



The Weisfeiler–Leman Dimension of Distance-Hereditary Graphs

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Received: 9 May 2023 / Accepted: 3 July 2023

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Abstract

A graph is said to be distance-hereditary if the distance function in every connected induced subgraph is the same as in the graph itself. We prove that the ordinary Weisfeiler–Leman algorithm tests the isomorphism of any two graphs if one of them is distance-hereditary; more precisely, the Weisfeiler–Leman dimension of the class of finite distance-hereditary graphs is equal to 2. The previously best known upper bound for the dimension was 7.

Keywords Distance-hereditary graph · Weisfeiler–Leman algorithm · Weisfeiler–Leman dimension · Graph isomorphism

Mathematics Subject Classification 05C60 · 05C17 · 68Q19

1 Introduction

Over the past few decades, the Weisfeiler–Leman algorithm (WL) has become one of the most studied tools for testing isomorphism of finite graphs [3]. This algorithm colors the arcs of the graphs in question and then compares the numerical invariants of the obtained colorings; the graphs are declared to be isomorphic if the corresponding invariants are equal, and nonisomorphic otherwise. In the general case, the output is

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not always true, for example, if the input graphs are nonisomorphic strongly regular graphs with the same parameters.

Stronger isomorphism invariants are obtained if, instead of coloring the arcs, one considers coloring the d -tuples of vertices, $d > 2$; the corresponding generalization is called the d -dimensional Weisfeiler–Leman algorithm or the d -dim WL for short. It was introduced by Babai (see also [15]) and played an essential role in his recent quasipolynomial algorithm testing isomorphism of arbitrary graphs [2]. For $d = 1$ and $d = 2$, the d -dim WL coincides with the naive refinement and ordinary WL, respectively.

It can be shown that, given a graph X , there exists a positive integer d_X such that if $d \geq d_X$, then the d -dim WL identifies X (i.e., tests isomorphism between X and any other graph); the smallest such d_X is called the WL-dimension of the graph X . (For the exact definitions, we refer the reader to Sect. 2.) An equivalent definition of the WL-dimension can also be stated in terms of first-order logic with counting quantifiers and a bounded number of variables; the interested reader is referred to the monograph [11].

A rather general problem can be formulated as follows: determine the maximum WL-dimension of a graph belonging to a given class \mathfrak{K} ; this number is called the WL-dimension of \mathfrak{K} (cf. [11, Definition 18.4.3]). Although the WL-dimension of the class of all graphs cannot be bounded by a constant [5], for many natural graph classes the situation is different. Among these classes are the interval graphs [9], the planar graphs [16], and many others (see, e.g., [13] and references therein).

In a recent paper [12], it was proved that the WL-dimension of the class of graphs of rank width at most r is less than or equal to $3r + 4$. From [19, Proposition 7.3], it follows that if $r = 1$, then the latter class coincides with the well-known class of distance-hereditary graphs introduced in [14] (see also [4]); a graph is said to be distance-hereditary if the distance function in every connected induced subgraph is the same as in the graph itself. Thus, according to [12], the WL-dimension of the class of distance-hereditary graphs is at most 7. The main result of the present paper shows that this upper bound is not tight. More precisely, the following theorem holds.

Theorem 1 *The WL-dimension of the class of finite distance-hereditary graphs is equal to 2.*

Note that when the dimension of a graph class \mathfrak{K} is bounded from above by a constant d , the graph isomorphism problem restricted to \mathfrak{K} is solved in polynomial time by the d -dim WL. Thus, Theorem 1 shows that this conclusion holds with $d = 2$ if \mathfrak{K} is the class of distance-hereditary graphs. An efficient algorithm for this particular graph isomorphism problem was constructed in [18]; see also [8].

Modulo a characterization of graphs that have WL-dimension 1 (see [1, 17]), the proof of Theorem 1 reduces to verifying that the WL-dimension of a distance-hereditary graph is at most 2. To this aim, we use the theory of coherent configurations, see Sect. 2. Namely, given a graph X the output coloring of the ordinary WL defines a coherent configuration \mathcal{X} , which preserves all information needed to test isomorphism between X and any other graph. Moreover, it provides a full invariant of \mathcal{X} with respect to algebraic isomorphisms. As was proved in [10], the WL-dimension of the graph X is at most 2 if and only if the coherent configuration \mathcal{X} is separable, i.e., every algebraic

isomorphism of \mathcal{X} is induced by a suitable combinatorial isomorphism. Thus, we only need to check that \mathcal{X} is separable if X is a distance-hereditary graph. The proof of the latter is based on an inductive characterization of the distance-hereditary graphs, see [4, Theorem 1].

To make the paper self-contained, we introduce relevant concepts and statements of the theory of coherent configurations in Sect. 2. A translation of graph-theoretical operations (used in the inductive characterization of the distance-hereditary graphs) to the language of coherent configurations occupies Sections 3 and 4. The proof of Theorem 1 is given in Sect. 5.

2 Rainbows, Coherent Configurations, Graphs

In our presentation of coherent configurations, we mainly follow the monograph [6], where all the details can be found.

2.1 Notation

Throughout the paper, Ω denotes a finite set. For $\Delta \subseteq \Omega$, the diagonal of the Cartesian product $\Delta \times \Delta$ is denoted by 1_Δ .

For a binary relation $r \subseteq \Omega \times \Omega$, we set $r^* = \{(\beta, \alpha) : (\alpha, \beta) \in r\}$, $\alpha r = \{\beta \in \Omega : (\alpha, \beta) \in r\}$ for an element $\alpha \in \Omega$, $\Omega_-(r) = \{\alpha \in \Omega : \alpha r \neq \emptyset\}$, $\Omega_+(r) = \Omega_-(r^*)$, and $r^f = \{(\alpha^f, \beta^f) : (\alpha, \beta) \in r\}$ for any bijection f from Ω to another set. The product of the relations $r, s \subseteq \Omega \times \Omega$, is denoted by $r \cdot s = \{(\alpha, \beta) : (\alpha, \gamma) \in r, (\gamma, \beta) \in s \text{ for some } \gamma \in \Omega\}$.

For a set S of relations on Ω , we denote by S^\cup the set of all unions of elements of S , put $S^* = \{s^* : s \in S\}$, and $S^f = \{s^f : s \in S\}$ for any bijection f from Ω to another set. For $r \subseteq \Omega \times \Omega$, we define $r \cdot S = \{r \cdot s : s \in S\}$, $S \cdot r = \{s \cdot r : s \in S\}$, and $\alpha S = \cup_{s \in S} \alpha s, \alpha \in \Omega$.

For a class Δ of a partition π , we set $\pi \setminus \Delta = \pi \setminus \{\Delta\}$.

2.2 Rainbows

Let Ω be a finite set and S a partition of $\Omega \times \Omega$. A pair $\mathcal{X} = (\Omega, S)$ is called a **rainbow** on Ω if

$$1_\Omega \in S^\cup, \text{ and } S^* = S. \tag{2.1}$$

A rainbow can be thought of as an arc-colored complete directed graph with loop at each vertex, such that the colors of loops are different from those of other arcs and each color class is symmetric or anti-symmetric.

The elements of the sets $\Omega, S =: S(\mathcal{X})$, and S^\cup are called the **points, basis relations, and relations** of \mathcal{X} , respectively. A unique basis relation containing a pair $(\alpha, \beta) \in \Omega \times \Omega$ is denoted by $r_{\mathcal{X}}(\alpha, \beta)$; we omit the subscript \mathcal{X} wherever it does not lead to misunderstanding.

A set $\Delta \subseteq \Omega$ is called a **fiber** of a rainbow \mathcal{X} if $1_\Delta \in S$; the set of all fibers is denoted by $F := F(\mathcal{X})$. The point set Ω is the disjoint union of the fibers. If Δ is a union of fibers, then the pair

$$\mathcal{X}_\Delta = (\Delta, S_\Delta)$$

is a rainbow, where S_Δ consists of all $s_\Delta = s \cap (\Delta \times \Delta)$, $s \in S$. In what follows, we set $\mathcal{X} \setminus \Delta = \mathcal{X}_{\Omega \setminus \Delta}$.

Let $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ be rainbows. A bijection $f: \Omega \rightarrow \Omega'$ is called a **combinatorial isomorphism** (or simply **isomorphism**) from \mathcal{X} to \mathcal{X}' if $S^f = S'$. When $\mathcal{X} = \mathcal{X}'$, the set of all these isomorphisms form a permutation group on Ω . This group has a (normal) subgroup

$$\text{Aut}(\mathcal{X}) = \{f \in \text{Sym}(\Omega) : s^f = s \text{ for all } s \in S\}$$

called the **automorphism group** of \mathcal{X} .

2.3 Coherent Configurations

A rainbow $\mathcal{X} = (\Omega, S)$ is called a **coherent configuration** if, for any $r, s, t \in S$, the number

$$c_{rs}^t := |\alpha r \cap \beta s^*|$$

does not depend on the choice of $(\alpha, \beta) \in t$; the numbers c_{rs}^t are called the **intersection numbers** of \mathcal{X} . In this case, the set S^U contains the relation $r \cdot s$ for all $r, s \in S^U$; this relation is obviously the (possibly empty) union of those $t \in S$ for which $c_{rs}^t \neq 0$.

Let \mathcal{X} be a coherent configuration. Then for any $s \in S$, the sets $\Omega_-(s)$ and $\Omega_+(s)$ are fibers of \mathcal{X} . In particular, the union

$$S = \bigcup_{\Delta, \Gamma \in F(\mathcal{X})} S_{\Delta, \Gamma}$$

is disjoint, where $S_{\Delta, \Gamma}$ consists of all $s \in S$, contained in $\Delta \times \Gamma$. The number $|\delta s|$ with $\delta \in \Delta$ equals the intersection number $c_{s s^*}^{1_\Delta}$, and hence does not depend on the choice of the point δ . It is called the **valency** of s and denoted by n_s .

2.4 Algebraic Isomorphisms and Separability

Let $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ be coherent configurations. A bijection $\varphi: S \rightarrow S'$, $r \mapsto r'$ is called an **algebraic isomorphism** from \mathcal{X} onto \mathcal{X}' if

$$c_{rs}^t = c_{r's'}^{t'}, \quad r, s, t \in S; \tag{2.2}$$

the set of all such φ is denoted by $\text{Iso}_{\text{alg}}(\mathcal{X}, \mathcal{X}')$.

Each isomorphism f from \mathcal{X} onto \mathcal{X}' induces an algebraic isomorphism between these configurations, which maps $r \in S$ to $r^f \in S'$. A coherent configuration \mathcal{X} is said to be **separable** if every algebraic isomorphism from \mathcal{X} to another coherent configuration \mathcal{X}' is induced by a bijection from Ω to Ω' (which in this case is an isomorphism of the configurations in question). In other words, a coherent configuration \mathcal{X} is separable when the tensor of its intersection numbers defines \mathcal{X} up to isomorphism; for example, this is the case when all the valencies of \mathcal{X} are equal to 1.

The algebraic isomorphism φ induces a bijection from S^{\cup} onto $(S')^{\cup}$: the union $r \cup s \cup \dots$ of basis relations of \mathcal{X} is taken to $r' \cup s' \cup \dots$. This bijection is also denoted by φ . It preserves the dot product, i.e., $\varphi(r \cdot s) = \varphi(r) \cdot \varphi(s)$ for all $r, s \in S$.

One can see that if $\Delta \in F(\mathcal{X})$, then $\varphi(1_{\Delta}) = 1_{\Delta'}$ for some $\Delta' \in F(\mathcal{X}')$ (and $|\Delta| = |\Delta'|$); we also denote such a Δ' by Δ^{φ} . This extends φ to a bijection $F(\mathcal{X}) \rightarrow F(\mathcal{X}')$ so that $(1_{\Delta})^{\varphi} = 1_{\Delta^{\varphi}}$ for all Δ .

2.5 Parabolics and Quotients

Let $\mathcal{X} = (\Omega, S)$ be a rainbow. An equivalence relation on Ω that is a union of basis relations of \mathcal{X} is called a **parabolic** of \mathcal{X} . The parabolic 1_{Ω} is said to be **trivial**. A parabolic is said to be **maximal** if it is inclusion-wise maximal. Suppose further that \mathcal{X} is a coherent configuration. An important property of a parabolic e is that if φ is an algebraic isomorphism from \mathcal{X} to a coherent configuration \mathcal{X}' , then $\varphi(e)$ is a parabolic of \mathcal{X}' and

$$|\alpha e| = |\alpha' e'|, \quad \text{for all } \alpha \in \Delta, \alpha' \in \Delta', \Delta \in F(\mathcal{X}), \tag{2.3}$$

where $e' = \varphi(e)$ and $\Delta' = \Delta^{\varphi}$. When $\mathcal{X} = \mathcal{X}'$ and φ is the identity map, this shows that the classes of the equivalence relation e restricted to $\Delta \in F(\mathcal{X})$ have the same cardinality.

Let e be an equivalence relation on Ω . Denote by Ω/e the set of all classes of e . The map

$$\rho_e: \Omega \rightarrow \Omega/e, \alpha \mapsto \alpha e, \tag{2.4}$$

is obviously a surjection. It naturally induces a surjection from the binary relations on Ω to those on Ω/e . We put

$$S/e := \rho_e(S) = \{\rho_e(s) : s \in S\}.$$

Given a partition π of Ω , we set π/e to be the partition of Ω/e with classes $\rho_e(\Delta)$, $\Delta \in \pi$.

Suppose that e is a parabolic of \mathcal{X} . Then the pair

$$\mathcal{X}/e = (\Omega/e, S/e)$$

is a coherent configuration. The mapping ρ_e induces a surjection from the parabolics (respectively, fibers) of \mathcal{X} on those of \mathcal{X}/e . Every algebraic isomorphism φ from \mathcal{X} onto a coherent configuration \mathcal{X}' induces a natural algebraic isomorphism from \mathcal{X}/e onto \mathcal{X}'/e' , taking $\rho_e(s)$ to $\rho_{e'}(\varphi(s))$ for all $s \in S$, where $e' = \varphi(e)$. Further details can be found in [6, Section 2.1.3].

2.6 Graphs

By a **graph** we mean a finite simple undirected graph, i.e., a pair $X = (\Omega, E)$ of a finite set Ω of vertices and an irreflexive symmetric relation $E \subseteq \Omega \times \Omega$, which represents the edge set of X . The elements of $E =: E(X)$, which are ordered pairs of vertices, are called **arcs**, and E is the **arc set** of the graph X . Two vertices $\alpha, \beta \in \Omega$ are said to be **adjacent** (in X) whenever $(\alpha, \beta) \in E$; we also say that β is an X -**neighbor** of α . A vertex is said to be **pendant**, if it has a unique X -neighbor. The graph X is **regular** if the number of X -neighbors of α is the same for all vertices $\alpha \in \Omega$. The **distance** between any two vertices of X is defined as usual to be the length of a shortest path in X from one to the other. For $\Delta \subseteq \Omega$, let $X \setminus \Delta$ denote the subgraph of X induced by $\Omega \setminus \Delta$.

Two vertices α and β of X are called **twins** (in X) if, for any vertex $\gamma \in \Omega \setminus \{\alpha, \beta\}$, the set γE contains either both α and β or none of them. The relation e “to be twins in X ” is an equivalence relation on Ω . An equivalence relation contained in e is called a **twin equivalence** of X .

Let e be a twin equivalence of X . Denote by X/e the graph with vertex set Ω/e , in which two distinct vertices $\alpha e, \beta e$ are adjacent whenever every two vertices, one in αe and the other in βe , are adjacent in X . We say that X/e is the **quotient graph** of the graph X .

Lemma 2.1 *Let e be a twin equivalence of a graph X . Then the quotient graph X/e is isomorphic to an induced subgraph of X .*

Proof By the definition of X/e , αe and βe are adjacent in X/e if and only if α' and β' are adjacent in X for all $\alpha' \in \alpha e, \beta' \in \beta e$. Thus, X/e can be seen as a graph obtained from X by removing from each equivalence class of e all but one (arbitrarily chosen) vertex. □

A graph X is called **distance-hereditary** if the distance between any two vertices in any connected induced subgraph of X is the same as it is in X . The lemma below immediately follows from the definition.

Lemma 2.2 *An induced subgraph of a distance-hereditary graph is distance-hereditary.*

Let us recall the three one-vertex extensions by means of which all finite connected distance-hereditary graphs (and only those) can be constructed (see Theorem 2.3 below). Let X be a graph and let α be any vertex of X . Extend X to a graph X' by adding a new vertex α' to X with new edges from α' to either

- only α ,

- α and all its X -neighbors,
- just all X -neighbors of α .

In the first case the new vertex α' has degree 1 in X' and we say that X' is obtained from X by **attaching a pendant vertex**, which is α' . In the remaining two cases the vertices α, α' are twins in X' , and we say that X' is obtained from X by **splitting a vertex**.

Theorem 2.3 [4, Theorem 1] *A finite connected graph is distance-hereditary if and only if it is obtained from the one-vertex graph by a sequence of one-vertex extensions: attaching pendant vertices and splitting vertices.*

Corollary 2.4 *A distance-hereditary graph with at least two vertices has either a pendant vertex or two distinct twins.*

2.7 Coherent Closure

There is a natural partial order \leq on the set of all coherent configurations on the same set Ω . Namely, given two coherent configurations $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega, S')$, we set

$$\mathcal{X} \leq \mathcal{X}' \Leftrightarrow S^\cup \subseteq (S')^\cup.$$

The **coherent closure** $WL(T)$ of a set T of relations on Ω is defined to be the smallest (with respect to \leq) coherent configuration \mathcal{X} on Ω such that each element of T belongs to $S(\mathcal{X})^\cup$. The operator WL is monotone in the sense that if $S^\cup \subseteq T^\cup$, then $WL(S) \leq WL(T)$ and, moreover, $WL(\rho_e(S)) \leq WL(\rho_e(T))$ if e is an equivalence relation on Ω . For a partition π of Ω , we denote by $WL(T)_\pi$ the coherent closure of the union of a set T and all $1_\Delta, \Delta \in \pi$.

Lemma 2.5 *Let $\mathcal{X} = (\Omega, S)$ and $\mathcal{Y} = (\Omega, T)$ be coherent configurations such that $\mathcal{X} \geq \mathcal{Y}$, and $\pi = F(\mathcal{X})$. Then $\pi = F(WL(T)_\pi)$.*

Proof Let $\sigma = F(WL(T)_\pi)$. By the definition of a coherent closure, $\sigma^\cup \supseteq \pi^\cup$. On the other hand, $\pi = F(\mathcal{X})$ and $\mathcal{X} \geq \mathcal{Y}$ implies $\mathcal{X} = WL(S)_\pi \geq WL(T)_\pi$ and hence $\sigma^\cup \subseteq \pi^\cup$. This yields $\sigma = \pi$, as σ and π are partitions of the same set Ω . \square

Let X be a graph and π be a partition of its vertex set Ω . The coherent configuration $WL(X)$ of X is defined to be the coherent closure of the set $T = \{E(X)\}$, and we set $WL(X)_\pi = WL(T)_\pi$. The following lemma is a special case of [6, Theorem 2.6.4].

Lemma 2.6 *In the above notation, denote by $Aut(X)_\pi$ the subgroup of $Aut(X)$ leaving each class of π fixed. Then $Aut(WL(X)_\pi) = Aut(X)_\pi$.*

In the present paper, we avoid (vertex) colored graphs, instead we prefer to speak of a graph X equipped with a partition π of the vertex set (of X). We say that π is **correct** if

$$\pi = F(WL(X)_\pi),$$

In what follows, we also say that π is a **correct partition** of the graph X .

The exact definition of the WL-dimension of a graph requires a discussion about the d -dimensional Weisfeiler–Leman algorithm, which is beyond the scope of the present paper; we refer the interested reader to the monograph [11]. In the theorem below, we cite a characterization of regular graphs of WL-dimension 1 and a characterization of graphs of the WL-dimension at most 2. The proofs can be found in [1, Lemma 3.1(a)], [17] and [10, Theorem 2.1], respectively.

Theorem 2.7 *Let X be a graph and d be the WL-dimension of X .*

- (1) *If X is regular, then $d = 1$ if and only if X or its complement is isomorphic to a complete graph, a cocktail party graph¹, or the 5-cycle.*
- (2) *$d \leq 2$ if and only if the coherent configuration $WL(X)$ is separable.*

3 Twins in Coherent Configurations

Let $\mathcal{X} = (\Omega, S)$ be a rainbow. Two points α and β are called \mathcal{X} -twins if

$$r(\gamma, \alpha) = r(\gamma, \beta) \text{ for all } \gamma \in \Omega \setminus \{\alpha, \beta\}.$$

(Note that if $r(\gamma, \alpha) = r(\gamma, \beta)$ holds for all $\gamma \in \Omega$, then necessarily $\alpha = \beta$.) Obviously, any two \mathcal{X} -twins belong to the same fiber of \mathcal{X} . Furthermore, the relation “being \mathcal{X} -twins” is an equivalence relation on Ω . We denote it by $e_{\mathcal{X}}$. Recall that if e is a relation and Δ is a union of fibers of \mathcal{X} , then e_{Δ} denotes $e \cap (\Delta \times \Delta)$ (see Sect. 2.2).

Lemma 3.1 *Let \mathcal{X} be a coherent configuration. Then $e := e_{\mathcal{X}}$ is a parabolic of \mathcal{X} . Moreover, for all irreflexive relations $r, s \in S(\mathcal{X})$, we have $r = s$ if and only if $\rho_e(r) = \rho_e(s)$.*

Proof The equivalence relation e is the union of e_{Δ} , $\Delta \in F(\mathcal{X})$. Thus, to prove the first statement, by [6, Proposition 2.1.18], it suffices to verify that e_{Δ} is a partial parabolic for every fiber Δ . To this end, for $\Delta \in F(\mathcal{X})$ and $s \in S := S(\mathcal{X})$ such that $\Delta = \Omega_-(s)$, put

$$e(s) := \{(\alpha, \beta) \in \Omega^2 : \alpha s = \beta s\},$$

and then e_{Δ} coincides with

$$\bigcap_{s \in S: \Omega_-(s) = \Delta} e(s). \tag{3.1}$$

Since $e(s)$ is a parabolic of \mathcal{X}_{Δ} by [7, Exercise 2.7.8(1)], it follows that e_{Δ} is a parabolic of \mathcal{X}_{Δ} .

¹ A matching graph in the terminology of [1].

To prove the second statement, it suffices to verify that if relations $r, s \in S$ are irreflexive and $\rho_e(r) = \rho_e(s)$, then $r = s$. Suppose on the contrary that $r \neq s$. Without loss of generality, we may assume that there exists a pair (α, β) belonging to $r \setminus s$. Since the pair $(\alpha e, \beta e)$ belongs to both $\rho_e(r)$ and $\rho_e(s)$, one can find points $\alpha' \in \alpha e$ and $\beta' \in \beta e$ such that $r(\alpha', \beta') = s$. On the other hand, since $e = e_{\mathcal{X}}$, this implies that α, α' are \mathcal{X} -twins and

$$r(\beta, \alpha) = r^* = r(\beta, \alpha')$$

and hence, as β, β' are \mathcal{X} -twins,

$$r = r(\alpha', \beta) = r(\alpha', \beta') = s,$$

whence $r = s$, a contradiction. □

In view of Lemma 3.1, the equivalence relation $e_{\mathcal{X}}$ is called the **twin parabolic** of the coherent configuration \mathcal{X} . A characterization of the twin parabolic among the other parabolics of a coherent configuration is given in the following statement.

Lemma 3.2 *Let e be a parabolic of a coherent configuration \mathcal{X} . Then $e = e_{\mathcal{X}}$ if and only if e is a maximal parabolic of \mathcal{X} satisfying the following two conditions²:*

- (1) $e \cdot s = s$ for all $s \in S(\mathcal{X})$ such that $s \cap e = \emptyset$;
- (2) $e_{\Delta} = 1_{\Delta}$ or $e_{\Delta} \setminus 1_{\Delta} \in S(\mathcal{X})$ for all $\Delta \in F(\mathcal{X})$.

Proof To prove the “only if” part, assume that $e = e_{\mathcal{X}}$. Then Condition (1) is obvious (note that the equality in (1) may not hold if $s \subseteq e$; for example, when, for some $\Delta \in F(\mathcal{X})$, $s \subseteq \Delta \times \Delta$ and $e_{\Delta} = \Delta \times \Delta$). To prove that Condition (2) holds, suppose on the contrary that $e_{\Delta} \setminus 1_{\Delta} \notin S(\mathcal{X})$. Then there are two distinct irreflexive basis relations r and s such that $r \cup s \subseteq e_{\Delta}$; in particular, this means that if $\alpha, \beta \in \Delta$ and $r(\alpha, \beta) \in \{r, s\}$, then $\rho_e(\alpha) = \rho_e(\beta)$. It follows that

$$\rho_e(r) = \rho_e(1_{\Delta}) = \rho_e(s),$$

which, by Lemma 3.1, implies that $r = s$, a contradiction. Finally, to prove the maximality of e , let $e \subsetneq e'$ for some parabolic e' of \mathcal{X} . Then there exist α and β that are not \mathcal{X} -twins but $(\alpha, \beta) \in e'$. It follows that the relations $r = r(\alpha, \gamma)$ and $s = r(\beta, \gamma)$ are distinct for some point γ . But then $e \cdot s \supseteq r \cup s \neq s$ and so Condition (1) is violated for e' .

To prove the “if” part, assume that e is a maximal parabolic of \mathcal{X} satisfying Conditions (1) and (2). Then Condition (1) implies that every two points α and β such that $(\alpha, \beta) \in e$ are \mathcal{X} -twins. Consequently, $e \subseteq e_{\mathcal{X}}$. Now the required statement follows from the maximality of e . □

² In fact, for the “if” part, Condition (1) already suffices.

Every algebraic isomorphism preserves the inclusion between the relations, the dot product and parabolics. Thus, the following corollary is an immediate consequence of Lemma 3.2.

Corollary 3.3 *Let $\varphi \in \text{Iso}_{\text{alg}}(\mathcal{X}, \mathcal{X}')$. Then $\varphi(e_{\mathcal{X}}) = e_{\mathcal{X}'}$.*

We complete the section with a statement showing a relationship between twins in graphs and in coherent configurations.

Lemma 3.4 *Let X be a graph, π a correct partition of X , and $\mathcal{X} = \text{WL}(X)_{\pi}$. Then*

- (1) every two \mathcal{X} -twins are twins in X ;
- (2) every two twins in X belonging to the same fiber of \mathcal{X} are \mathcal{X} -twins.

Proof Part (1) follows by $E(X) \in S(\mathcal{X})^{\cup}$. Let Ω be the vertex set of X , and let α, β be distinct twins in X . Then the transposition $(\alpha, \beta) \in \text{Sym}(\Omega)$ is an automorphism of X fixing pointwise the set $\Omega \setminus \{\alpha, \beta\}$ and permuting α and β . If α, β are contained in the same fiber of \mathcal{X} , then this transposition preserves the partition $\pi = F(\mathcal{X})$ of Ω , hence it is an automorphism of \mathcal{X} by Lemma 2.6. This immediately proves Part (2). \square

4 Two Operations

In this section we consider two operations on graphs: **removing a matching** and **reducing twins**, which consist in removing subsets of vertices satisfying certain conditions. In both cases, the resulting graph is an induced subgraph of the original one. Our goal is to show how these operations affect the coherent configurations of graphs.

Throughout this section $X = (\Omega, E)$ is a graph.

4.1 Matchings

Let $\mathcal{X} = (\Omega, S)$ be a coherent configuration. A basis relation $m \in S$ is called a **matching** if m is irreflexive and $n_m = n_{m^*} = 1$. If this is the case, then, for every $\alpha \in \Omega_-(m)$, the set αm is a singleton, whose unique element is also denoted by αm by abuse of notation. Note that a matching m defines a bijection from $\Omega_-(m)$ to $\Omega_+(m)$. Furthermore, one can see that if a relation $m \cdot s$ (similarly, $s \cdot m$) with $s \in S$ is nonempty, then it is a basis one.

Suppose further that $E \in S^{\cup}$ and let m be a matching of \mathcal{X} . If $\Omega_-(m) \neq \Omega_+(m)$ and for any $\delta \in \Omega_-(m)$ the vertex δm is a unique X -neighbor of δ (i.e., the vertices of $\Omega_-(m)$ are all pendant), then m is called a **pendant matching** of \mathcal{X} . If, for any $\delta \in \Omega_-(m)$, the vertex δm is a twin of δ in X , then m is called a **twin matching** of \mathcal{X} . It is more precise to speak about pendant or twin matchings of \mathcal{X} with respect to the graph X ; in what follows we omit X if it is clear from the context.

Proposition 4.1 (Removing matching) *Let X be a graph, π a correct partition of X , m a matching of $\text{WL}(X)_{\pi}$, and $\Delta := \Omega_-(m)$. If m is pendant or twin, then*

$$\text{WL}(X)_{\pi} \setminus \Delta = \text{WL}(X \setminus \Delta)_{\pi \setminus \Delta}, \tag{4.1}$$

and $\pi \setminus \Delta$ is a correct partition of $X \setminus \Delta$.

Proof Put

$$\mathcal{X} := (\Omega, S) = \text{WL}(X)_\pi, \quad Y = X \setminus \Delta, \quad \mathcal{Y} := (\Omega \setminus \Delta, S') = \text{WL}(X \setminus \Delta)_{\pi \setminus \Delta},$$

so that Eq. (4.1) can be rewritten as $\mathcal{X} \setminus \Delta = \mathcal{Y}$, and we aim to prove this equality.

Since π is a correct partition, one has $1_\Gamma \in S(\mathcal{X})$, for all $\Gamma \in \pi \setminus \Delta$, and thus $1_\Gamma \in S(\mathcal{X} \setminus \Delta)$. Furthermore, since $E(Y) = E(X \setminus \Delta) \in S(\mathcal{X} \setminus \Delta)^\cup$ and \mathcal{Y} is the minimal coherent configuration containing $E(Y)$ and 1_Γ , for all $\Gamma \in \pi \setminus \Delta$, among its relations, we have $S(\mathcal{Y})^\cup \subseteq S(\mathcal{X} \setminus \Delta)^\cup$, i.e.,

$$\mathcal{X} \setminus \Delta \geq \mathcal{Y}. \tag{4.2}$$

Let T be the set of nonempty binary relations on Ω belonging to the set

$$S' \cup m \cdot S' \cup S' \cdot m^* \cup m \cdot S' \cdot m^*.$$

One can see that T is a partition of $\Omega \times \Omega$ that satisfies Eq. (2.1); this allows us to define an auxiliary rainbow $\mathcal{X}' = (\Omega, T)$. Moreover, observe that

$$T_{\Delta, \Delta} = m \cdot S' \cdot m^*, \tag{4.3}$$

$$T_{\Delta, \Omega \setminus \Delta} = T_{\Omega \setminus \Delta, \Delta}^* = m \cdot S', \tag{4.4}$$

$$T_{\Omega \setminus \Delta, \Omega \setminus \Delta} = S'. \tag{4.5}$$

Claim 4.2 *Suppose that \mathcal{X}' is a coherent configuration and $E \in T^\cup$. Then the conclusion of the proposition holds.*

Proof By $E \in T^\cup$, we see that $\mathcal{X}' \geq \text{WL}(X)$. Furthermore, the partition $F(\mathcal{X}')$ of Ω is a refinement of $\pi = F(\mathcal{X})$, since $F(\mathcal{X}') = F(\mathcal{Y}) \cup \{\Delta\}$ by the construction of \mathcal{X}' and $F(\mathcal{Y})$ is a refinement of $\pi \setminus \Delta$ by the definition of \mathcal{Y} . This implies that $\mathcal{X}' \geq \text{WL}(X)_\pi = \mathcal{X}$. Hence, we obtain

$$\mathcal{X} \setminus \Delta \leq \mathcal{X}' \setminus \Delta = \mathcal{Y}, \tag{4.6}$$

which, together with Eq. (4.2), yields $\mathcal{X} \setminus \Delta = \mathcal{Y}$, as required. Moreover, then $F(\mathcal{Y}) = F(\mathcal{X} \setminus \Delta) = \pi \setminus \Delta$ holds, i.e., $\pi \setminus \Delta$ is a correct partition of the graph $X \setminus \Delta$. \square

We need the following two claims to complete the proof of the proposition by using Claim 4.2.

Claim 4.3 *\mathcal{X}' is a coherent configuration.*

Proof We need to verify that the number

$$a = |\alpha r \cap \beta s^*|$$

does not depend on the choice of $(\alpha, \beta) \in t$ for all $r, s, t \in T$. Let $c_{r',s'}^t$ with $r', s', t' \in S'$ stand for the intersection numbers of \mathcal{Y} .

Obviously, $a = 0$ for all $(\alpha, \beta) \in t$, if $\Omega_+(r) \neq \Omega_-(s)$; otherwise, Eqs. (4.3)–(4.5) show that, up to replacing any of $r, s, t \in T$ by r^*, s^*, t^* , respectively, we need to consider the following cases:

- $r, s, t \in S'$: here clearly $a = c_{r,s}^t$ holds;
- $r \in S' \cdot m^*, s \in m \cdot S'$ and $t \in S'$: here $r = r' \cdot m^*, s = m \cdot s'$ for some $r', s' \in S'$, and hence

$$a = |\alpha r \cap \beta s^*| = |\alpha(r' \cdot m^*) \cap \beta(s' \cdot m^*)| = |\alpha r' \cap \beta s'^*| = c_{r',s'}^t;$$

- $r \in m \cdot S', s \in S'$ and $t \in m \cdot S'$: here $r = m \cdot r', t = m \cdot t'$ for some $r', t' \in S'$, and hence

$$a = |\alpha(m \cdot r') \cap \beta s^*| = |(\alpha m)r' \cap \beta s^*| = c_{r',s}^{t'};$$

- $r \in m \cdot S' \cdot m^*, s \in m \cdot S'$ and $t \in m \cdot S'$: here $r = m \cdot r' \cdot m^*, s = m \cdot s', t = m \cdot t'$ for some $r', s', t' \in S'$, and hence

$$a = |\alpha(m \cdot r' \cdot m^*) \cap \beta(s' \cdot m^*)| = |(\alpha m)r' \cap \beta s'^*| = c_{r',s'}^{t'};$$

- $r \in m \cdot S', s \in S' \cdot m^*$ and $t \in m \cdot S' \cdot m^*$: here $r = m \cdot r', s = s' \cdot m^*, t = m \cdot t' \cdot m^*$ for some $r', s', t' \in S'$, and hence

$$a = |\alpha(m \cdot r') \cap \beta(m \cdot s'^*)| = |(\alpha m)r' \cap (\beta m)s'^*| = c_{r',s'}^{t'};$$

- $r, s, t \in m \cdot S' \cdot m^*$: here $r = m \cdot r' \cdot m^*, s = m \cdot s' \cdot m^*, t = m \cdot t' \cdot m^*$ for some $r', s', t' \in S'$, and, as above, $a = c_{r',s'}^{t'}$ holds.

Thus, in any case the number a does not depend on the choice of $(\alpha, \beta) \in t$, and we are done. □

Claim 4.4 $E \in T^U$ holds.

Proof We first note that $E_{\Omega \setminus \Delta} = E(Y) \in T^U$ holds, since $E(Y) \in (S')^U$ by the definition of \mathcal{Y} and $S' \subseteq T$ by the definition of \mathcal{X}' . If m is a pendant matching, then obviously $E = m \cup m^* \cup E(Y)$, so we are done by $m \in T$.

Suppose that m is a twin matching. Choose an arbitrary $s \in T$ with $m \cdot s \neq \emptyset$. Observe that, for any $\delta \in \Delta, \gamma := \delta m$, and $\beta \in \gamma s$, we have that $r_{\mathcal{X}'}(\delta, \beta) = m \cdot s$. As δ and γ are twins in X , i.e., $(\gamma, \beta) \in E \Leftrightarrow (\delta, \beta) \in E$, we see that

$$s \subseteq E \Leftrightarrow m \cdot s \subseteq E \quad \text{and} \quad s \cap E = \emptyset \Leftrightarrow (m \cdot s) \cap E = \emptyset. \tag{4.7}$$

If $\Omega_+(s) \subseteq \Omega \setminus \Delta$, then $s \in S'$ by Eq. (4.5). Since s is a basis relation of \mathcal{Y} and $E(Y) \subseteq E$, it follows that $(s \cap E \neq \emptyset \Rightarrow s \subseteq E)$ holds. By Eq. (4.7), we obtain

$$s' \in S', (m \cdot s') \cap E \neq \emptyset \quad \Rightarrow \quad m \cdot s' \subseteq E. \tag{4.8}$$

If $\Omega_+(s) = \Delta$, then $s = s' \cdot m^*$ for $s' = r_{\mathcal{X}'(\gamma, \beta m)}$, $s' \in S'$ by Eq. (4.3). As β and βm are twins in X , we see that

$$s \subseteq E \Leftrightarrow s' \subseteq E \quad \text{and} \quad s \cap E = \emptyset \Leftrightarrow s' \cap E = \emptyset. \tag{4.9}$$

As above, it follows that $(s' \cap E \neq \emptyset \Rightarrow s' \subseteq E)$ holds. By Eqs. (4.7), (4.9), this implies that

$$s'' \in S', (m \cdot s'' \cdot m^*) \cap E \neq \emptyset \quad \Rightarrow \quad m \cdot s'' \cdot m^* \subseteq E. \tag{4.10}$$

By Eqs. (4.8) and (4.10), we conclude that

$$E_{\Delta, \Omega} = \bigcup_{\Gamma \in \pi} E_{\Delta, \Gamma} = E_{\Delta} \cup \left(\bigcup_{\Gamma \in \pi \setminus \Delta} E_{\Delta, \Gamma} \right) \in (m \cdot S' \cup m \cdot S' \cdot m^*)^{\cup} \subseteq T^{\cup},$$

which, together with $E_{\Delta, \Omega}^* = E_{\Omega, \Delta}$, implies that $E = E_{\Delta, \Omega} \cup E_{\Delta, \Omega}^* \cup E_{\Omega \setminus \Delta} \in T^{\cup}$, as required. □

Claims 4.3 and 4.4 show that \mathcal{X}' satisfies the assumption of Claim 4.2, whence the proposition follows. □

Proposition 4.5 *In the notation of Proposition 4.1, $WL(X)_{\pi} \setminus \Delta$ is separable if and only if $WL(X)_{\pi}$ is separable.*

Proof The result follows from [10, Lemma 3.3(1)]. □

4.2 Twins

Let π be a correct partition of the graph X and e the twin parabolic of $WL(X)_{\pi}$. Recall (see Sect. 2.5) that $\pi/e := \rho_e(\pi)$, where ρ_e is the mapping defined by Eq. (2.4). Since ρ_e maps fibers to fibers, we have that

$$\pi/e = F(WL(X)_{\pi}/e). \tag{4.11}$$

The next proposition is an analogue of Proposition 4.1 in regard to twins in a graph.

Proposition 4.6 (Reducing twins) *Let X be a graph, π a correct partition of X , and e the twin parabolic of $WL(X)_{\pi}$. Then e is a twin equivalence of X ,*

$$WL(X)_{\pi}/e = WL(X/e)_{\pi/e}, \tag{4.12}$$

and π/e is a correct partition of X/e .

Proof The statement about e being a twin equivalence of the graph X follows from Lemma 3.4 (1). Next, we put

$$\mathcal{X} := (\Omega, S) = WL(X)_{\pi}, \quad Y = X/e, \quad \mathcal{Y} := (\Omega/e, \bar{T}) = WL(Y)_{\pi/e},$$

so that Eq. (4.12) can be rewritten as $\mathcal{X}/e = \mathcal{Y}$, and we aim to prove this equality.

By the definition of \mathcal{X}/e (see Sect. 2.5), it follows that $E(Y) = \rho_e(E) \in S(\mathcal{X}/e)^\cup$ and $1_\Gamma \in S(\mathcal{X}/e)$ for every $\Gamma \in \pi/e = F(\mathcal{X}/e)$ by Eq. (4.11). Since \mathcal{Y} is the minimal coherent configuration containing $E(Y)$ and 1_Γ , for all $\Gamma \in \pi/e$, among its relations, we have

$$\mathcal{X}/e \geq \mathcal{Y}. \tag{4.13}$$

In order to prove $\mathcal{X}/e \leq \mathcal{Y}$, we first note that since $F(\mathcal{X}/e) = \pi/e$ and $\mathcal{Y} = \text{WL}(Y)_{\pi/e}$, Lemma 2.5 implies that $F(\mathcal{Y}) = \pi/e$. Then we need to define an auxiliary rainbow. To this end, put $\rho := \rho_e$ and define a set T of binary relations on Ω as follows:

$$\{\rho^{-1}(\bar{t}) : \bar{t} \in \bar{T} \text{ is irreflexive}\} \cup \{e_\Delta \setminus 1_\Delta : \Delta \in F\}^\sharp \cup \{1_\Delta : \Delta \in F\}, \tag{4.14}$$

where $F = F(\mathcal{X})$ and, for any set S , S^\sharp stands for $S \setminus \{\emptyset\}$. Then $\rho(t) \in \bar{T}$ for any irreflexive $t \in T$, and $\rho(e_\Delta \setminus 1_\Delta) = \rho(1_\Delta) = 1_{\rho(\Delta)}$

for all $\Delta \in F(\mathcal{X})$. Further, one can see that T is a partition of $\Omega \times \Omega$ that satisfies Eq. (2.1). Therefore, $\mathcal{X}' = (\Omega, T)$ is a rainbow. Every $r \in T$ is a union of some relations of S ; in particular, $r \in \Delta \times \Gamma$ for some $\Delta, \Gamma \in F$. Moreover, $e \in T^\cup$ and hence it is a parabolic of \mathcal{X}' .

Claim 4.7 *Suppose that \mathcal{X}' is a coherent configuration and $E \in T^\cup$. Then the conclusion of the proposition holds.*

Proof Since $E \in T^\cup$, we see that $\mathcal{X}' \geq \text{WL}(X)$. By Eq. (4.14), $F(\mathcal{X}')$ coincides with $\pi = F(\mathcal{X})$ and hence $\mathcal{X}' \geq \text{WL}(X)_\pi = \mathcal{X}$.

Furthermore, since $\rho(t) \in \bar{T}$ for all $t \in T$, it follows $\mathcal{X}'/e = \mathcal{Y}$. Therefore,

$$\mathcal{X}/e \leq \mathcal{X}'/e = \mathcal{Y}, \tag{4.15}$$

which, together with Eq. (4.13), yields $\mathcal{X}/e = \mathcal{Y}$, as required. Finally, π/e is a correct partition of the graph X/e , since $F(\mathcal{Y}) = \pi/e$. □

We need the following two claims.

Claim 4.8 *\mathcal{X}' is a coherent configuration.*

Proof It suffices to verify that the number

$$a = |\alpha r \cap \beta s^*|$$

does not depend on the choice of $(\alpha, \beta) \in t$ for all $r, s, t \in T$. To this end, we set $\bar{\delta} = \rho(\delta)$ for every $\delta \in \Omega$, and note that $\bar{r} = \rho(r)$, $\bar{s} = \rho(s)$, and $\bar{t} = \rho(t)$ are basis relations of \mathcal{Y} . Therefore,

$$|\bar{\alpha} r \cap \bar{\beta} s^*| = c_{\bar{r}\bar{s}}^{\bar{t}}.$$

and this number does not depend on $(\alpha, \beta) \in t$. Furthermore, if $a = 0$ for some $(\alpha, \beta) \in t$, then $c_{\overline{r,s}} = 0$, because $e \subseteq e_{\mathcal{X}'}$ (by the definition of T). But then obviously $a = 0$ for all $(\alpha, \beta) \in t$. Thus, without loss of generality, we may assume that $a \neq 0$.

Let $\gamma \in \alpha r \cap \beta s^*$. Note that $\alpha r \cap \beta s^* \subseteq \Gamma$ for some $\Gamma \in F$. By Eq. (2.3) (applied to $\Delta = \Delta' = \Gamma$), this implies that the number $k = |\gamma e|$ does not depend on $\gamma \in \alpha r \cap \beta s^*$. It follows that

$$a = \begin{cases} kc_{\overline{r,s}} & \text{if } r \not\subseteq e \wedge s \not\subseteq e, \\ k - 1 & \text{if } (r \not\subseteq e \wedge s \subseteq e') \vee (r \subseteq e' \wedge s \not\subseteq e), \\ k - 2 & \text{if } r \subseteq e' \wedge s \subseteq e' \wedge t \subseteq e', \\ 1 & \text{otherwise,} \end{cases}$$

where $e' = e \setminus 1_\Omega$; here, we made use of the fact that if $x \in T$ is contained in e , then $x \subseteq e'$ or $x \subseteq 1_\Omega$. Thus, the number a does not depend on the choice of $(\alpha, \beta) \in t$, as required. □

Claim 4.9 $E \in T^U$ holds.

Proof It suffices to verify that E contains each irreflexive $t \in T$ such that $t \cap E \neq \emptyset$. Note that the latter condition implies that $\rho(t) \cap \rho(E) \neq \emptyset$ and hence $\rho(t) \subseteq \rho(E)$, since $\rho(t)$ is a basis relation of \mathcal{Y} and $\rho(E)$ is a relation of \mathcal{Y} . Thus,

$$\rho^{-1}(\rho(t)) \subseteq \rho^{-1}(\rho(E)).$$

On the other hand, the relation $\rho^{-1}(\rho(t))$ is equal to t if $\rho(t)$ is irreflexive, or e_Δ for some $\Delta \in F(\mathcal{X})$ otherwise (see Eq. (4.14)). In any case, $t \subseteq \rho^{-1}(\rho(t))$. Furthermore,

$$\rho^{-1}(\rho(E)) \subseteq E \cup 1_\Omega,$$

because e is a twin equivalence of X (see above). Thus,

$$t \subseteq \rho^{-1}(\rho(t)) \subseteq \rho^{-1}(\rho(E)) \subseteq E \cup 1_\Omega.$$

Since t is irreflexive, this implies that $t \subseteq E$, as required. □

Claims 4.8, 4.9 show that \mathcal{X}' satisfies the assumption of Claim 4.7, whence the proposition follows. □

The following proposition, which holds for any coherent configuration, together with Proposition 4.6 shows that $WL(X)_\pi$ is separable if $WL(X/e)_{\pi/e}$ is separable, and this fact will be used in the proof of Theorem 1 in Sect. 5.

Proposition 4.10 *A coherent configuration \mathcal{X} is separable if $\mathcal{X}/e_{\mathcal{X}}$ is separable.*

Proof Let $\mathcal{X} = (\Omega, S)$ and $e := e_{\mathcal{X}}$. Assume that the coherent configuration \mathcal{X}/e is separable. We need to verify that given a coherent configuration $\mathcal{X}' = (\Omega', S')$, any algebraic isomorphism $\varphi : s \mapsto s'$ from \mathcal{X} to \mathcal{X}' is induced by a bijection (see Sect. 2.4). We note that by Corollary 3.3, $e' := \varphi(e)$ is the twin parabolic of \mathcal{X}' .

Given $\Delta \in F(\mathcal{X})$, we choose a full system $\overline{\Delta}$ of distinct representatives of the classes of the equivalence relation e_Δ , and let $\overline{\Omega}$ be the union of all $\overline{\Delta}$, $\Delta \in F(\mathcal{X})$ and $\overline{S} = S_{\overline{\Omega}}$ be the set of the relations of S restricted to $\overline{\Omega}$. Then the pair $\overline{\mathcal{X}} = (\overline{\Omega}, \overline{S})$ is obviously a rainbow. Since the parabolic e is twin, the natural bijection

$$\overline{h}: \overline{\Omega} \rightarrow \Omega/e, \quad \alpha \mapsto \alpha e$$

is a rainbow isomorphism from $\overline{\mathcal{X}}$ to \mathcal{X}/e (see the second part of Lemma 3.1). In particular, $\overline{\mathcal{X}}$ is a coherent configuration. In a similar way, one can define the sets $\overline{\Delta}'$, $\Delta' \in F(\mathcal{X}')$, and $\overline{\Omega}'$, the rainbow $\overline{\mathcal{X}'}$, the isomorphism

$$\overline{h}': \overline{\Omega}' \rightarrow \Omega'/e', \quad \alpha' \mapsto \alpha' e',$$

and check that $\overline{\mathcal{X}'}$ is a coherent configuration.

The algebraic isomorphism φ induces an algebraic isomorphism $\overline{\varphi} \in \text{Iso}_{\text{alg}}(\mathcal{X}/e, \mathcal{X}'/e')$ (see Sect. 2.5). By the proposition assumption, the coherent configuration \mathcal{X}/e is separable. Consequently, $\overline{\varphi}$ is induced by a bijection, say h . It follows that the composition mapping $\overline{f} = \overline{h} \circ h \circ (h')^{-1}$ induces the restriction of φ to \overline{S} , i.e.,

$$r(\alpha, \beta)' = r(\alpha^{\overline{f}}, \beta^{\overline{f}}), \quad \alpha, \beta \in \overline{\Omega}. \tag{4.16}$$

Let us extend \overline{f} to a bijection $f: \Omega \rightarrow \Omega'$. To do so, given $\alpha \in \overline{\Omega}$, we choose an arbitrary bijection $f_\alpha: \alpha e \rightarrow \alpha' e$ that takes α to $\alpha' = \alpha^{\overline{f}}$ (such a bijection exists because $|\alpha e| = |\alpha' e'|$ in view of Eq. (4.16)). Since the union of αe , $\alpha \in \overline{\Omega}$, equals Ω , the desired bijection \overline{f} is defined uniquely by the condition $f|_{\alpha e} = f_\alpha$.

To complete the proof it suffices to verify that $r(\alpha, \beta)^f = r(\alpha, \beta)'$ for all $\alpha, \beta \in \Omega$, i.e., the algebraic isomorphism φ is induced by the bijection f . Denote by $\overline{\alpha}$ and $\overline{\beta}$ the unique points of $\overline{\Omega}$, lying in αe and βe , respectively. Then α and β are \mathcal{X} -twins of $\overline{\alpha}$ and $\overline{\beta}$, respectively. Moreover, from the definition of f , it follows that α^f and β^f are \mathcal{X}' -twins of $\overline{\alpha}^f$ and $\overline{\beta}^f$, respectively. Thus, by Eq. (4.16), we have

$$r(\alpha, \beta)^f = r(\alpha^f, \beta^f) = r(\overline{\alpha}^f, \overline{\beta}^f) = r(\overline{\alpha}, \overline{\beta})' = r(\alpha, \beta)',$$

as required. □

5 Proof of Theorem 1

To prove Theorem 1, we need the following two auxiliary lemmas.

Lemma 5.1 *The WL-dimension of the class of distance-hereditary graphs is greater than 1.*

Proof It follows from Theorem 2.7(1) that a regular graph X has WL-dimension 1 if and only if X or its complement is isomorphic to a complete graph, a cocktail party

graph, or the 5-cycle. Since $K_{n,n}$, a complete bipartite graph with parts of size n , is regular, it has WL-dimension greater than 1 if $n > 2$. As $K_{n,n}$ is distance-hereditary by Theorem 2.3, the lemma follows. \square

Lemma 5.2 *Let X be a distance-hereditary graph with at least two vertices, π a correct partition of X , and $\mathcal{X} = \text{WL}(X)_\pi$. Then \mathcal{X} has a twin matching or a pendant matching, or the twin parabolic $e_{\mathcal{X}}$ is nontrivial.*

Proof Let $X = (\Omega, E)$ and suppose first that there are no twins in X . Then X has pendant vertices by Corollary 2.4. No two of them share the same X -neighbor, for otherwise they are twins in X , a contradiction. It follows that if α is a pendant vertex and β is a unique X -neighbor of α , then $m = r(\alpha, \beta)$ is a matching in \mathcal{X} . Moreover, $\Omega_-(m) \neq \Omega_+(m)$, for otherwise the vertices α and β form a connected component of X and hence are twins. Thus, m is a pendant matching.

Let X have two distinct twins α and β . If they belong to the same fiber of \mathcal{X} , then the twin parabolic $e_{\mathcal{X}}$ is nontrivial by Lemma 3.4(2) and we are done. Thus, we may assume that no two distinct twins in X belong to the same fiber of \mathcal{X} . To complete the proof, it suffices to verify that the relation $m = r(\alpha, \beta)$ is a (twin) matching. Assume on the contrary that m or m^* has valency at least 2. Without loss of generality, we may assume that there exists $\beta' \in \alpha m$ other than β .

Suppose that there exists an X -neighbor γ of β , which is not an X -neighbor of β' . Then the relation $r = r(\alpha, \gamma)$ is contained in E (because $r(\beta, \gamma) \subseteq E$ and α and β are twins in X), whereas $t = r(\beta', \gamma)$ is not. On the other hand,

$$r(\alpha, \beta) = m = r(\alpha, \beta') \subseteq r(\alpha, \gamma) \cdot r(\gamma, \beta') = r \cdot t^*.$$

It follows that there exists $\gamma' \in \Omega$ such that $(\alpha, \gamma') \in r$ and $(\beta, \gamma') \in t^*$. Since $r \subseteq E$ and $t \cap E = \emptyset$, this contradicts the fact that α and β are twins in X . Thus, the point γ does not exist and hence $\beta E \subseteq \beta' E$. Since β' and β lie in the same fiber of \mathcal{X} and E is a relation of \mathcal{X} , this inclusion is the equality. Consequently, β and β' are distinct twins in X , lying in the same fiber, a contradiction. \square

We are now in a position to prove Theorem 1. Let X be a distance-hereditary graph. By Lemma 5.1, it suffices to prove that the WL-dimension of X is at most 2, or, equivalently, the coherent configuration $\text{WL}(X)$ is separable (see Theorem 2.7(2)). We shall prove a more general statement that, for a correct partition π of X , the coherent configuration $\text{WL}(X)_\pi$ is separable, which implies the result by $\text{WL}(X) = \text{WL}(X)_\pi$, where $\pi = F(\text{WL}(X))$.

We use induction on the number n of vertices of X .

Without loss of generality, we may assume that $n \geq 2$ and the statement holds for all distance-hereditary graphs with at most $n - 1$ vertices and their correct partitions. By Lemma 5.2, the coherent configuration $\mathcal{X} = \text{WL}(X)_\pi$ has a twin matching or a pendant matching m , or the twin parabolic $e := e_{\mathcal{X}}$ is nontrivial.

In the former case, let Δ denote $\Omega_-(m)$. By Proposition 4.1, $\pi \setminus \Delta$ is a correct partition of the graph $X \setminus \Delta$. Since this graph is distance-hereditary by Lemma 2.2, the coherent configuration $\text{WL}(X \setminus \Delta)_{\pi \setminus \Delta}$ is separable by induction. Thus, \mathcal{X} is separable by Propositions 4.1 and 4.5.

In the latter case, π/e is a correct partition of the quotient graph X/e by Proposition 4.6. By Lemmas 2.1 and 2.2, X/e is distance-hereditary. Hence, by induction, the coherent configuration $WL(X/e)_{\pi/e}$ is separable. Thus, \mathcal{X} is separable by Propositions 4.6 and 4.10.

Acknowledgements The research of Alexander Gavriluk is supported by JSPS KAKENHI Grant Number 22K03403. Roman Nedela is supported by the Czech Science Foundation, grant GACR 20-15576S and by the Slovak Research and Development Agency, Grant No. APVV-15-0220.

Funding Japan Society for the Promotion of Science London, 22K03403, Alexander Gavriluk. Czech Science Foundation, GACR 20-15576S, Roman Nedela. Slovak Research and Development Agency, APVV-15-0220, Roman Nedela.

Data Availability The manuscript has no associated data.

Declarations

Conflict of Interest The authors have no relevant financial or non-financial interests to disclose.

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