_0, \\ x_0 &= \tfrac{1}{2}.\end{aligned}$$

Because $x_0 = \frac{1}{2}$ and u is a strictly increasing function on $\mathbb{R}$, we get $u(x) \neq 0$.

Now the first condition of the boundary value problem (2.1) can be manipulated since $u(0) \neq 0$ and $\cos c \neq 0$

$$\begin{aligned}\tan c &= \frac{u'(0)}{u(0)}, \\ &= \frac{A\sqrt{-\lambda}\cosh\left(-\sqrt{-\lambda}x_0\right)}{A\sinh\left(-\sqrt{-\lambda}x_0\right)}, \\ &= \sqrt{-\lambda}\coth\left(\frac{-\sqrt{-\lambda}}{2}\right),\end{aligned}$$

which is exactly (2.23).

It remains to study the solvability of (2.23), i.e. the solvability of the nonlinear equation

$$h(l) = \tan c, \tag{2.24}$$

where we denoted $l := -\sqrt{-\lambda}$ and $h(l) := -l \cdot \coth\left(\frac{l}{2}\right)$.

Since we have

$$\lim_{l\to 0-} h(l) = \lim_{l\to 0-} -l\cdot\coth\frac{l}{2} = \lim_{l\to 0-} -l\frac{\cosh\frac{l}{2}}{\sinh\frac{l}{2}} = -2,$$

$$\lim_{l\to -\infty} h(l) = \lim_{l\to -\infty} -l\cdot\coth\frac{l}{2} = -\infty,$$

and the function h is strictly increasing for $l \in (-\infty, 0)$, there is exactly one negative solution of the equation (2.24) for $\tan c < -2$ and there is no solution of the equation (2.24) for $\tan c \geq -2$. □

At the end of this chapter, we summarize results concerning the boundary value problem (2.1) in the following theorem.

Theorem 1. *There are infinitely many eigenvalues for the boundary value problem* (2.1)*.*

1. *For $c = \frac{\pi}{2}$, all these eigenvalues are positive and form the sequence $(\lambda_k^{\mathtt{I}})_{k=1}^{+\infty}$, where $\lambda_k^{\mathtt{I}} = 4k^2\pi^2, k \in \mathbb{N}$.*
2. *For $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, all eigenvalues form two sequences $(\lambda_k^{\mathtt{I}})_{k=1}^{+\infty}$ and $(\lambda_k^{\mathtt{II}})_{k=1}^{+\infty}$ such that*

$$\lambda_1^{\mathtt{II}} < \lambda_1^{\mathtt{I}} < \lambda_2^{\mathtt{II}} < \lambda_2^{\mathtt{I}} < \ldots < \lambda_k^{\mathtt{II}} < \lambda_k^{\mathtt{I}} < \ldots$$

Moreover, we have that $\lambda_k^{\mathtt{I}} = 4k^2\pi^2$ and $\lambda_k^{\mathtt{II}} = l_k^2$, where l_k for $k \geq 2$ is the unique solution of the nonlinear equation

$$-\,l\cdot\cot\frac{l}{2} = \tan c \tag{2.25}$$

on the interval $(2(k-1)\pi, 2k\pi)$. For $\tan c > -2$, the first eigenvalue $\lambda_1^{\mathtt{II}} = l_1^2$ is positive and l_1 is the unique solution of (2.25) *on* $(0, 2\pi)$*. For $\tan c = -2$, the first eigenvalue $\lambda_1^{\mathtt{II}}$ is zero. Finally, for $\tan c < -2$, the first eigenvalue $\lambda_1^{\mathtt{II}} = -l_1^2$ is negative and l_1 is unique solution of the nonlinear equation*

$$-l\cdot\coth\frac{l}{2} = \tan c$$

on the interval $(-\infty, 0)$.

Proof. For $c = \frac{\pi}{2}$, the assertion follows from Lemma 1. For $c \in (-\frac{\pi}{2}, \frac{\pi}{2})$, using Lemma 5, we obtain two sequences $(\lambda_k^{\mathtt{I}})$ and $(\lambda_k^{\mathtt{II}})$ of eigenvalues for the problem (2.1). Moreover, the first element $\lambda_1^{\mathtt{II}}$ is discussed in Lemmas 6 and 7.

□

Chapter 3

Implicit description of the Fučík spectrum

This chapter is devoted to studying the structure of the Fučík spectrum Σ_c for the problem

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, & x \in (0,1), \\ u(0) \cdot \sin c = u'(0) \cdot \cos c, & \int_0^1 u(x)\,\mathrm{d}x = 0, \end{cases} \tag{3.1}$$

where $\alpha, \beta \in \mathbb{R}$ and $-\frac{\pi}{2} < c \leq \frac{\pi}{2}$. More precisely, for fixed paremeter $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$, our goal is to describe the Fučík spectrum Σ_c as the set of all pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that the problem (3.1) has a non-trivial solution u. As in the previous chapter, by a non-trivial solution of (3.1) we mean a function $u \in C^2(0,1) \cap C^1[0,1]$ which is not identically zero on $(0,1)$, satisfies the differential equation in (3.1) pointwise on this interval and also satisfies both boundary conditions in (3.1).

The implicit description of the Fučík spectrum Σ_c for the problem (3.1) is provided in [5], but only for $-\frac{\pi}{2} < c < 0$. In the following section, we focus on $\alpha, \beta > 0$ and show how to get the implicit description of the Fučík spectrum Σ_c in a new compact form for $-\frac{\pi}{2} < c \leq \frac{\pi}{2}$. Presented compact description of Σ_c in the first quadrant extends known results in [3] concerning only the special case of $c = \frac{\pi}{2}$.

In the second section of this chapter, we provide the implicit description of some parts of the Fučík spectrum Σ_c for $\alpha \cdot \beta < 0$ (i.e. in the second and fourth quadrants of $\alpha\beta-$plane). Let us note that similar description can be found in [5] (see Theorem 4 on page 511).

3.1 Compact description in the first quadrant

In this part, we deal with the problem (3.1) for $\alpha, \beta > 0$, i.e. we investigate the following problem

$$\begin{cases} u''(x) + a^2 u^+(x) - b^2 u^-(x) = 0, & x \in (0,1), \\ u(0) \cdot \sin c = u'(0) \cdot \cos c, & \int_0^1 u(x)\,\mathrm{d}x = 0, \end{cases} \tag{3.2}$$

where we denoted $a = \sqrt{\alpha}$ and $b = \sqrt{\beta}$. Our goal is to describe all pairs $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that the problem (3.2) has a non-trivial solution u. According to the basic facts stated in [3] (see pages 182 and 183), it is enought to find all pairs $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that the solution u of the following initial value problem

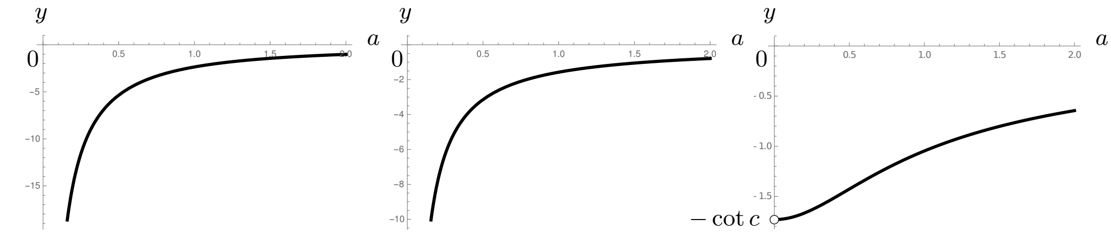
$$\begin{cases} u''(x) + a^2 u^+(x) - b^2 u^-(x) = 0, & x \in \mathbb{R}, \\ u(p(a,c)) = 0, & u'(p(a,c)) = a \cdot b > 0, \end{cases} \tag{3.3}$$

satisfies the integral condition

$$\int_0^1 u(x)\,\mathrm{d}x = 0, \tag{3.4}$$

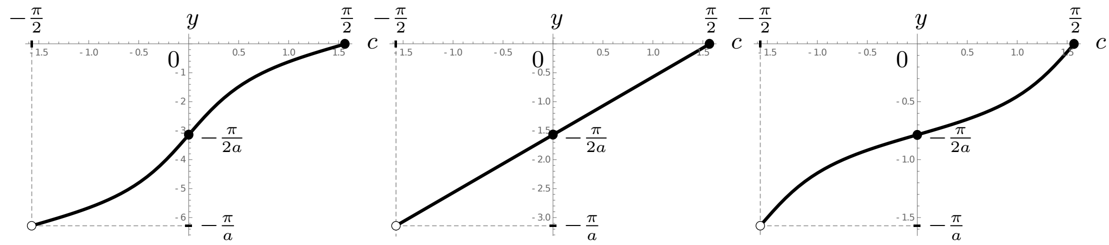
where the function $p : \mathbb{R}^+ \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ is defined in (2.14) as

$$p(a,c) := \begin{cases} -\frac{1}{a}\operatorname{arccot}\left(\frac{1}{a}\tan c\right) & \text{for } c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 0 & \text{for } c = \frac{\pi}{2}. \end{cases} \tag{3.5}$$



(a) $c = -\frac{\pi}{4}$ (b) $c = 0$ (c) $c = \frac{\pi}{6}$

Fig. 3.1: The graph of the function $p(\cdot, c) : a \mapsto p(a, c)$ for different values of c.



(a) $a = 0.5$ (b) $a = 1$ (c) $a = 2$

Fig. 3.2: The graphs of the function $p(a, \cdot) : c \mapsto p(a, c)$ for different values of a.

It is straightforward to verify that the function p is a continuous function and its range is the interval $(-\infty, 0]$. Moreover, for fixed $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the function $p = p(\cdot, c)$ as a function of the first variable $a > 0$ is strictly increasing and has the range (see Figure 3.1)

$$\operatorname{Ran}(p(\cdot, c)) = \begin{cases} (-\cot c, 0) & \text{for } c \in (0, \frac{\pi}{2}), \\ (-\infty, 0) & \text{for } c \in \left(-\frac{\pi}{2}, 0\right]. \end{cases}$$

For fixed $a > 0$, the function $p = p(a, \cdot)$ as a function of the second variable $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is also strictly increasing (see Figure 3.2).

The solution u of the initial value problem (3.3) is T-periodic, where $T = \frac{\pi}{a} + \frac{\pi}{b} > 0$, and has the following form on the interval $(0, T]$

$$u(x) = \begin{cases} b \sin(a(x - p)) & \text{for } x \in \left(0, p + \frac{\pi}{a}\right], \\ -a \sin\left(b\left(x - \left(p + \frac{\pi}{a}\right)\right)\right) & \text{for } x \in \left(p + \frac{\pi}{a},\ p + \frac{\pi}{a} + \frac{\pi}{b}\right], \\ b \sin\left(a\left(x - \left(p + \frac{\pi}{a} + \frac{\pi}{b}\right)\right)\right) & \text{for } x \in \left(p + \frac{\pi}{a} + \frac{\pi}{b},\ T\right], \end{cases} \tag{3.6}$$

where we shortened the notation of $p(a, c)$ as p. For better clarity, we use the abbreviated notation p instead of $p(a, c)$ in the following text.

Now, let us define the set

$$\mathcal{M}_c := \Big\{(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+ \colon \text{the solution } u \text{ of the initial value problem (3.3) satisfies } \textstyle\int_0^1 u(x)\,\mathrm{d}x = 0\Big\}. \tag{3.7}$$

We have the following link between the set $\mathcal{M}_c$ and the Fučík spectrum Σ_c (see Figure 3.5 and 3.6):

1. If $(a, b) \in \mathcal{M}_c$ then $(a^2, b^2) \in \Sigma_c$ and $(b^2, a^2) \in \Sigma_c$.
2. If $(\alpha, \beta) \in \Sigma_c$ with $\alpha, \beta > 0$ then $(\sqrt{\alpha}, \sqrt{\beta}) \in \mathcal{M}_c$ or $(\sqrt{\beta}, \sqrt{\alpha}) \in \mathcal{M}_c$.

This implies that

$$\Sigma_c \cap \mathbb{R}^+ \times \mathbb{R}^+ = \left\{(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+ : \left(\sqrt{\alpha}, \sqrt{\beta}\right) \in \mathcal{M}_c \quad \vee \quad \left(\sqrt{\beta}, \sqrt{\alpha}\right) \in \mathcal{M}_c\right\}$$

and thus, it is enough to describe only the set $\mathcal{M}_c$.

Definition 1. *For $a, b > 0$, let us define*

$$F(x) := \int_0^x u(t)\ \mathrm{d}t,\ \ x \in \mathbb{R},$$

where u is the solution of the initial value problem (3.3).

Using this definition, the integral condition (3.4) can be equivalently written as $F(1) = 0$.

Lemma 8. *The function F satisfies the following equation*

$$\forall x \in \mathbb{R}: \quad F(x+T) = F(x) + F(T), \tag{3.8}$$

where $T = \frac{\pi}{a} + \frac{\pi}{b}$.

Proof. Since u is the T-periodic solution of the initial value problem (3.3), we get for all $x \in \mathbb{R}$ that

$$F(x+T) = \int_0^{x+T} u(t)\,\mathrm{d}t = \int_0^{x} u(t)\,\mathrm{d}t + \int_x^{x+T} u(t)\,\mathrm{d}t = F(x) + \int_0^{T} u(t)\,\mathrm{d}t = F(x) + F(T).$$

□

Lemma 9. *The function F is T-periodic, where $T = \frac{\pi}{a} + \frac{\pi}{b}$, if and only if $a = b$.*

Proof. Using the Definition 1 and (3.6), we have

$$F(x) = \begin{cases} -\frac{b}{a}\cos(a(x-p)) + \frac{b}{a}\cos(ap) & \text{for } x \in \left(0,\ p+\frac{\pi}{a}\right], \\ \frac{a}{b}\cos\left(b\left(x-\left(p+\frac{\pi}{a}\right)\right)\right) + \frac{b}{a}\cos(ap) + \frac{b}{a} - \frac{a}{b} & \text{for } x \in \left(p+\frac{\pi}{a},\ p+\frac{\pi}{a}+\frac{\pi}{b}\right], \\ -\frac{b}{a}\cos\left(a\left(x-\left(p+\frac{\pi}{a}+\frac{\pi}{b}\right)\right)\right) + \frac{b}{a}\cos(ap) + 2\frac{b}{a} - 2\frac{a}{b} & \text{for } x \in \left(p+\frac{\pi}{a}+\frac{\pi}{b},\ T\right]. \end{cases} \tag{3.9}$$

Using (3.6) we get

$$\begin{aligned} F(T) &= \frac{2b}{a} - \frac{2a}{b} + \frac{b}{a}\cos(ap) - \frac{b}{a}\cos\left(a\left(\frac{\pi}{a} + \frac{\pi}{b} - p - \frac{\pi}{a} - \frac{\pi}{b}\right)\right) \\ &= \frac{2b}{a} - \frac{2a}{b} \\ &= \frac{2T}{\pi}(b-a). \end{aligned} \tag{3.10}$$

Therefore, F(T) = 0 if and only if $a = b$ and the assertion follows from Lemma 8. □

Definition 2. *For $a, b > 0$, let us define*

$$G(x) := 1 - \int_0^x (u(t) - \bar{u})\,\mathrm{d}t, \quad x \in \mathbb{R}, \qquad \bar{u} := \frac{1}{T}\int_0^T u(t)\,\mathrm{d}t,$$

where $T = \frac{\pi}{a} + \frac{\pi}{b}$ and u is the solution of the initial value problem (3.3).

Now, it is straightforward to verify that for all $x \in \mathbb{R}$ we have

$$G(x) = 1 - F(x) + \frac{F(T)}{T}x = 1 - F(x) + \frac{2}{\pi}(b-a)x, \tag{3.11}$$

and also the function G can be evaluated using (3.9). For $x \in \left(0,\ p+\frac{\pi}{a}\right]$, we have $G(x) = \frac{b}{a}\cos(a(x-p)) - \frac{b}{a}\cos(ap) + \frac{2}{\pi}(b-a)x + 1$, for $x \in \left(p+\frac{\pi}{a},\ p+\frac{\pi}{a}+\frac{\pi}{b}\right]$, we get $G(x) = -\frac{a}{b}\cos\left(b\left(x-\left(p+\frac{\pi}{a}\right)\right)\right) - \frac{b}{a}\cos(ap) - \frac{b}{a} + \frac{a}{b} + \frac{2}{\pi}(b-a)x + 1$ and for $x \in \left(p+\frac{\pi}{a}+\frac{\pi}{b},\ T\right]$, we obtain $G(x) = \frac{b}{a}\cos\left(a\left(x-\left(p+\frac{\pi}{a}+\frac{\pi}{b}\right)\right)\right) - \frac{b}{a}\cos(ap) - 2\frac{b}{a} + 2\frac{a}{b} + \frac{2}{\pi}(b-a)x + 1$, where $T = \frac{\pi}{a} + \frac{\pi}{b}$.
Moreover, the integral condition (3.4) can be written as

$$G(1) = 1 + \frac{F(T)}{T}. \tag{3.12}$$

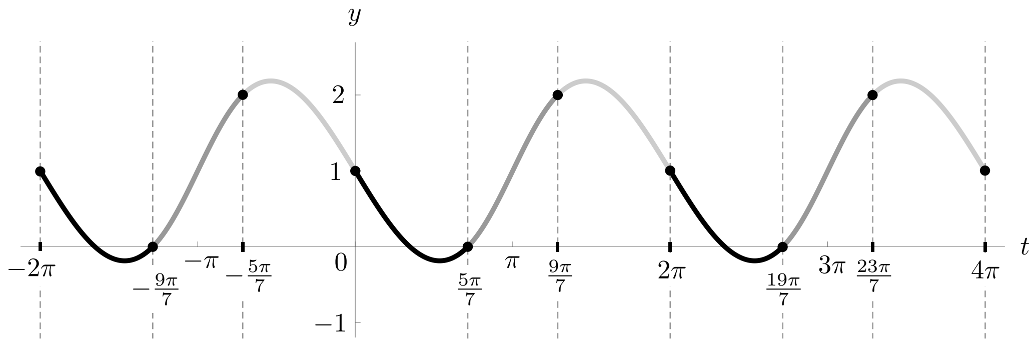


Fig. 3.3: Graph of the 2π-periodic function $\mathcal{G}(a,b,\cdot) : t \mapsto \mathcal{G}(a,b,t)$ for $a = 2, b = 5$ and $c = 0$.

Lemma 10. *The function G is T-periodic, where $T = \frac{\pi}{a} + \frac{\pi}{b}$.*

Proof. For all $x \in \mathbb{R}$, using (3.11) and (3.8), we obtain

$$G(x+T) = 1 - F(x+T) + \frac{F(T)}{T}(x+T) = 1 - F(x) + \frac{F(T)}{T}x = G(x). \tag{3.13}$$

□

In the following definition, we define a new function $\mathcal{G} = \mathcal{G}(a,b,t)$. This function is 2π-periodic in the third variable and is given by parts (see Figure 3.3 for $a = 2$, $b = 5$, $c = 0$). Another examples of graphs of the function $\mathcal{G}(a,b,\cdot)$ can be seen in Figure 3.4.

Definition 3. *Let us define the function $P : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ as*

$$P(a,b,t) := \left(\frac{b}{a} - \frac{a}{b}\right)\frac{t}{\pi} + 1, \quad a > 0, \quad b > 0, \quad t \in \mathbb{R}, \tag{3.14}$$

and the function $\mathcal{G} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, which is 2π-periodic in the third variable

$$\forall a > 0\ \forall b > 0\ \forall t \in \mathbb{R} : \ \mathcal{G}(a,b,t+2\pi) = \mathcal{G}(a,b,t),$$

and is define for $a > 0,\ b > 0,\ t \in (0, 2\pi]$ as

$$\mathcal{G}(a,b,t) := \begin{cases} \frac{b}{a}\cos\left(\frac{a+b}{2b}t - ap\right) - \frac{b}{a}\cos(ap) + P(a,b,t) & \text{for } t \in I_1, \\ \frac{a}{b}\cos\left(\frac{a+b}{2a}(t-2\pi) - bp\right) - \frac{b}{a}\cos(ap) + P(a,b,t-\pi) & \text{for } t \in I_2, \\ \frac{b}{a}\cos\left(\frac{a+b}{2b}(t-2\pi) - ap\right) - \frac{b}{a}\cos(ap) + P(a,b,t-2\pi) & \text{for } t \in I_3, \end{cases} \tag{3.15}$$

where

$$I_1 := \left(0, \frac{2b(\pi+ap)}{a+b}\right], \quad I_2 := \left(\frac{2b(\pi+ap)}{a+b}, 2\pi + \frac{2abp}{a+b}\right], \quad I_3 := \left(2\pi + \frac{2abp}{a+b}, 2\pi\right],$$

and p stands for the value of $p(a,c)$ defined in (3.5).

Theorem 2. *The pair $(a,b) \in \mathcal{M}_c$ if and only if $a, b > 0$ and*

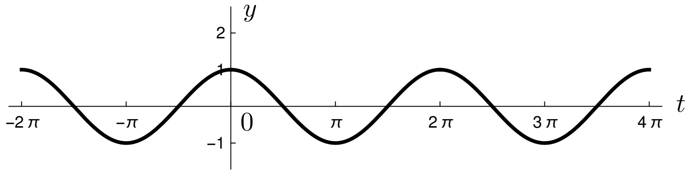
$$\mathcal{G}\left(a, b, \frac{2ab}{a+b}\right) = 1 + \frac{2}{\pi}(b-a). \tag{3.16}$$

Proof. Let u be the solution of the initial value problem (3.3), where $a, b > 0$. The integral condition (3.4) can be rewritten as (3.12).

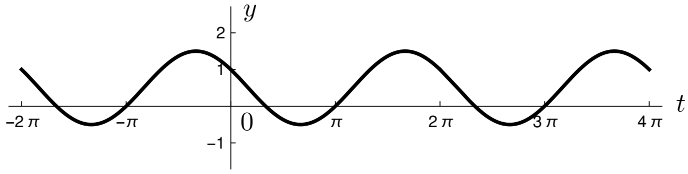
Now, we prove that

$$\forall x \in \mathbb{R} : \quad G(x) = \mathcal{G}\left(a, b, \frac{2ab}{a+b}x\right). \tag{3.17}$$

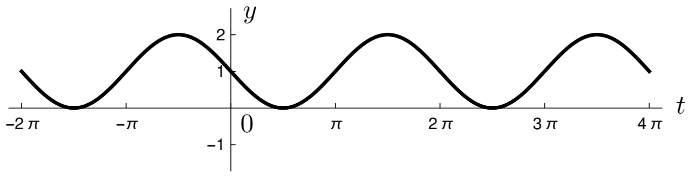
Let us define $t(x) := \frac{2ab}{a+b}x$ and split the proof according to the value of x.



(a) $a = 1,\, b = 1,\, c = \frac{\pi}{2}$



(b) $a = 1,\, b = 1,\, c = \frac{\pi}{6}$



(c) $a = 1,\, b = 1,\, c = 0$

Fig. 3.4: Graph of the 2π-periodic function $\mathcal{G}(a, b, \cdot) : t \mapsto \mathcal{G}(a, b, t)$ for $a = b = 1$ and different values of the parameter c.

1. Firstly, let us consider $x \in \left(0, p + \frac{\pi}{a}\right]$. Using the definition of t, we get $t \in \left(0, \frac{2b(ap+\pi)}{a+b}\right]$ and

$$
\begin{aligned}
G(x) &= \frac{b}{a}\cos\left(a\left(x-p\right)\right) - \frac{b}{a}\cos\left(ap\right) + \frac{2}{\pi}\left(b-a\right)x + 1 \\
&= \frac{b}{a}\cos\left(ax-ap\right) - \frac{b}{a}\cos\left(ap\right) + \frac{2}{\pi}\left(b-a\right)\frac{\left(a+b\right)t}{2ab} + 1 + \\
&\quad \left(\frac{b}{a}-\frac{a}{b}\right)\frac{t}{\pi} - \left(\frac{b}{a}-\frac{a}{b}\right)\frac{t}{\pi} - \frac{b}{a} + \frac{a}{b} + \frac{b}{a} - \frac{a}{b} \\
&= \frac{b}{a}\cos\left(a\frac{\left(a+b\right)t}{2ab} - ap\right) - \frac{b}{a}\cos\left(ap\right) + \frac{\left(b-a\right)\left(a+b\right)t}{\pi ab} + \\
&\quad P\left(a,b,t-\pi\right) + \frac{b}{a} - \frac{a}{b} - \frac{b^2-a^2}{ab}\cdot\frac{t}{\pi} \\
&= \frac{b}{a}\cos\left(\frac{\left(a+b\right)t}{2b} - ap\right) - \frac{b}{a}\cos\left(ap\right) + \frac{b}{a} - \frac{a}{b} + P\left(a,b,t-\pi\right) \\
&= \mathcal{G}\left(a,b,t\right).
\end{aligned}
$$

2. Secondly, if $x \in \left(p + \frac{\pi}{a}, p + \frac{\pi}{a} + \frac{\pi}{b}\right]$ then $t \in \left(\frac{2b(ap+\pi)}{a+b}, 2\pi + \frac{2abp}{a+b}\right]$ and we get

$$\begin{aligned} G(x) &= -\frac{a}{b}\cos\left(b\left(x - \left(p + \frac{\pi}{a}\right)\right)\right) - \frac{b}{a}\cos(ap) - \frac{b}{a} + \frac{a}{b} + \frac{2}{\pi}(b-a)x + 1 \\ &= -\frac{a}{b}\cos\left(bx - bp - \pi\frac{b}{a}\right) - \frac{b}{a}\cos(ap) + \frac{2}{\pi}(b-a)x - \\ &\quad \frac{b}{a} + \frac{a}{b} + 1 + \left(\frac{b}{a} - \frac{a}{b}\right)\frac{t}{\pi} - \left(\frac{b}{a} - \frac{a}{b}\right)\frac{t}{\pi} \\ &= -\frac{a}{b}\cos\left(\frac{a+b}{2a}t - bp - \frac{b}{a}\pi\right) - \frac{b}{a}\cos(ap) + \\ &\quad \frac{2}{\pi}(b-a)\frac{a+b}{2ab}t + P(a,b,t-\pi) - \left(\frac{b^2-a^2}{ab}\right)\frac{t}{\pi} \\ &= \mathcal{G}(a,b,t). \end{aligned}$$

3. And finally, if $x \in \left(p + \frac{\pi}{a} + \frac{\pi}{b}, T\right]$ then $t \in \left(2\pi + \frac{2ab\pi}{a+b}, 2\pi\right]$, and we obtain

$$\begin{aligned} G(x) &= \frac{b}{a}\cos\left(ax - ap - \pi - \frac{a}{b}\pi\right) - \frac{b}{a}\cos(ap) - 2\frac{b}{a} + 2\frac{a}{b} + \frac{2}{\pi}(b-a)x + 1 \\ &= \frac{b}{a}\cos\left(\frac{a+b}{2b}t - ap - \pi - \frac{a}{b}\pi\right) - \frac{b}{a}\cos(ap) - \frac{b}{a} + \frac{a}{b} + \\ &\quad \frac{2}{\pi}(b-a)\frac{a+b}{2ab}t + 1 - \frac{b}{a} + \frac{a}{b} + \left(\frac{b}{a} - \frac{a}{b}\right)\frac{t}{\pi} - \left(\frac{b^2-a^2}{ab}\right)\frac{t}{\pi} \\ &= \frac{b}{a}\cos\left(\frac{a+b}{2b}t - ap - \pi - \frac{a}{b}\pi\right) - \frac{b}{a}\cos(ap) - \frac{b}{a} + \frac{a}{b} + \\ &\quad \left(\frac{b^2-a^2}{ab}\right)\frac{t}{\pi} + P(a,b,t-\pi) - \left(\frac{b^2-a^2}{ab}\right)\frac{t}{\pi} = \\ &= \mathcal{G}(a,b,t). \end{aligned}$$

Thus, the equality in (3.17) holds for all $x \in \mathbb{R}$, which means that the equality (3.12) is the same as the equality (3.16) due to (3.10). □

Corollary 1. *We have that $(a,b) \in \mathcal{M}_c$ if and only if $a, b > 0$ and*

$$\mathcal{G}\left(a, b, \frac{2ab}{a+b}\right) = P\left(a, b, \frac{2ab}{a+b}\right). \tag{3.18}$$

Proof. According to the definition (3.14), we get

$$P\left(a, b, \frac{2ab}{a+b}\right) = \frac{b^2-a^2}{ab} \cdot \frac{2ab}{(a+b)\pi} + 1 = \frac{2}{\pi}(b-a) + 1.$$

Therefore, the assertion is a direct consequence of Theorem 2. □

Remark 2. *Let us point out that for $a = b = \sqrt{\lambda} > 0$, we have*

$$\begin{aligned} &P(\sqrt{\lambda}, \sqrt{\lambda}, t) = 1, \\ &\mathcal{G}(\sqrt{\lambda}, \sqrt{\lambda}, t) = \cos(t - \sqrt{\lambda}p) - \cos(\sqrt{\lambda}p) + 1. \end{aligned}$$

Moreover, in this case, the equation (3.18) reduces to

$$\begin{aligned} \mathcal{G}(\sqrt{\lambda}, \sqrt{\lambda}, \sqrt{\lambda}) &= P(\sqrt{\lambda}, \sqrt{\lambda}, \sqrt{\lambda}), \\ \cos(\sqrt{\lambda} - \sqrt{\lambda}p) - \cos(\sqrt{\lambda}p) + 1 &= 1, \end{aligned}$$

which is exactly the nonlinear equation (2.13). Recall that the solvability of the equation (2.13) is provided in the proof of Lemma 5.

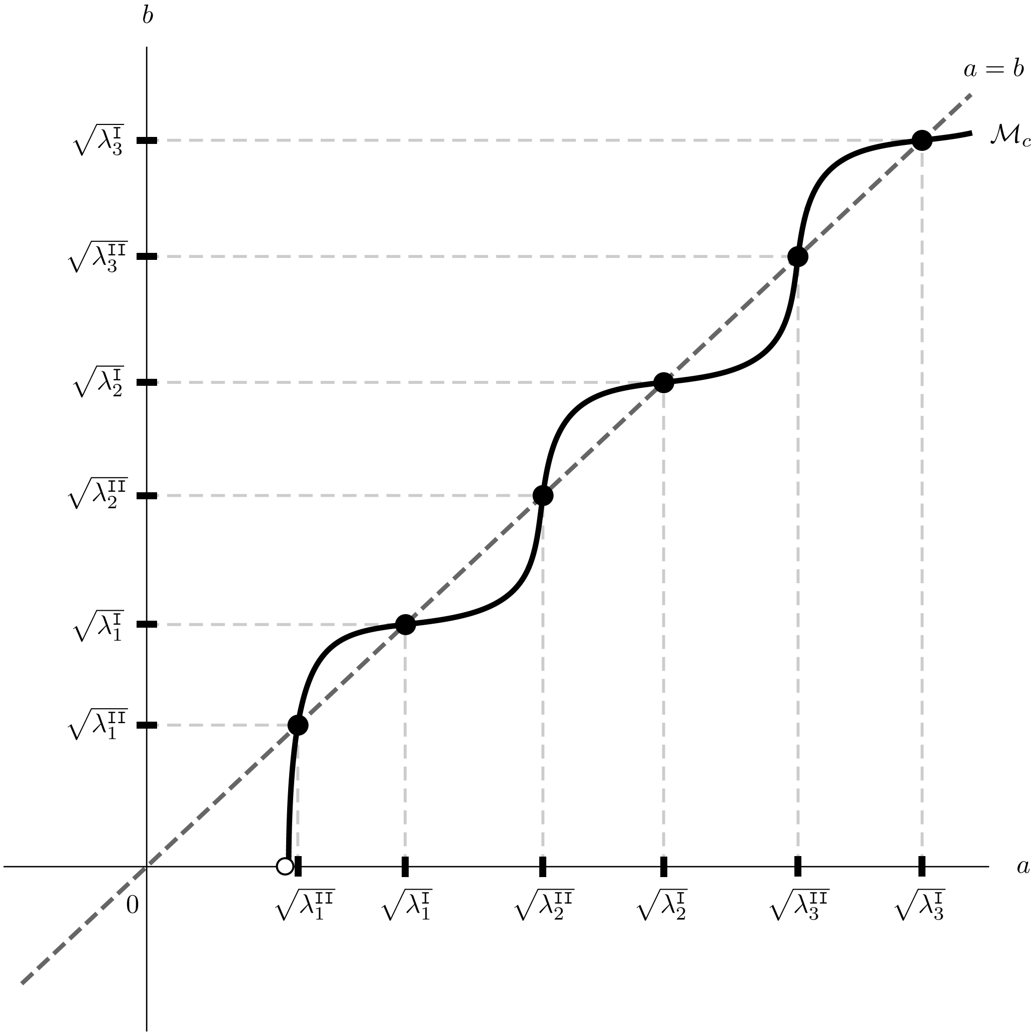


Fig. 3.5: The set $\mathcal{M}_c$ in the first quadrant of the ab-plane for $c = \frac{\pi}{4}$.

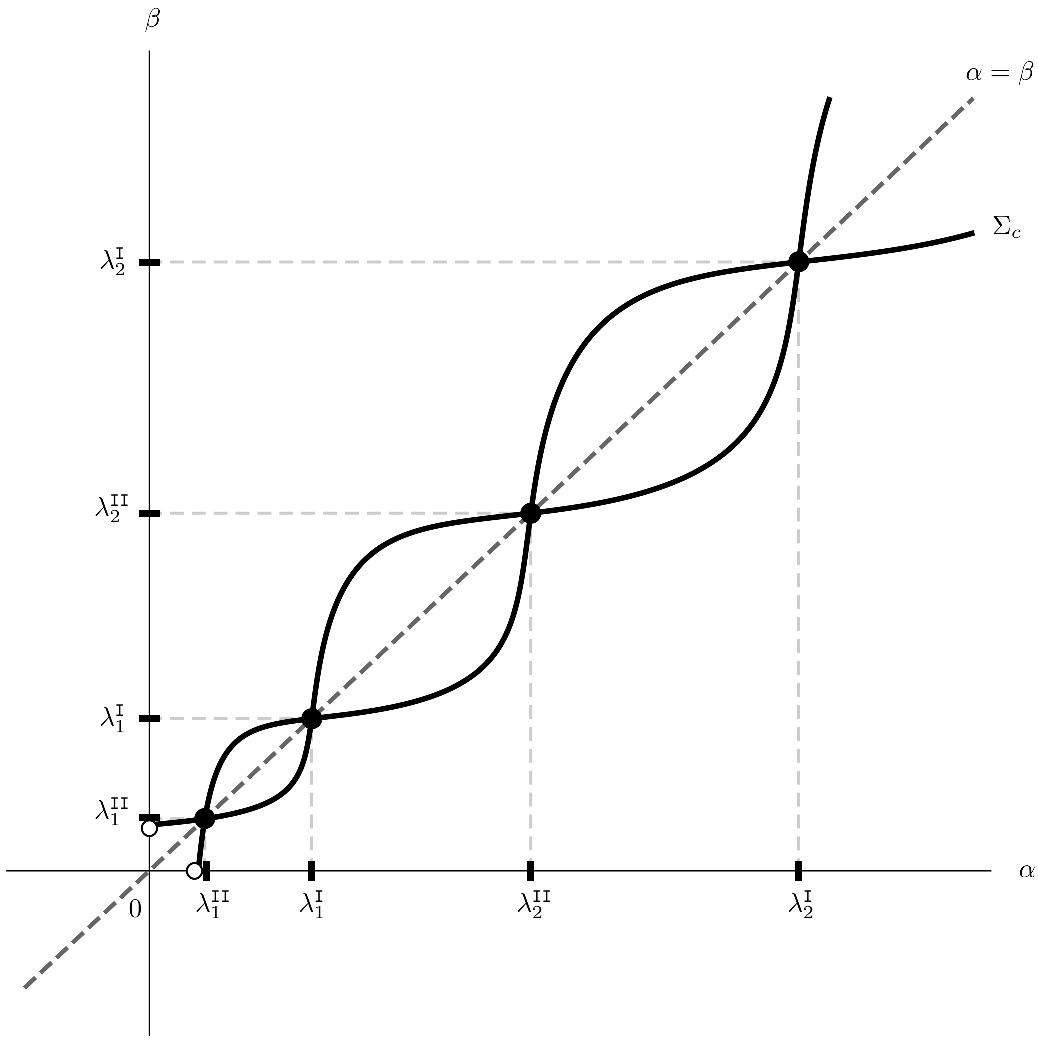


Fig. 3.6: The Fučík spectrum Σ_c in the first quadrant of the $\alpha\beta$-plane for $c = \frac{\pi}{4}$.

3.2 Description in the fourth quadrant

In this section, we investigate the problem (3.1) for $\alpha > 0$ and $\beta < 0$, i.e. we study the following problem

$$\begin{cases} u''(x) + a^2u^+(x) + b^2u^-(x) = 0, & x \in (0,1)\,, \\ u(0)\cdot \sin c = u'(0)\cdot \cos c, & \int_0^1 u(x)\,\mathrm{d}x = 0, \end{cases} \tag{3.19}$$

where we denoted $a = \sqrt{\alpha}$ and $b = -\sqrt{-\beta}$. As in the previous section, let us introduce the corresponding initial value problem

$$\begin{cases} u''(x) + a^2u^+(x) + b^2u^-(x) = 0, & x \in \mathbb{R}, \\ u(p(a,c)) = 0, \quad u'(p(a,c)) = 1, \end{cases} \tag{3.20}$$

where the function p is given by (2.14), and define the set

$$\mathcal{N}_c := \Big\{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^- \colon \text{the solution } u \text{ of the initial value problem (3.20) satisfies } \int_0^1 u(x)\ \mathrm{d}x = 0\Big\}. \tag{3.21}$$

We have the following link between the set $\mathcal{N}_c$ and the Fučík spectrum Σ_c (see Figure 3.7 and 3.8): If $(a,b) \in \mathcal{N}_c$ then $(a^2, -b^2) \in \Sigma_c$ and $(-b^2, a^2) \in \Sigma_c$.

Theorem 3. *The set $\mathcal{N}_c$ consists of all pairs $(a,b) \in \mathbb{R}^+ \times \mathbb{R}^-$ such that*

$$a > \frac{\pi}{1-p(a,c)} \tag{3.22}$$

and

$$\cosh\left(b - bp(a,c) - \frac{b}{a}\pi\right) = 1 + \frac{b^2}{a^2} + \frac{b^2}{a^2}\cos\left(ap(a,c)\right). \tag{3.23}$$

Proof. The solution u to the initial value problem (3.20) can be written as

$$u(x) = \begin{cases} \frac{1}{a}\sin\left(a\left(x-p\right)\right) & \text{for } 0 \le x \le p + \frac{\pi}{a}, \\ -\frac{1}{b}\sinh\left(b\left(x - \left(p + \frac{\pi}{a}\right)\right)\right) & \text{for } x > p + \frac{\pi}{a}. \end{cases} \tag{3.24}$$

At first, let us examine the case of $p + \frac{\pi}{a} \ge 1$. The solution u of the initial value problem (3.20) is only positive on the interval $(0,1)$ and therefore, the integral condition $\int_0^1 u(x)\ \mathrm{d}x = 0$ cannot be satisfied.
At second, in the case of $p + \frac{\pi}{a} < 1$, we have

$$a > \frac{\pi}{1-p}$$

and using (3.24), we evaluate the integral $\int_0^1 u(x)\,\mathrm{d}x$ in the following way

$$\begin{aligned} \int_0^1 u\left(x\right)\,\mathrm{d}x &= \int_0^{p+\frac{\pi}{a}} \frac{1}{a}\sin\left(a\left(x-p\right)\right)\,\mathrm{d}x + \int_{p+\frac{\pi}{a}}^1 -\frac{1}{b}\sinh\left(b\left(x - \left(p + \frac{\pi}{a}\right)\right)\right)\,\mathrm{d}x \\ &= \frac{1}{a^2}\Big[-\cos\left(a\left(x-p\right)\right)\Big]_0^{p+\frac{\pi}{a}} - \frac{1}{b^2}\cdot\left[\cosh\left(b\left(x - \left(p+\frac{\pi}{a}\right)\right)\right)\right]_{p+\frac{\pi}{a}}^1 \\ &= \frac{1}{a^2}\left(1+\cos\left(ap\right)\right) - \frac{1}{b^2}\left(\cosh\left(b - bp - \frac{b}{a}\pi\right) - 1\right) \\ &= \frac{1}{a^2} + \frac{1}{a^2}\cos\left(ap\right) + \frac{1}{b^2}\left(1 - \cosh\left(b - bp - \frac{b}{a}\pi\right)\right). \end{aligned}$$

And thus, we obtain that the integral condition $\int_0^1 u(x)\,\mathrm{d}x = 0$ is satisfied if and only if

$$\cosh\left(b - bp - \frac{b}{a}\pi\right) = 1 + \frac{b^2}{a^2} + \frac{b^2}{a^2}\cos\left(ap\right).$$

□

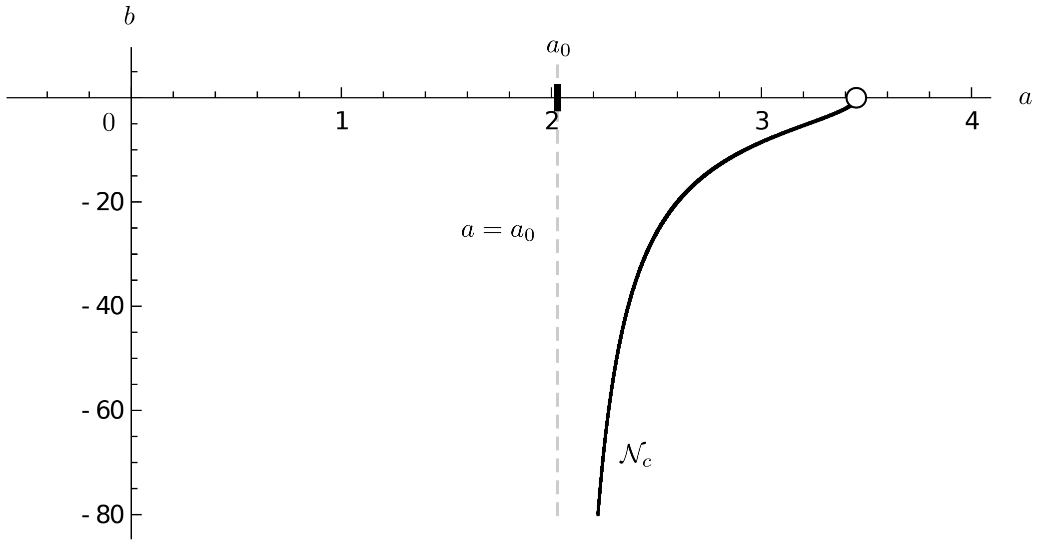


Fig. 3.7: The set $\mathcal{N}_c$ in the fourth quadrant of the ab-plane for $c = \frac{\pi}{4}$, where a_0 is given by $a_0 = \frac{\pi}{1-p(a_0,c)}$.

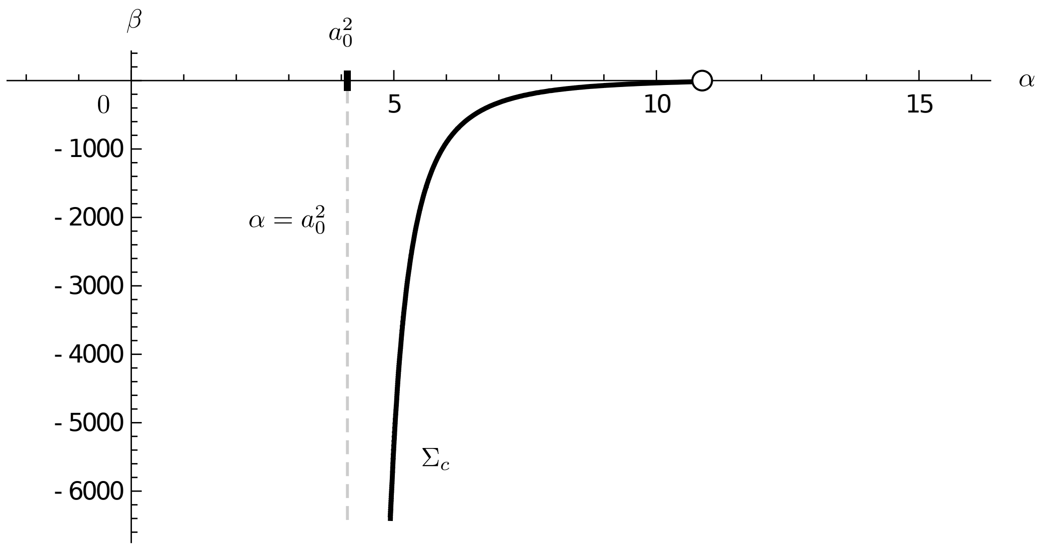


Fig. 3.8: The Fučík spectrum Σ_c in the fourth quadrant of the $\alpha\beta$-plane for $c = \frac{\pi}{4}$, where a_0 is given by $a_0 = \frac{\pi}{1-p(a_0,c)}$.

Chapter 4

The Fučík spectrum as parametrized curves

In this chapter, we investigate the problem (1.1) for $c = 0$, i.e. we study the following problem

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, & x \in (0,1), \\ u'(0) = 0, & \int_0^1 u(x)\,\mathrm{d}x = 0, \end{cases} \tag{4.1}$$

and our goal is to find the parametrization of its Fučík spectrum Σ_0. According to results from the previous chapter, we proceed such that we find parametrizations of sets $\mathcal{M}_0$ and $\mathcal{N}_0$ (recall (3.7) and (3.21) for $c = 0$).

4.1 The parametrization of the set $\mathcal{N}_0$

The set $\mathcal{N}_0$ is described by (3.22) and (3.23) in Theorem 3 for $c = 0$. For $c = 0$, we have $p(a, 0) = -\frac{\pi}{2a}$ and thus, $\mathcal{N}_0$ is the set of all pairs $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^-$ (see Figure 4.1) such that

$$a > \frac{\pi}{2} \quad \text{and} \quad \cosh\left(b - \frac{b\pi}{2a}\right) = 1 + \frac{b^2}{a^2}. \tag{4.2}$$

Theorem 4. *The set $\mathcal{N}_0$ is a continuous curve $\eta : (-\infty, 0) \to \mathbb{R}^2$ with the parametrization $\eta(s) := (\eta_1(s), \eta_2(s))$, where functions $\eta_1, \eta_2 : (-\infty, 0) \to \mathbb{R}$ are defined as*

$$\eta_1(s) = \frac{\pi}{2} - \frac{s}{\sqrt{\cosh s - 1}}, \qquad \eta_2(s) = s - \frac{\pi}{2}\sqrt{\cosh s - 1}.$$

Proof. First of all, let us denote

$$k := \frac{b}{a}, \quad s := b - \frac{b\pi}{2a}. \tag{4.3}$$

Using the inequality in (4.2), we get that $\frac{\pi}{2a} < 1$ and that $k, s < 0$. Moreover, the condition (4.2) can be equivalentely written as

$$\cosh s = 1 + k^2, \quad k < 0,\ s < 0,$$

or as

$$k = -\sqrt{\cosh s - 1}, \quad s < 0. \tag{4.4}$$

Now, the inverse transformation to (4.3) reads

$$a = \frac{\pi}{2} + \frac{s}{k}, \quad b = s + \frac{\pi}{2}k,$$

and thus, using (4.4), we obtain

$$a = \frac{\pi}{2} - \frac{s}{\sqrt{\cosh s - 1}} = \eta_1(s), \qquad b = s - \frac{\pi}{2}\sqrt{\cosh s - 1} = \eta_2(s).$$

The continuity of the curve η is straightforward to verify, which finishes the proof. □

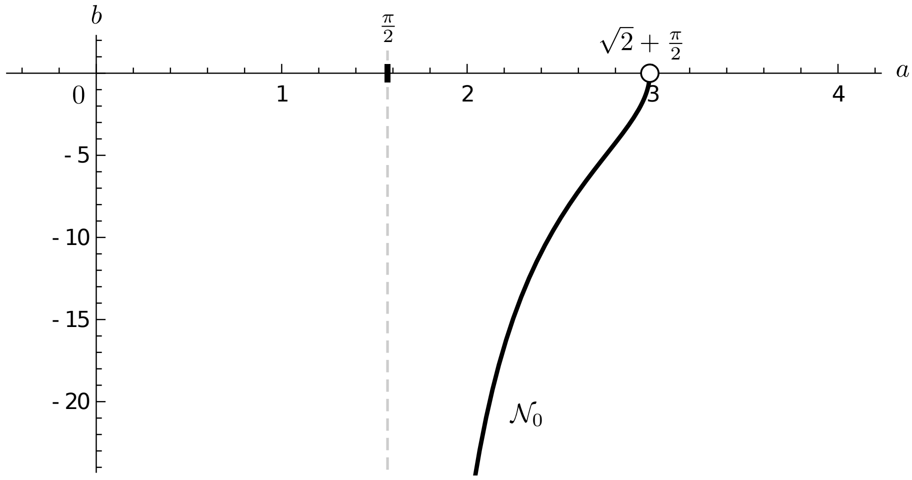


Fig. 4.1: The set $\mathcal{N}_0$ in the fourth quadrant of the ab-plane.

Remark 3. *It is straightforward to verify that*

$$\lim_{s\to 0-} (\eta_1(s), \eta_2(s)) = \left(\sqrt{2}+\tfrac{\pi}{2}, 0\right).$$

Moreover, it is possible to show that both points $\left(\left(\sqrt{2}+\frac{\pi}{2}\right)^2, 0\right)$ *and* $\left(0, \left(\sqrt{2}+\frac{\pi}{2}\right)^2\right)$ *belong to the Fučík spectrum* Σ_0.

4.2 The parametrization of the set $\mathcal{M}_0$

The set $\mathcal{M}_0$ is described by (3.18) in Corollary 1 for $c = 0$. For $c = 0$, we have that $p(a, 0) = -\frac{\pi}{2a}$ and thus, $\mathcal{M}_0$ is the set of all pairs $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$ (see Figure 4.2) such that

$$\mathcal{G}\left(a, b, \frac{2ab}{a+b}\right) = P\left(a, b, \frac{2ab}{a+b}\right), \tag{4.5}$$

where the function P is defined in (3.14) as $P(a, b, t) = \left(\frac{b}{a} - \frac{a}{b}\right)\frac{t}{\pi} + 1$, the function $\mathcal{G}$ reads

$$\mathcal{G}(a, b, t) = \begin{cases} \frac{b}{a}\cos\left(\frac{a+b}{2b}t + \frac{\pi}{2}\right) + P(a, b, t) & \text{for } t \in I_1, \\ \frac{a}{b}\cos\left(\frac{a+b}{2a}(t-2\pi) + \frac{b\pi}{2a}\right) + P(a, b, t-\pi) & \text{for } t \in I_2, \\ \frac{b}{a}\cos\left(\frac{a+b}{2b}(t-2\pi) + \frac{\pi}{2}\right) + P(a, b, t-2\pi) & \text{for } t \in I_3, \end{cases} \tag{4.6}$$

and

$$I_1 = \left(0, \frac{b\pi}{a+b}\right], \quad I_2 = \left(\frac{b\pi}{a+b}, 2\pi - \frac{b\pi}{a+b}\right], \quad I_3 = \left(2\pi - \frac{b\pi}{a+b}, 2\pi\right].$$

Theorem 5. *The set* $\mathcal{M}_0$ *is a continuous curve* $\mu : (0, +\infty) \to \mathbb{R}^2$ *with the parametrization* $\mu(s) := (\mu_1(s), \mu_2(s))$*, where functions* $\mu_1, \mu_2 : (0, +\infty) \to \mathbb{R}$ *are defined as*

$$\mu_1(s) := \begin{cases} \pi n - \frac{\pi}{2} + (s + \pi - \pi n)\sqrt{\frac{1-2n}{\cos s - 2n + 1}} & \textit{for } s \in (2\pi(n-1), 2\pi(n-1)+\pi], \\ & n \in \mathbb{N}, \\ s - \pi n + \frac{\pi}{2} + \pi n\sqrt{\frac{2n+\cos s}{2n}} & \textit{for } s \in (2\pi n - \pi, 2\pi n], \\ & n \in \mathbb{N}, \end{cases}$$

$$\mu_2(s) := \begin{cases} s + \pi - \pi n + \left(\pi n - \frac{\pi}{2}\right)\sqrt{\frac{\cos s - 2n + 1}{1-2n}} & \textit{for } s \in (2\pi(n-1), 2\pi(n-1)+\pi], \\ & n \in \mathbb{N}, \\ \pi n + \left(s - \pi n + \frac{\pi}{2}\right)\sqrt{\frac{2n}{2n+\cos s}} & \textit{for } s \in (2\pi n - \pi, 2\pi n], \\ & n \in \mathbb{N}. \end{cases}$$

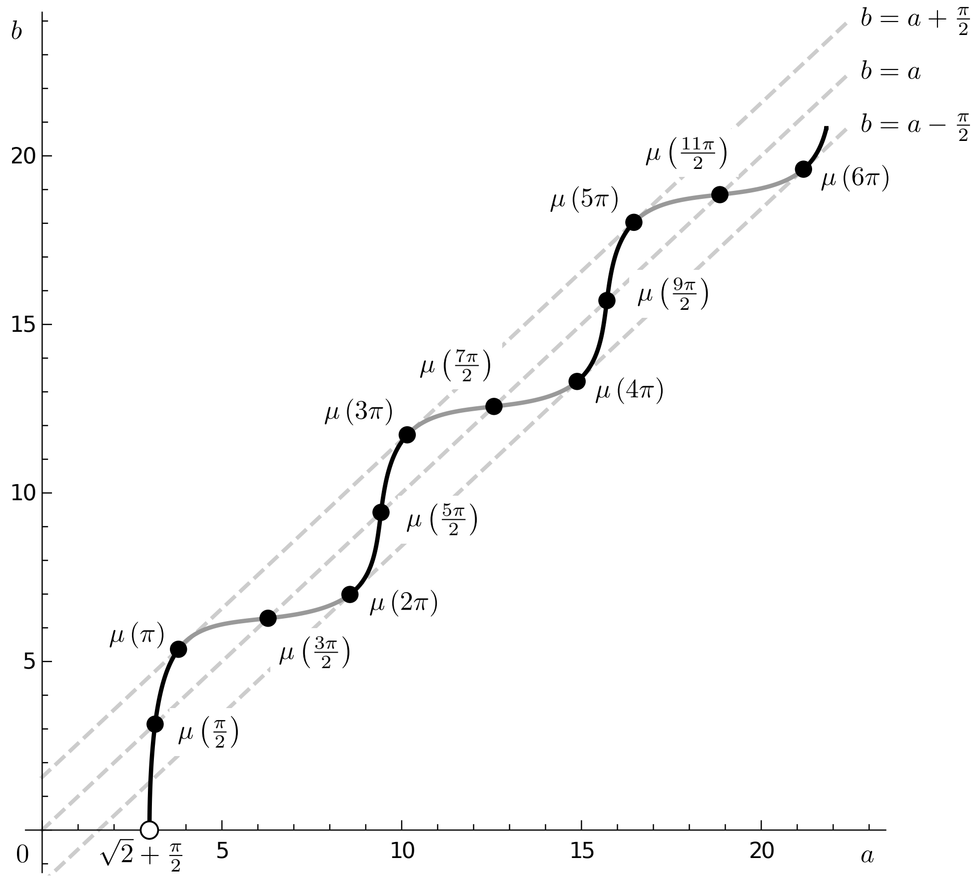


Fig. 4.2: The set $\mathcal{M}_0$ in the first quadrant of the ab-plane.

Proof. First of all, let us rewrite (4.6) in the following form

$$\mathcal{G}(a,b,t) = \begin{cases} \frac{b}{a}\cos\left(\frac{at}{2b} + \frac{t}{2} + \frac{\pi}{2}\right) + P(a,b,t) & \text{for } t \in \left(0,\ \frac{b\pi}{a+b}\right], \\ -\frac{a}{b}\cos\left(\frac{bt}{2a} + \frac{t}{2} - \frac{b\pi}{2a}\right) + P(a,b,t-\pi) & \text{for } t \in \left(\frac{b\pi}{a+b},\ 2\pi - \frac{b\pi}{a+b}\right], \\ \frac{b}{a}\cos\left(\frac{at}{2b} + \frac{t}{2} - \frac{\pi}{2} - \frac{a\pi}{b}\right) + P(a,b,t-2\pi) & \text{for } t \in \left(2\pi - \frac{b\pi}{a+b},\ 2\pi\right], \end{cases} \tag{4.7}$$

Now, let us denote

$$k := \frac{b}{a} > 0, \quad t := \frac{2ab}{a+b} > 0, \tag{4.8}$$

and rewrite the equation (4.5) (where $\mathcal{G}(a,b,t)$ is given by (4.7)) in the following equivalent form

$$\tilde{\mathcal{G}}(k,t) = \tilde{P}(k,t), \tag{4.9}$$

where

$$\tilde{P}(k,t) = \left(k - \frac{1}{k}\right)\frac{t}{\pi} + 1$$

and

$$\tilde{\mathcal{G}}(k,t) = \begin{cases} k\cos\left(\frac{t}{2k} + \frac{t}{2} + \frac{\pi}{2}\right) + \tilde{P}(k,t) & \text{for } t \in \left(0,\ \frac{k\pi}{k+1}\right], \\ -\frac{1}{k}\cos\left(\frac{t}{2} + \frac{kt}{2} - \frac{k\pi}{2}\right) + \tilde{P}(k,t-\pi) & \text{for } t \in \left(\frac{k\pi}{k+1},\ 2\pi - \frac{k\pi}{k+1}\right], \\ k\cos\left(\frac{t}{2k} + \frac{t}{2} - \frac{\pi}{2} - \frac{\pi}{k}\right) + \tilde{P}(k,t-2\pi) & \text{for } t \in \left(2\pi - \frac{k\pi}{k+1},\ 2\pi\right]. \end{cases} \tag{4.10}$$

Let us point out that the inverse transformation to (4.8) has the following form

$$a = \frac{t\,(k+1)}{2k} > 0, \quad b = \frac{t\,(k+1)}{2} > 0. \tag{4.11}$$

Now, let us split the proof according to the value of $t > 0$.

1. Let us consider $t \in \left(0,\ \frac{k\pi}{k+1}\right]$. In this case, the equation (4.9) reduces to

$$k \cos\left(\frac{t}{2k} + \frac{t}{2} + \frac{\pi}{2}\right) + \tilde{P}\,(k,t) = \tilde{P}\,(k,t)\,,$$

which can be equivalently written as

$$k \sin \frac{t(k+1)}{2k} = 0. \tag{4.12}$$

The equality in (4.12) cannot be satisfied since $k > 0$ and $0 < \frac{t(k+1)}{2k} \le \frac{\pi}{2}$. Thus, for given $k > 0$, there is no $t \in \left(0, \frac{k\pi}{k+1}\right]$ such that (4.9) holds.

2. For

$$t \in \left(\frac{k\pi}{k+1} + 2\pi\,(n-1)\,,\, 2\pi n - \frac{k\pi}{k+1}\right],\ n \in \mathbb{N}, \tag{4.13}$$

the condition (4.9) can be rewritten using the 2π-periodicity in the second argument

$$\tilde{\mathcal{G}}\,(k, t - 2\pi\,(n-1)) = \tilde{P}\,(k,t)\,,$$

$$-\frac{1}{k}\cos\left(\frac{t}{2} - \pi\,(n-1) + \frac{kt}{2} - k\pi\,(n-1) - \frac{k\pi}{2}\right) + \tilde{P}\,(k, t - 2\pi\,(n-1) - \pi) = \tilde{P}\,(k,t)\,. \tag{4.14}$$

Now, let us denote

$$s := \frac{t}{2} - \pi\,(n-1) + \frac{kt}{2} - k\pi\,(n-1) - \frac{k\pi}{2} + 2\pi\,(n-1) \tag{4.15}$$

and express t in terms of k and s as

$$t = \frac{2s - 2\pi n + 2\pi + 2kn\pi - k\pi}{k+1}. \tag{4.16}$$

According to (4.13), we have that $s \in (2\pi\,(n-1)\,,\ 2\pi\,(n-1) + \pi]$ and the condition (4.14) can be written as $-\cos\,(s) + k^2 - 1 - 2nk^2 + 2n = 0$ or equivalently (since $k > 0$)

$$k = \sqrt{\frac{\cos s - 2n + 1}{1 - 2n}} \tag{4.17}$$

Finally, let us combine (4.16), (4.17) and (4.11) to obtain

$$a = \pi n - \frac{\pi}{2} + (s + \pi - \pi n)\sqrt{\frac{1-2n}{\cos s - 2n + 1}} = \mu_1\,(s)\,,$$
$$b = s + \pi - \pi n + \left(\pi n - \frac{\pi}{2}\right)\sqrt{\frac{\cos s - 2n + 1}{1 - 2n}} = \mu_2\,(s)\,,$$

where $s \in (2\pi\,(n-1)\,,\ 2\pi\,(n-1) + \pi]$.

3. For

$$t \in \left(2\pi n - \frac{k\pi}{k+1},\ 2\pi n\right],\ n \in \mathbb{N}, \tag{4.18}$$

the condition (4.9) can be rewritten as

$$\tilde{\mathcal{G}}\,(k, t - 2\pi\,(n-1)) = \tilde{P}\,(k,t)\,,$$

$$k\cos\left(\frac{t-2\pi(n-1)}{2k}+\frac{t-2\pi(n-1)}{2}-\frac{\pi}{2}-\frac{\pi}{k}\right)+\frac{2}{k}-2k+\tilde{P}(k,t-2\pi(n-1))=\tilde{P}(k,t). \tag{4.19}$$

now, let us denote

$$s:=\frac{t}{2k}-\frac{\pi(n-1)}{k}\frac{t}{2}-\pi(n-1)-\frac{\pi}{2}-\frac{\pi}{k}+\pi(2n-1) \tag{4.20}$$

and express t as

$$t=\frac{2ks+2\pi n-2kn\pi+k\pi}{k+1}. \tag{4.21}$$

Using (4.18), we have $s\in\left(-\pi+2\pi n,\ -\frac{\pi}{2}+2\pi n\right]$ and the condition (4.9) reads $-k\cos(s)-2nk+2n\frac{1}{k}=0$ or, (since $k>0$)

$$k=\sqrt{\frac{2n}{\cos s+2n}}. \tag{4.22}$$

Now, let us combine (4.11), (4.21) and (4.22) to obtain

$$a=s-\pi n+\frac{\pi}{2}+\pi n\sqrt{\frac{2n+\cos s}{2n}}=\mu_1(s),$$
$$b=\pi n+\left(s-\pi n+\frac{\pi}{2}\right)\sqrt{\frac{2n}{2n+\cos s}}=\mu_2(s),$$

where $s\in\left(-\pi+2\pi n,\ -\frac{\pi}{2}+2\pi n\right]$.

4. Finally, let us consider

$$t\in\left(2\pi n,\ \frac{k\pi}{k+1}+2\pi n\right],\ n\in\mathbb{N}. \tag{4.23}$$

In this case, using 2π periodicity of the function $\tilde{\mathcal{G}}$, the condition (4.9) reads

$$\tilde{\mathcal{G}}(k,t-2\pi n)=\tilde{P}(k,t),$$

which can be rewritten as

$$k\cos\left(\frac{t-2\pi n}{2k}+\frac{t-2\pi n}{2}+\frac{\pi}{2}\right)+\tilde{P}(k,t-2\pi n)=\tilde{P}(k,t). \tag{4.24}$$

Now, let us denote

$$s:=\frac{t}{2k}-\frac{\pi n}{k}+\frac{t}{2}-\pi n+\frac{\pi}{2}+\pi(2n-1). \tag{4.25}$$

Using (4.23), the values of s are in the interval $s\in\left(2\pi n-\frac{\pi}{2},\ 2\pi n\right]$. Using (4.25), we get

$$t=\frac{k(2s-2\pi n+\pi)+2\pi n}{k+1}. \tag{4.26}$$

The condition (4.24) reads $-k\cos s-2nk+2n\frac{1}{k}=0$ and thus, we obtain ($k>0$)

$$k=\sqrt{\frac{2n}{2n+\cos s}}. \tag{4.27}$$

And finally, using (4.11), (4.26) and (4.27), we get

$$a=s-\pi n+\frac{\pi}{2}+\pi n\sqrt{\frac{2n+\cos s}{2n}}=\mu_1(s),$$
$$b=\pi n+\left(s-\pi n+\frac{\pi}{2}\right)\sqrt{\frac{2n}{2n+\cos s}}=\mu_2(s),$$

where $s\in\left(2\pi n-\frac{\pi}{2},\ 2\pi n\right]$.

It is straightforward to verify the continuity of functions μ_1 and μ_2, which finishes the proof. □

Remark 4. *According to Theorem 4, Theorem 5 and Remark 3, we have two continuous curves which belong to the Fučík spectrum* Σ_0 *and are symmetric with respect to the diagonal* $\alpha = \beta$ *and one of them is the curve* $\gamma : \mathbb{R} \to \mathbb{R}^2$ *given by*

$$\gamma(s) := (\gamma_1(s), \gamma_2(s)),$$

$$\gamma_1(s) := \begin{cases} \eta_1^2(s) & \text{for } s < 0, \\ (\sqrt{2} + \frac{\pi}{2})^2 & \text{for } s = 0, \\ \mu_1^2(s) & \text{for } s > 0, \end{cases}$$

$$\gamma_2(s) := \begin{cases} -\eta_2^2(s) & \text{for } s < 0, \\ 0 & \text{for } s = 0, \\ \mu_2^2(s) & \text{for } s > 0, \end{cases}$$

where η_1, η_2 *and* μ_1, μ_2 *are given in Theorem 4 and Theorem 5, respectively.*

Chapter 5

Conclusion

In this thesis, we obtained the following main results:

1. The description of all eigenvalues for the linear boundary value problem (2.1) in Theorem 1.
2. The compact implicit description of the Fučík spectrum Σ_c in the first quadrant in Theorem 2.
3. The parametrization of the Fučík spectrum Σ_0 by two continuous curves (see Remark 4).

Despite all our efforts, there are still some unanswered questions concerning the problem (1.1) and its Fučík spectrum Σ_c. Thus, at the end of this thesis, let us formulate at least some conjectures (see Figure 1.1):

1. The Fučík spectrum Σ_c and the set $\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \cdot \beta < 0\}$ (the union of the second and fourth quadrants) are disjoint sets if and only if $\tan c = -2$.
2. The set $\mathcal{N}_c$ introduced in (3.21) is an empty set if and only if $\tan c \leq -2$.
3. For $-2 < \tan c \neq 0$, the set $\mathcal{N}_c$ is a continuous curve in the fourth quadrant of ab-plane.
4. There exists some $q \in (-2, 0)$ such that for $-2 < \tan c < q < 0$, the set $\mathcal{N}_c$ as a continuous curve in the fourth quadrant continues to the third quadrant of ab-plane.

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