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ALGEBRAICKÁ ANALÝZA KONVOLUCÍ NADPLOCH

Jan Vršek

Disertační práce k získaní akademického titulu doktor v oboru

Aplikovaná matematika

Školitel:

RNDr. Miroslav Lávička, Ph.D.

Plzeň, 2011

FACULTY
OF APPLIED SCIENCES
UNIVERSITY
OF WEST BOHEMIA

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ALGEBRAIC ANALYSIS OF CONVOLUTIONS OF HYPERSURFACES

Jan Vršek

A thesis for the degree of Doctor in the subject of

Applied Mathematics

Advisor:

RNDr. Miroslav Lávička, Ph.D.

Plzeň, 2011

Acknowledgement

I would like to thank all the people who have supported me during my studies. Especially many thanks belong to my family for their moral and material support, my advisor RNDr. Miroslav Lávička, Ph.D. for his guidance and constant support and my brother Mgr. Michal Bizzarri for his advices on corrections of the thesis.

I hereby declare that this Ph.D. thesis is completely my own work and that I used only the cited sources.

Plzeň, January 20, 2011, ...

Anotace

V posledních letech se studium konvolucí nadploch (zejména křivek a ploch) stalo aktivní oblastí výzkumu. Například operace ofsetu, tj. jedna z fundamentálních vlastností v počítačově podporovaném designu (CAD) není nic jiného než konvoluce s kružnicí/kulovou plochou. Hlavním cílem předkládané práce je poskytnout teoretickou analýzu konvolucí nadploch z algebraického úhlu pohledu.

Té bude věnována zejména první část práce. Přestože dokážeme, že konvoluce ireducibilních nadploch je téměř vždy ireducibilní, může se v některých případech rozpadnout na více komponent. Horní odhad jejich počtu nalezneme s využitím tzv. konvolučního stupně. Pro ten bude v případě křivek odvozena formule, vyjadřující konvoluční stupeň v závislosti na algebraickém stupni a rodu křivky. Detailní analýze budou podrobeny speciální a degenerované komponenty. Dále věnujeme speciální pozornost racionálním nadplochám a racionálním komponentám jejich konvolucí.

V druhé části práce se zaměříme na dvě nejjednodušší třídy algebraických nadploch vzhledem k operaci konvoluce, konkrétně na nadplochy s konvolučním stupněm jedna a dva. Zatímco první jmenovaná třída se ukáže být totožná s již známou třídou LN nadploch, významným zástupcem druhé třídy jsou nadsféry. Racionalita konvolucí s těmito nadplochami bude detailně prozkoumána. Navíc pro křivky odvodíme formuli umožňující vypočítat rod jejich konvoluce s obecnou křivkou. Závěrem bude nalezen rozklad křivek nízkého konvolučního stupně na konvoluci konečně mnoha jednoduchých fundamentálních křivek.

Klíčová slova

Konvoluce, incidenční varieta, varieta parametrů, opěrná funkce, konvoluční stupeň, LN nadplocha, QN nadplocha, koherentní forma

Annotation

In recent years, studying convolutions of hypersurfaces (especially of curves and surfaces) has become an active research area. For instance, one of the fundamental operations in Computer Aided Design, i.e., offsetting, can be expressed as the convolution with a circle/sphere. The main goal of the thesis is to provide the theoretical analysis of convolutions of hypersurfaces from the algebraic point of view.

This goal will be accomplished in the first part of the thesis. Although we will prove that the convolution of irreducible algebraic hypersurfaces is generically irreducible, it can still decomposes into more irreducible components. The upper bound for the number of components, in the terms of the so-called convolution degrees of the hypersurfaces, will be given. Further, a formula expressing the convolution degree of a plane curve using the algebraic degree and the genus of the curve will be derived. In addition, a detailed analysis of the so-called special and degenerated components is provided. The special attention will be devoted to rational hypersurfaces and rational components

The second part of the thesis will focus on the two simplest classes of algebraic hypersurfaces with respect to the operation of convolution, namely on the hypersurfaces with the convolution degree one and two. The former case turns out to coincide with the well-known LN hypersurfaces, i.e., hypersurfaces with Linear Normals, and the most prominent example of later hypersurfaces are hyperspheres. The problem of rationality of convolutions with these hypersurfaces will be studied in more detail. In the curve case, the genus formula is derived. Moreover the decomposition of curves with low convolution degree into the convolution of finite number of simple fundamental ones will be provided.

Keywords

Convolution, incidence variety, parameter variety, support function, convolution degree, LN hypersurface, QN hypersurface, coherent form

Glossary of notations

$\sharp X$	Cardinality of the set X
$\mathbb{C}^n$	Affine n-space
$\mathbf{x} = (x_1, \ldots, x_n)$	Affine point
$P^n\mathbb{C}$	Projective n-space
$\mathbf{x} = (x_0 : \cdots : x_n)$	Projective point
ω	Ideal hyperplane ($x_0 = 0$)
$\mathrm{cl}\,(X)$	Topological closure of a set X
$\mathcal{X}, \mathcal{Y} \ldots$	Algebraic varieties
$\mathcal{X}^P, \mathcal{Y}^P \ldots$	Projective algebraic varieties
$\mathcal{X}^\vee, \mathcal{Y}^\vee \ldots$	Dual algebraic varieties
$\mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[\mathbf{x}]$	Set of polynomials in variables $x_1, \ldots, x_n$
$\mathbf{x}(s), \mathbf{y}(t), \ldots$	Parameterizations
$\mathbf{n_x}(s), \mathbf{n_y}(t), \ldots$	Normal vector fields
$\deg \varphi$	Degree of the mapping φ
$\mathrm{T}_\mathbf{q}\mathcal{X}$	Tangent space
$\mathrm{T}^A_\mathbf{q}\mathcal{X}$	Affine tangent space
$\mathcal{X}_{\mathrm{Reg}}$	Set of regular points
$\mathcal{X}_{\mathrm{Sing}}$	Set of singular points
$m_\mathbf{q}(\mathcal{X})$	Multiplicity of the point $\mathbf{q}$
$I_\mathbf{q}(\mathcal{X}, \mathcal{Y})$	Intersection multiplicity
$\mu_\mathbf{q}(\mathcal{X})$	Milnor number
$r_\mathbf{q}(\mathcal{X})$	Number of branches
$\delta_\mathbf{q}(\mathcal{X})$	Delta invariant
$\deg \mathcal{X}$	Degree of $\mathcal{X}$
$\dim \mathcal{X}$	Dimension of $\mathcal{X}$
$\mathrm{g}\,(\mathcal{X})$	Genus of a curve $\mathcal{X}$
$\mathrm{Gr}(k, n)$	Set of k-planes in $P^n\mathbb{C}$
$\mathrm{Gr}^0(k, n)$	Set of k-planes in $\mathbb{C}^n$ passing through the origin
$(\mathbf{v}, \mathcal{V}) \sim_\star (\mathbf{w}, \mathcal{W})$	Coherent points
$\mathcal{V} \star \mathcal{W}$	Convolution
$\mathcal{I}(\mathcal{V}, \mathcal{W})$	Incidence variety
$\pi_\mathcal{V}$	Projection $\mathcal{I}(\mathcal{V}, \mathcal{W}) \to \mathcal{V}$
$\pi_\mathcal{W}$	Projection $\mathcal{I}(\mathcal{V}, \mathcal{W}) \to \mathcal{W}$
σ	Mapping $\mathcal{I}(\mathcal{V}, \mathcal{W}) \to \mathcal{V} \star \mathcal{W}$
$\mathbf{v}(t) \sim_\star \mathbf{w}(t)$	Coherent parameterizations
$\mathcal{P}(\mathbf{v}, \mathbf{w})$	Parameter variety
$\Omega_\mathcal{V}$	ω-correction
$\kappa_\mathcal{V}$	Convolution degree
$i^\mathcal{V}_\mathcal{X}$	Index of $\mathcal{X}$ w.r.t. $\mathcal{V}$
X^-	Set centrally symmetric to X
$D_\mathcal{V}(\mathbf{x}), D_\Delta(\mathbf{x})$	Coherent forms
$\mathcal{B}_\Delta$	Generalized Blaschke hypercylinder

Contents

Chapter 1

Introduction

The notion of convolution can be found in literature in two different ways. In computer vision, image and signal processing, electrical engineering, etc., a convolution curve/surface is introduced through $f(x) = g(x) \star h(x) = \int_{\mathbb{R}^n} g(t)h(x-t)dt$ $(n = 2, 3)$, called the convolution of the geometry function g and the kernel function h, cf. Bloomenthal and Shoemake (1991). In geometric modelling, the convolution hypersurface $\mathcal{V} \star \mathcal{W}$ of two hypersurfaces $\mathcal{V}$, $\mathcal{W}$ in Euclidean space is taken as the envelope of $\mathcal{V}$ under the translations defined by vectors $\mathbf{v} \in \mathcal{W}$. In this thesis we will deal with the convolution in the second sense only.

In recent years, studying convolutions of hypersurfaces has been an active research area. For instance, one of the fundamental operations in Computer Aided Design, i.e., offsetting, can be expressed as the convolution with a circle/sphere. Applying operation of convolution with other hypersurfaces, we arrive at so-called general offsets. Many interesting problems related to this topic have arisen, e.g. analysis of (geometric and algebraic) properties, determining number and kind of their components, computing the convolution degrees of hypersurfaces and mainly a construction of rational parameterizations of convolution hypersurfaces (if they exist). In addition, the construction of convolutions is closely related to another basic geometric operation – the Minkowski sum. The boundary of Minkowski sum of two objects is a subset of the convolution of the corresponding boundary curves. Hence, by eliminating all redundant parts in the convolution curve, one can generate the Minkowski sum boundary. The Minkowski sum is used in various important geometric computations, especially for collision detection among planar curved objects. Reader who is interested in this topis is kindly referred to Peternell and Steiner (2007); Šír et al. (2007).

The main goal of this thesis is to provide an algebraic analysis of the operation. The thesis is divided into three chapters. After introduction, in Chapter 2 we introduce the fundamentals of algebraic geometry needed to read the following chapters. Next the convolution of two hypersurfaces is defined and its elementary properties are mentioned. For instance convolutions with hypersurfaces with degenerated Gauss image are discussed in brief and some applications of convolution in CAGD are presented.

The main part of the thesis is contained in Chapters 3 and 4. We start with review of methods used for studying the convolutions. In particular, we recall the approach based on the so-called incidence varieties introduced in Vršek and Lávička (2010b), methods used for finding rational parameterizations developed in Lávička and Bastl (2007); Lávička et al. (2010) and finally dual representation of convolutions used e.g. in Sabin (1974); Šír et al. (2008); Aigner et al. (2009).

The second part of Chapter 3 is based on paper Vršek and Lávička (2010b). We present a complete algebraic analysis of degeneration and existence of simple and special components of convolutions of irreducible hypersurfaces. The main characterization of a hypersurface from the point of view convolutions, deeply studied in the thesis and firstly defined in Lávička and Bastl (2007), is the so-called convolution degree which reflects the behavior of a hypersurface in the process of convolution construction with a generic algebraic hypersurface. In the curve case, a relation of the convolution degree to the genus of an algebraic curve is analyzed. Although we will present a relation between convolution degrees of input hypersurfaces and the upper bound on the number of irreducible components of their convolution, we will prove that the convolution tends to be irreducible in general. Special attention is devoted to rational hypersurfaces and their convolutions.

Since computing the convolutions is a non-linear problem, it is not easy to decide whether the resulting hypersurface is irreducible, rational etc. Even more it takes a lot of effort just to compute the defining equation of the convolution hypersurface. For these reasons the last chapter arises naturally as the attempt to identify the class of hypersurfaces whose convolution could be handled easily. Since the invariant associated to each hypersurface which measures its complexity with respect to the operation of convolution is the convolution degree, we study hypersurfaces of low convolution degree.

We prove that hypersurfaces with convolution degree one are exactly the hypersurfaces with Linear field of Normal vectors (LN hypersurfaces), introduced in Jüttler (1998). The study of LN hypersurfaces has its own history, as in Peternell and Manhart (2003); Sampoli et al. (2006) it was proved that they admit rational convolution with an arbitrary rational hypersurface. Another approach based on Gröbner basis computation and discussion of the associated convolution degree of parameterized hypersurfaces was used in Lávička and Bastl (2007) where a special class of GRC parameterizations of hypersurfaces (Generally admitting Rational Convolutions with any arbitrary rational hypersurface) was introduced. As a by-product, it was proved that all non-developable polynomial quadratic surfaces belong to the class of LN surfaces. Nevertheless, the detailed algebraic analysis of convolution hypersurfaces has not been at disposal up to the present day.

The second part of Chapter 4 is devoted to the hypersurfaces with convolution degree two. In spite of higher convolution degree we are still able to provide a full algebraic analysis of their convolutions with an arbitrary hypersurfaces. Moreover we identify the special class of the so-called QN hypersurfaces (where QN stands for quadratic normals). Prominent QN hypersurfaces are hyperspheres because the convolution with a hypersphere is nothing but the celebrated offset. It is not surprising that the offsets were analyzed firstly as they possess the highest application potential and the rationality of classical offset curves and surfaces has been studied for many years.

In the case of planar curves, Farouki and Sakkalis (1990) introduced the class of Pythagorean Hodograph (PH) curves as polynomial curves possessing rational offset curves and polynomial arc-length functions. The deep analysis of PH curves has followed – see e.g. Farouki and Neff (1995); Farouki (2008). Later, the concept of polynomial planar PH curves was generalized to rational PH curves (Pottmann (1995); Peternell and Pottmann (1998)). The notion of rational surfaces with rational offsets, called Pythagorean Normal vector (PN) surfaces, was introduced in Pottmann (1995) – more details about PN surfaces can be found e.g. in Lü (1996); Peternell and Pottmann (1998). Consequently an algebraic analysis of offsets was provided in Arrondo et al. (1997); Sendra and Sendra (2000); Alcazar and Sendra (2007). Since the QN hypersurfaces shares a lot of properties with hyperspheres with respect to convolution, we will be able to generalize some mentioned results on the class of QN hypersurfaces.

Chapter 2

Preliminaries

We roughly explain basics of algebraic geometry needed in this thesis. The notions of algebraic variety, its dimension, tangent spaces, etc. are defined. The existence of a rational parametrization of curves and surfaces is discussed. Finally, the notions of dual variety and variety with degenerated Gauss image are explained.
The convolution of two arbitrary varieties in $\mathbb{C}^n$ is defined and some of its very basic properties are discussed. Moreover, since hypersurfaces with degenerated Gauss image are quite special with respect to the operation of convolution, we give a brief analysis of their behavior – at least in low dimensional cases. We conclude this section exposing some applications of convolutions in CAGD.

2.1 Basics of algebraic geometry

Curves and surfaces in CAGD are mostly represented by their rational parameterizations. Since any such an object is an algebraic variety, the algebraic geometry could be a proper tool for a theoretical investigation. Although in practical applications we want to deal with curves and surfaces in real two and three-dimensional space, we work over the field of complex numbers $\mathbb{C}$ thorough this thesis. This allow us to prove global results. Moreover we do not limit ourself to the curves and surfaces, but we deal with varieties of an arbitrary dimension. In spite of this the most attention is devoted to these low dimensional cases.

The main goal of this section is to introduce notations used thorough this thesis. All the statements in this section are well known and thus we formulate them here without proofs or references. More details concerning fundamentals of algebraic geometry can be found e.g. in Brieskorn and Knörer (1986); Cox et al. (2005); Hartshorne (1977); Harris (1992); Walker (1950).

2.1.1 Algebraic varieties and their properties

A given set of polynomials $f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[\mathbf{x}]$ defines an *affine variety* as the set

$$\mathbb{V}(f_1, \ldots, f_k) := \{\mathbf{x} \in \mathbb{C}^n \,|\, f_1(\mathbf{x}) = \cdots = f_k(\mathbf{x}) = 0\}. \tag{2.1}$$

Affine varieties which can be described by a single nonconstant equation $f(\mathbf{x}) = 0$ are called *hypersurfaces*. Moreover if $\hat{f}$ is the squarefree part of f then $\mathbb{V}(\hat{f}) = \mathbb{V}(f)$. Hence in what follows, we will consider the defining polynomial f of the hypersurface $\mathbb{V}(f)$ to be squarefree.

Affine varieties form the closed sets of the *Zariski topology* on $\mathbb{C}^n$. For any set $X \subset \mathbb{C}^n$ we denote $\operatorname{cl} X$ its *closure* in this topology, i.e., $\operatorname{cl} X$ is the smallest affine variety containing X. A variety is called *irreducible* if it cannot be written as a union of two proper varieties. The Zariski topology gives rise to the definition of the dimension as follows. If $\mathcal{X}$ is an irreducible variety then its *dimension* $\dim \mathcal{X}$ is defined to be the biggest integer k such that there is a chain $\emptyset \neq \mathcal{X}_0 \subsetneq \mathcal{X}_1 \subsetneq \cdots \subsetneq \mathcal{X}_k = \mathcal{X}$ of irreducible subvarieties of $\mathcal{X}$. The variety of which components have the same dimensions is called *equidimensional*. Unless stated otherwise, we mean by variety an equidimensional variety. A variety of dimension one is called a *curve* and a variety of dimension two a *surface*. An affine space $\mathbb{C}^n$ has dimension n and it is well known that a variety $\mathcal{X} \subset \mathbb{C}^n$ is a hypersurface if and only if $\dim \mathcal{X} = n - 1$.

The *degree* of an variety $\mathcal{X} \subset \mathbb{C}^n$, denoted by $\deg \mathcal{X}$ is defined as the number of points in the intersection of $\mathcal{X}$ and a generic $(n - \dim \mathcal{X})$–dimensional plane. In the case that $\mathcal{X}$ is hypersurface with a defining polynomial f, then $\deg \mathcal{X} = \deg f$.

For any variety $\mathcal{X}$ the set

$$I(\mathcal{X}) := \{f \in \mathbb{C}[\mathbf{x}] \,|\, \forall \mathbf{x} \in \mathcal{X} : f(\mathbf{x}) = 0\} \tag{2.2}$$

is called the *ideal of the affine variety* $\mathcal{X}$. The *coordinate ring* of $\mathcal{X}$ is defined as $\mathbb{C}[\mathcal{X}] := \mathbb{C}[\mathbf{x}]/I(\mathcal{X})$. Any $\varphi \in \mathbb{C}[\mathcal{X}]$ is called a *regular function* on $\mathcal{X}$. The quotient of two regular functions φ/ψ, where ψ is not a zero divisor in $\mathbb{C}[\mathcal{X}]$, is called a *rational function*. In the case the variety $\mathcal{X}$ is irreducible then the quotient field $\mathbb{C}(\mathcal{X})$ of $\mathbb{C}[\mathcal{X}]$ is called the *field of rational functions* on $\mathcal{X}$.

The mapping $\varphi : \mathcal{X} \to \mathcal{Y}$ given by $\varphi : \mathbf{p} \mapsto (\varphi_1(\mathbf{p}), \ldots, \varphi_n(\mathbf{p}))$, where $\varphi_i \in \mathbb{C}(\mathcal{X})$, is called a *rational mapping*. This rational mapping is said to be *dominant* if $\operatorname{cl}\varphi(\mathcal{X}) = \mathcal{Y}$. If φ is a dominant mapping between varieties with the same dimensions, then there exists a positive constant k such that for a generic $\mathbf{q} \in \mathcal{Y}$ the fibre $\varphi^{-1}(\mathbf{q})$ contains exactly k points. This constant k is called the *degree* of φ and we denote it by $\deg\varphi$. If $\deg\varphi = 1$ then there exists a rational inverse and such φ is called *birational*. Some important examples of rational mappings are *rational parameterizations*, i.e., dominant mapping $\mathbb{C}^n \to \mathcal{X}$. The parameterizations will be denoted e.g. by $\mathbf{x}(s)$, where $s = (s_1, \ldots, s_n)$. With a little abuse of notation we will write $s = s_1$ in the curve case. A parameterization given by a birational mapping is called *proper*. A variety which admits a rational parameterization is called *unirational*. In addition, if it admits a proper parameterization we speak about a *rational* variety. The unirationality of a variety does not imply its rationality in general. However, in the most important cases for CAGD, i.e., for curves and surfaces, these notions coincide. Moreover there are birational invariants (i.e., integers associated to the variety which do not change under birational transformations), which provide criteria of rationality. These are *genus* $\mathrm{g}\,(\mathcal{X})$ of a curve and e.g. the *irregularity* $\mathrm{q}\,(\mathcal{X})$ and *second plurigenus* $\mathrm{P}_2(\mathcal{X})$ of a surface. The definition of genus will be given in Subsection 2.1.2. The definition of remaining two invariants is beyond the scope of this introduction – reader is kindly refereed to e.g. Iskovskikh and Shafarevich (1989)

Theorem 2.1. (Lüroth) *A curve $\mathcal{X}$ is rational if and only if $\mathrm{g}\,(\mathcal{X}) = 0$.*

Theorem 2.2. (Castelnuovo – Enriques rationality criterion) *A surface $\mathcal{X}$ is rational if and only if $\mathrm{q}\,(\mathcal{X}) = \mathrm{P}_2(\mathcal{X}) = 0$.*

The concept of *projective algebraic varieties* can be introduced in a similar way. Throughout this paper we will assume that the affine space $\mathbb{C}^n$ is included in the projective space $P^n\mathbb{C}$ via the mapping $(x_1, \ldots, x_n) \hookrightarrow (1 : x_1 : \cdots : x_n)$. Projective points of the hyperplane $x_0 = 0$ have no preimages in this mapping and they are called *points at infinity*. The *ideal hyperplane* containing these points will be denoted by ω. For any variety $\mathcal{X}$ we denote its projective closure by $\mathcal{X}^P$, specially if $\mathcal{X}$ is a hypersurface defined by the equation $f = 0$ then $\mathcal{X}^P$ is defined by the equation $F = 0$ where F is the homogenization of f.

2.1.2 Local properties of algebraic varieties

Since the tangent space is of local nature, it suffices to define it for affine varieties only. The natural extension for projective varieties can be done by an affine cover of the projective variety. Let $\mathcal{X} \subset \mathbb{C}^n$ be a variety and $f_1, \ldots, f_\ell \in \mathbb{C}[\mathbf{x}]$ be the basis of its ideal. Then we define the *tangent space* $T_{\mathbf{p}}\mathcal{X}$ at the point $\mathbf{p} \in \mathcal{X}$ to be a subspace of $\mathbb{C}^n$ defined by the linear equations

$$\nabla f_1(\mathbf{p}) \cdot \mathbf{x} = \cdots \nabla f_\ell(\mathbf{p}) \cdot \mathbf{x} = 0, \tag{2.3}$$

where ∇f_i are vectors consisting of all the partial derivatives. The more usual way of defining tangent space is to define it as an affine subspace of the ambient space $\mathbb{C}^n$ passing through the point $\mathbf{p}$ with the vector direction $T_{\mathbf{p}}\mathcal{X}$. We call such a subspace the *affine tangent space* and denote it by $T^A_{\mathbf{p}}\mathcal{X}$, see Fig. 2.1.

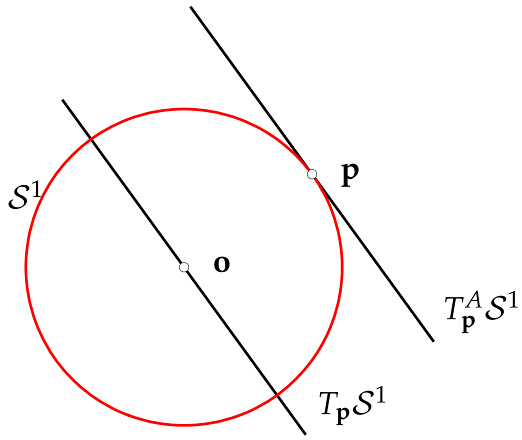


Figure 2.1: The difference between placing the tangent space and the affine tangent space.

It can be shown that $\Phi(\mathbf{x}) := \dim T_{\mathbf{x}}\mathcal{X}$ is an upper-semicontinuous function whose minimum equals $\dim \mathcal{V}$. A point $\mathbf{p}$ such that $\Phi(\mathbf{p}) > \dim \mathcal{V}$ is called *singular*, otherwise it is called *regular*. The nonempty open set of regular points is denoted $\mathcal{X}_{\text{Reg}}$ and the set of singular points by $\mathcal{X}_{\text{Sing}}$. A variety without singular points is usually called *smooth*. Let us note that if $\mathcal{X} = \mathbb{V}(f)$ is a hypersurface, then the singular locus is given as the solution of the system of equations

$$f(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_1} = \cdots \frac{\partial f(\mathbf{x})}{\partial x_n} = 0. \tag{2.4}$$

Next, we will focus on affine plane curves, i.e., on hypersurfaces in the affine plane. For this purpose we set $\mathcal{X} = \mathbb{V}(f)$, $\mathcal{Y} = \mathbb{V}(g)$, where $f, g \in \mathbb{C}[x_1, x_2]$. The *intersection multiplicity* of two affine curves at their common point $\mathbf{p} \in \mathcal{X} \cap \mathcal{Y}$ is defined

$$I_{\mathbf{p}}(\mathcal{X}, \mathcal{Y}) := \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,\mathbf{p}} / \langle f, g \rangle, \tag{2.5}$$

where

$$\mathcal{O}_{\mathbb{C}^2,\mathbf{p}} := \left\{ \frac{\varphi}{\psi} \in \mathbb{C}(x_1, x_2) |\, \psi(\mathbf{p}) \neq 0 \right\} \tag{2.6}$$

and $\langle f, g \rangle$ is the ideal generated by f and g in $\mathcal{O}_{\mathbb{C}^2,\mathbf{p}}$. In terms of the introduced intersection multiplicity, it can be shown that the affine tangent $T^A_{\mathbf{p}}\mathcal{X}$ space is the union of all straight lines $\mathcal{L}$ going through $\mathbf{p}$ such that $I_{\mathbf{p}}(\mathcal{X}, \mathcal{L}) > 1$. If $\mathbf{p} \in \mathcal{X}$ is singular then for every straight line the intersection multiplicity is greater then one. Moreover the minimum of all the multiplicities is called the *multiplicity of the point* $\mathbf{p}$ and denoted by $m_{\mathbf{p}}(\mathcal{X})$. Other important invariants associated with singularities are the *Milnor numbers* defined by

$$\mu_{\mathbf{p}}(\mathcal{X}) = I_{\mathbf{p}}\left(\mathbb{V}\left(\frac{\partial f}{\partial x_1}\right), \mathbb{V}\left(\frac{\partial f}{\partial x_2}\right)\right) \tag{2.7}$$

and *delta invariants* given by the Milnor-Jung formula

$$\delta_{\mathbf{p}}(\mathcal{X}) = \frac{1}{2}\left(\mu_{\mathbf{p}}(\mathcal{X}) + r_{\mathbf{p}}(\mathcal{X}) - 1\right), \tag{2.8}$$

where $r_{\mathbf{p}}(\mathcal{X})$ expresses the number of branches of a plane curve going through the point $\mathbf{p}$.

Each smooth projective curve over $\mathbb{C}$ can be viewed as a compact, orientable real two-dimensional manifold, i.e., a real surface. Then the *genus* $\mathrm{g}\,(\mathcal{X}^P)$ of a curve is defined to be the genus of the real surface. It is a birational invariant of the curve and because every algebraic curve is birationally equivalent to a smooth one, one may naturally extend the definition of the genus to singular curves, too. Using delta invariants, one can write the formula for the genus of a plane curve.

Theorem 2.3. (MAX NOETHER'S FORMULA) *The* genus *of a plane curve* $\mathcal{X}^{\mathcal{P}}$ *of degree* d *can be computed by the formula*

$$\mathrm{g}\,(\mathcal{X}^{\mathcal{P}}) = \frac{1}{2}(d-1)(d-2) - \sum_{\mathbf{p}\in\mathcal{X}^P} \delta_{\mathbf{p}}(\mathcal{X}^{\mathcal{P}}). \tag{2.9}$$

The following theorem describes the relation between genera of two non-singular curves using the ramification divisor of a corresponding dominant rational mapping, cf. Brieskorn and Knörer (1986); Hartshorne (1977).

Theorem 2.4. (RIEMANN-HURWITZ FORMULA) *Let* $\mathcal{X}^P$ *and* $\mathcal{Y}^P$ *be two smooth projective curves and* $\varphi : \mathcal{X}^P \to \mathcal{Y}^P$ *a dominant rational mapping. Then the following formula holds*

$$2\mathrm{g}\,(\mathcal{X}^P) - 2 = \deg(\varphi)\left(2\mathrm{g}\,(\mathcal{Y}^P) - 2\right) + \deg D_R, \tag{2.10}$$

where $\deg D_R \geq 0$ *is the degree of the ramification divisor.*

Riemann-Hurwitz formula serves as a very useful tool also for curves with singularities since we are always able to find a non-singular curve birationally equivalent to a given one, and then apply (2.10).

2.1.3 Dual hypersurfaces

Let $\mathrm{Gr}(k,n)$ denote the set of k-planes (i.e., varieties of dimension k and degree 1) in projective space $P^n\mathbb{C}$. We call this set a *Grassmannian* and it may be shown that it admits a structure of an algebraic variety. We are not going to more details here, but a nice introduction to Grassmannians can be found e.g. in Harris (1992). Since any hyperplane in $P^n\mathbb{C}$ is determined by equation $a_0x_0 + a_1x_1 + \cdots + a_nx_n = 0$ where $(n+1)$-tuple $(a_0,\ldots,a_n)$ is given uniquely up to multiplication by a nonzero constant, it is obvious that $\mathrm{Gr}(n-1,n) \simeq P^n\mathbb{C}$. This projective space is often denoted by $P^n\mathbb{C}^\vee$ and called the *dual space*.

Next, let $\mathbb{C}^n \hookrightarrow P^n\mathbb{C}$ be a natural inclusion. Then the image of the set of k-dimensional planes passing through the origin of $\mathbb{C}^n$ forms a closed subset in $\mathrm{Gr}(k,n)$. We denote it by $\mathrm{Gr}^0(k,n)$. Since any $(n-1)$-dimensional subspace of $\mathbb{C}^n$ is determined by equation $a_1x_1 + \cdots + a_nx_n = 0$ then analogously to the previous case there is a canonical isomorphism $\mathrm{Gr}^0(n-1,n) \simeq P^{n-1}\mathbb{C}$.

Due to the two "different" tangent spaces we have defined, we are going to study two mappings related to them.

If $\mathcal{X}^P$ is a projective hypersurface then the mapping $\mathcal{X}^P \to P^n\mathbb{C}^\vee$ which assigns to each point its affine tangent hyperplane is obviously well defined besides the singular locus of the variety. A *dual variety* $\mathcal{X}^\vee$ is then defined to be the closure of the image of $\mathcal{X}^P$ under this mapping. Since the natural setting for convolutions is affine space, we will work mainly with affine hypersurfaces $\mathcal{X} \subset \mathbb{C}^n$. The dual variety of an affine hypersurface $\mathcal{X}$ is constructed naturally via the composed mapping $\mathcal{X} \hookrightarrow \mathcal{X}^P \to P^n\mathbb{C}^\vee$.

It is always possible to pass between dual and primal representation. We bring in here only one way, which will be often used in what follows. The original hypersurface is then viewed as the envelope of the affine tangents in $\mathcal{X}^{\mathcal{P}}$. If the dual hypersurface is unirational, then $(n-1)$-parameter family of affine tangent hyperplanes can be written in the form

$$\Sigma : \mathbf{n}(s) \cdot \mathbf{x} = h(s), \tag{2.11}$$

From this, the parameterization of hypersurface $\mathcal{X}$ is obtained by solving the system of equations

$$\Sigma, \quad \frac{\partial \Sigma}{\partial s_1}, \quad \ldots, \quad \frac{\partial \Sigma}{\partial s_{n-1}}. \tag{2.12}$$

This is system of n linear equation in n variables $\mathbf{x} = (x_1,\ldots,x_n)$ with coefficients in $\mathbb{C}(s)$, and thus it can be solved with standard tools of linear algebra.

In differential geometry, the Gauss mapping is defined as a mapping of a smooth hypersurface $\mathcal{X} \subset \mathbb{C}^n$ to the unit sphere $\mathcal{S}^{n-1}$ by associating to each point the corresponding unit normal vector. To follow this, it could be natural to define Gauss mapping for hypersurface $\mathcal{X} : f(\mathbf{x}) = 0$ by $\mathbf{x} \mapsto \nabla f(\mathbf{x})/\|\nabla f(\mathbf{x})\|$

(cf. (2.4)). Because of the norm in the denominator this mapping is not rational in general. To excise this difficulty we define the *Gauss mapping* $\gamma_{\mathcal{X}} : \mathcal{X} \to \mathrm{Gr}^0(n-1,n)$ by assigning to each regular point $\mathbf{x} \in \mathcal{X}$ its tangent space $T_{\mathbf{x}}\mathcal{X}$.[1] This is again a rational mapping well defined outside the singular locus. Generally it holds that the Gauss mapping $\gamma_{\mathcal{X}} : \mathcal{X} \to \mathrm{Gr}^0(n-1,n)$ has finite fibres and thus the dimension of the Gauss image $\gamma_{\mathcal{X}}(\mathcal{X})$ is equal to the dimension $\mathcal{X}$. If this is not the case, i.e., $\dim \gamma_{\mathcal{X}}(\mathcal{X}) < \dim \mathcal{X}$ then the hypersurface is said to has a *degenerated Gauss image*.

2.2 Convolutions of hypersurfaces

This section is devoted to the first introduction to the convolutions of algebraic hypersurfaces. The notion of convolution of curves and surfaces was at first introduced to CAGD by M. Sabin, cf. Sabin (1974). In recent years it experienced a rebirth thanks e.g. to the papers Peternell and Manhart (2003); Sampoli et al. (2006); Lávička and Bastl (2007); Šír et al. (2008).

In the first part of this section, we will define convolution of arbitrary two varieties. Although only convolutions of hypersurfaces will be analyzed, this general definition will turn out to be useful at least when we will study the so-called degenerated components, cf. Subsection 3.2.3. The rest of the section will be devoted to some applications of convolutions to CAGD.

2.2.1 Basic definitions

Let $A, B \subset \mathbb{C}^n$ be two subspaces then we write $A + B$ for the subspace spanned by A and B.

Definition 2.5. Two points $\mathbf{v}$, $\mathbf{w}$ on algebraic varieties $\mathcal{V}$ and $\mathcal{W}$, respectively are said to be *coherent*, denoted by $(\mathbf{v}, \mathcal{V}) \sim_\star (\mathbf{w}, \mathcal{W})$, if $T_{\mathbf{v}}\mathcal{V} + T_{\mathbf{w}}\mathcal{W} \neq \mathbb{C}^n$.

Now, the convolution is defined as the sum of the pairs of coherent points.

Definition 2.6. Let $\mathcal{V}, \mathcal{W} \subset \mathbb{C}^n$ be two algebraic varieties, then the algebraic variety

$$\boxed{\mathcal{V} \star \mathcal{W} = \mathrm{cl}\,\{\mathbf{v} + \mathbf{w} \in \mathbb{C}^n \mid \ (\mathbf{v}, \mathcal{V}) \sim_\star (\mathbf{w}, \mathcal{W})\}} \tag{2.13}$$

is called the *convolution* of varieties $\mathcal{V}$ and $\mathcal{W}$.

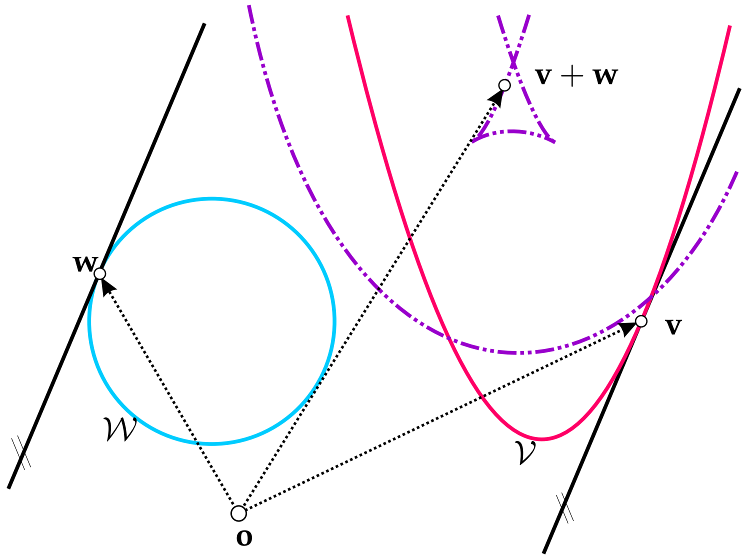


Figure 2.2: Construction of the convolution $\mathcal{V} \star \mathcal{W}$.

An example of the construction of the convolution of two plane curves is seen in Fig. 2.2. From Definition 2.5 it is obvious that two points on hypersurfaces are coherent if and only if they are regular and the corresponding tangent hyperplanes are parallel. Thus it agrees with the usual definition for the hypersurface case. As the second limit case consider two varieties $\mathcal{V}, \mathcal{W} \subset \mathbb{C}^n$ such that $\dim \mathcal{V} + \dim \mathcal{W} < n$. Then for an arbitrary pair of regular points $\mathbf{v} \in \mathcal{V}$ and $\mathbf{w} \in \mathcal{W}$ we have $\dim(T_{\mathbf{v}}\mathcal{V} + T_{\mathbf{w}}\mathcal{W}) \leq \dim \mathcal{V} + \dim \mathcal{W} < n$, and hence any such points are coherent. The resulting variety $\mathcal{V} \star \mathcal{W}$ is thus obtained by sweeping one variety along the other one.

Let $H \subset \mathbb{C}^n$ be a subspace of dimension $n-1$ and $\mathcal{X}$ be a hypersurface. Then we will write

$$\mathcal{X}_H = \{\mathbf{p} \in \mathcal{X} | T_{\mathbf{p}}\mathcal{X} = H\} \tag{2.14}$$

If we consider H to be an element of $\mathrm{Gr}^0(n-1, n)$ then $\mathcal{X}_H = \gamma_{\mathcal{X}}^{-1}(H) \cap \mathcal{X}_{\mathrm{Reg}}$, where $\gamma_{\mathcal{X}}$ is the Gauss mapping. Under the assumption, that $\mathcal{X}$ does not have the degenerated Gauss image, the mapping $\gamma_{\mathcal{X}}$ is dominant. Thus for a hypersurface $\mathcal{Y}$ and a generic point $\mathbf{q}$ on it, there exists a point $\mathbf{p} \in \mathcal{X}_{T_{\mathbf{q}}\mathcal{Y}}$. The point $\mathbf{p} + \mathbf{q}$ then lies on the convolution $\mathcal{X} \star \mathcal{Y}$. On the other hand if $\mathcal{X}$ has the degenerated Gauss image, then for a generic point $\mathbf{q} \in \mathcal{Y}$ there exists no point $\mathbf{p} \in \mathcal{X}$ coherent with $\mathbf{q}$. For this reason we will omit such hypersurfaces from our further analysis. Regardless we will briefly discuss the convolution with these hypersurfaces at least for curves and surfaces at this place.

First let $\mathcal{V}$ be a generic curve and $\mathcal{D}$ be a curve with the degenerated Gauss

[1]Let us note that this is slightly different from the standard definition as $\mathrm{Gr}^0(n-1, n) \simeq P^{n-1}\mathbb{C} \simeq S^{n-1}/\{\pm\mathbf{x}\}$.

image. Then $\mathcal{D}$ has to be a line and let us denote the only tangent line to $\mathcal{D}$ by H. Then $\mathcal{V}_H$ is finite and after setting $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\} = \mathcal{V}_H$ we arrive at the convolution $\mathcal{V} \star \mathcal{D}$ consisting of ℓ copies of the line $\mathcal{D}$ translated by $\mathbf{v}_i$, i.e.,

$$\mathcal{V} \star \mathcal{D} = \bigcup_{i=1}^{\ell} \mathbf{v}_i \star \mathcal{D}. \tag{2.15}$$

If $\mathcal{D}$ is a surface with the degenerated Gauss image, then two cases must be distinguished. First if $\dim \gamma_{\mathcal{D}}(\mathcal{D}) = 0$ then $\mathcal{D}$ is a plane and the situation is completely analogous to the curve case. If the Gauss image of $\mathcal{D}$ is one-dimensional then the set of points on $\mathcal{V}$ coherent with some point on $\mathcal{D}$ forms a curve, say $\mathcal{B}$ – this need not to be irreducible. Then the convolution $\mathcal{V} \star \mathcal{D}$ consists of the same number of component as the $\mathcal{B}$ has. All of them are again surfaces with degenerated Gauss image. This is illustrated by the next example.

Example 2.7. Let $\mathcal{V} = \mathcal{S}^2$ be the unit sphere and $\mathcal{D}$ be a surface consisting of tangents to the twisted cubic $\mathbf{a}(s) = (s, 3s^2, 6s^3)$. Then $\mathcal{D}$ is the so-called tangent developable surface and it admits a parameterization

$$\mathbf{d}(s,t) = \mathbf{a}(s) + t\mathbf{a}'(s) = (s, 3s^2, 6s^3) + t(1, 6s, 18s^2). \tag{2.16}$$

The tangent planes $T_{\mathbf{d}(s,t)}\mathcal{D}$ depend only on the parameter s and may be expressed by the equation

$$(\mathbf{a}' \times \mathbf{a}'') \cdot \mathbf{x} = 6(18s^2, -6s, 1) \cdot \mathbf{x} = 0, \tag{2.17}$$

and it is easy to prove that $\mathcal{B} \subset \mathcal{S}^2$ decomposes into two curves possessing rational parameterizations

$$\mathbf{b}_{\pm}(s) = \pm \frac{1}{1 + 18s^2}\big(18s^2, -6s, 1\big). \tag{2.18}$$

Since $(\mathbf{b}(s), \mathcal{V}) \sim_{\star} (\mathbf{d}(s,t), \mathcal{D})$ for all $s \in \mathbb{C}$ and $t \in \mathbb{C}\backslash\{0\}$, the convolution $\mathcal{V} \star \mathcal{D}$ consists of two components $\mathcal{U}^{\pm}$ parameterized by $\mathbf{b}_{\pm}(s) + \mathbf{d}(s,t)$.

Both of these components are again tangent developable surfaces, i.e., there should exist their parameterization in the form $\mathbf{c}_{\pm}(s) + t\mathbf{c}'_{\pm}(s)$. Simple computations reveal that $\mathbf{c}_{\pm}(s)$ can be expressed as

$$\mathbf{c}_{\pm}(s) = (\pm 1 + s, 3s^2, \pm 1 + 6s^3), \tag{2.19}$$

see Fig. 3.2.

Assumption 2.8. From now, by a variety we mean a variety with non-degenerated Gauss image.

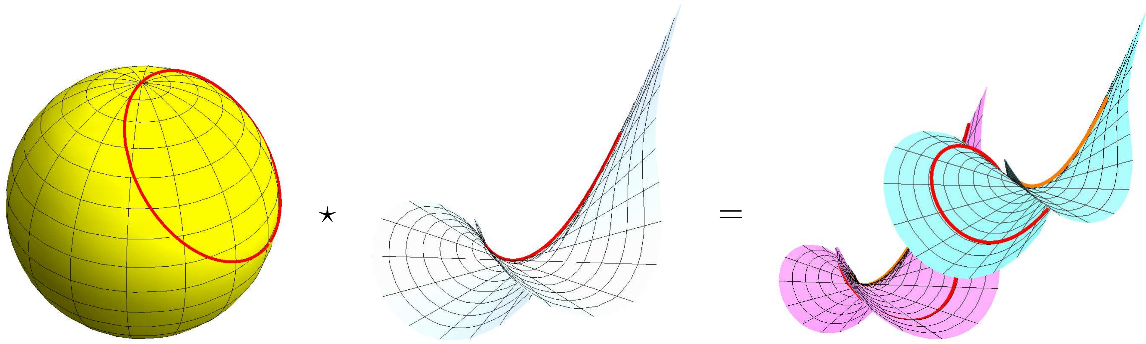


Figure 2.3: The convolution of sphere and tangent developable. Curves on surfaces: the red one on sphere is $\mathcal{B}$, on tangent developable it is twisted cubic and on the convolution they are curves parameterized by $\mathbf{b}_{\pm}(s)\mathbf{a}(s)$. Finally orange curves on the convolution are the those with parameterization $\mathbf{c}_{\pm}(s)$.

Obviously $(\mathcal{U} \cup \mathcal{V}) \star \mathcal{W} = (\mathcal{U} \star \mathcal{W}) \cup (\mathcal{V} \star \mathcal{W})$ and hence in what follows, we can consider convolutions of irreducible hypersurfaces only. Contrariwise, the convolution of two irreducible varieties need not to be irreducible. In general, almost every point on some component of $\mathcal{V} \star \mathcal{W}$ can be written as $\mathbf{v} + \mathbf{w}$ for exactly one pair of coherent points $\mathbf{v}$ and $\mathbf{w}$. However there can exist a component which does not fulfil this property.

Definition 2.9. Let $\mathcal{V}$, $\mathcal{W}$ be two algebraic hypersurfaces. An irreducible component $\mathcal{X} \subset \mathcal{V} \star \mathcal{W}$ will be called *simple*, *special*, or *degenerated* respectively, if there exists a nonempty open set $X \subset \mathcal{X}$, such that $\forall \mathbf{u} \in X$ exist(s) exactly one, more than one but finitely many, or infinitely many pair(s) $\mathbf{v} \in \mathcal{V}$, $\mathbf{w} \in \mathcal{W}$ such that $(\mathbf{v}, \mathcal{V}) \sim_{\star} (\mathbf{w}, \mathcal{W})$, respectively, and $\mathbf{u} = \mathbf{v} + \mathbf{w}$.

The origin of degenerated and special components and their detailed algebraic analysis will be given in Subsection 3.2.3.

2.2.2 Applications

This subsection is devoted to two most important applications where convolutions can be found. Since these applications are motivated by real world problems, they are not usually solved over $\mathbb{C}$. For this reason we will formulate them over $\mathbb{R}$, too.

Minkowski sum The *Minkowski sum* of two sets $A, B \subset \mathbb{R}^2$ $(\mathbb{R}^3)$ is defined as

$$A \oplus B := \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A,\ \mathbf{b} \in B\}. \tag{2.20}$$

It is one of the fundamental operation in CAGD with applications in nesting and packing problems, mathematical morphology, motion planing, computer graphics etc., see de Berg et al. (1997)

If the sets A and B are represented by a polygonal boundary, the computation of the Minkowski sum is well known. Let $A \subset \mathbb{R}^2$ be a convex[2] set with polygonal boundary determined by vertices $\mathbf{a}_1, \ldots, \mathbf{a}_n$. And analogously, B is considered to be a convex set with polygonal boundary given by $\mathbf{b}_1, \ldots, \mathbf{b}_m$. Then the Minkowski sum is convex too and can be computed as

$$A \oplus B = \text{ConvexHull} \left\{ \mathbf{a}_i + \mathbf{b}_j \right\}_{i,j=1}^{m,n}, \tag{2.21}$$

The advantage of this "naive algorithm" is that it may be directly generalized to three-dimensional space. On the other hand its complexity in planar case is $O(mn \log mn)$, which is far away form the optimal one. There exists a plenty of more effective algorithms, but we are not going into such a detail here.

The situation becomes more complicated whenever the sets A and B are bounded by smooth curves. At this moment, there is no way how to use previous algorithm and the convolution appears as a useful tool because of formula

$$\partial (A \oplus B) \subset \partial A \star \partial B, \tag{2.22}$$

where ∂X denotes the boundary of a set X, see Peternell and Steiner (2007). Although this formula is usually formulated for smooth boundaries ∂A and ∂B only, it holds for most of boundaries with singularities too. For the only class of counterexamples see Remark 4.10.

We conclude this paragraph by summarizing the computation of Minkowski sum of two "curved" objects into the following algorithm (see Fig 2.4 too).

Algorithm 1 Minkowski sum

Input: $A, B \subset \mathbb{R}^n$ regions.
Output: $A \oplus B$.
1: *Identify* boundary hypersurfaces $\mathcal{V} := \partial A$, $\mathcal{W} := \partial B$;
2: *Compute* $\mathcal{U} = \mathcal{V} \star \mathcal{W}$;
3: *Trim* $\mathcal{U}$ to obtain $\widetilde{\mathcal{U}}$ as the boundary of Minkowski sum;
4: **return** $A \oplus B$ is object bounded by $\widetilde{\mathcal{U}}$.

Since the areas A and B are the most usually given by their boundaries the first step of algorithm is trivial. As we will see in Chapter 3 the convolution $\partial A \star \partial B$

[2] This condition is not as restrictive as it could seem, because any set may be written as a union of convex sets.

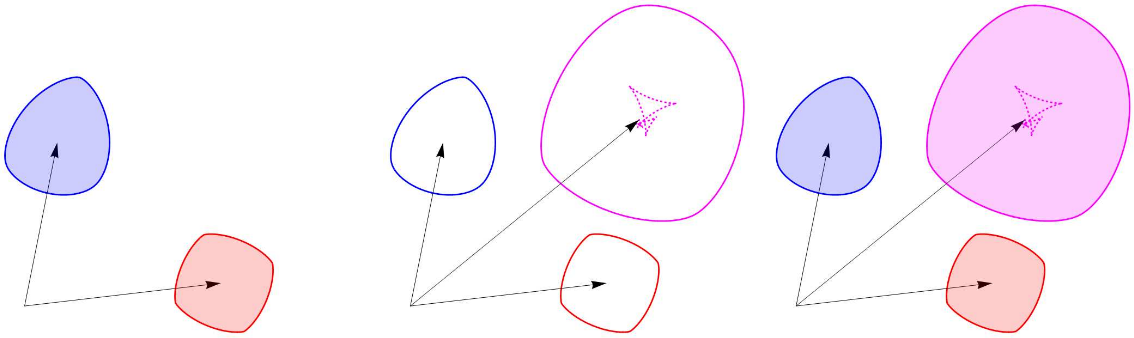


Figure 2.4: Two regions in $\mathbb{R}^2$ (left), convolution of their boundaries (middle) and Minkowski sum (right).

can be quite complicated and moreover the rationality of the boundary hypersurfaces does not ensure the rationality of the resulting hypersurface. Thus it is usual to approximate the boundary by some simpler hypersurfaces which behaves better with respect to the operation of convolution. Such hypersurfaces are studied in Chapter 4.

Offsets The offset hypersurface is defined as a locus of points of constant normal distance from the generator hypersurface. Formally if $\mathbf{x}(s)$ is parameterization of $\mathcal{X}$ and $\mathbf{n}(s)$ the associated unit normal vector field, then the parameterization of δ-offset $\mathcal{O}_\delta(\mathcal{X})$ is given by

$$\mathbf{x}(s) \pm \delta\mathbf{n}(s). \tag{2.23}$$

The offsets are widely used in applications, such as 2.5D pocket machining, 3D NC machining, definition of tolerence regions, acces space representations in robotics, curved plate (shell) representation in solid modeling, rapid prototyping where materials are solidified in succesive two-dimensional layers, brush stroke representation and in feature recognition through construction of skeletons or medial axes of geometric models – see Maekawa (1999).

There is an obvious problem when one want to represent offset e.g. in Bézier form. Since $\mathbf{n}(s)$ is required to be the unit vector field, there is a square-root involved in the expression of $\mathbf{n}(s)$. Thus the offset if not generally rational even when the generator hypersurface was. This led Farouki and Sakkalis (1990) to introduce a class of curves with rational offsets – the so-called *PH curves*. These are curves given by a polynomial parameterization $\mathbf{x}(s) = (x(s), y(s))$, whose hodograph fulfills the so-called *Pythagorean Hodograph property*

$$\left(x'(s)\right)^2 + \left(y'(s)\right)^2 \equiv \sigma^2(s), \tag{2.24}$$

for some polynomial $\sigma(t)$. The three polynomials $x'(s)$, $y'(s)$, $\sigma(s)$ form the so-called Pythagorean triple and they have an elegant explicit description

$$\begin{aligned} x'(s) &= w(s)\big(u^2(s) - v^2(s)\big), \\ y'(s) &= 2w(s)u(s)v(s), \\ \sigma(s) &= w(s)\big(u^2(s) + v^2(s)\big), \end{aligned} \tag{2.25}$$

for three real polynomials $u(s)$, $v(s)$ and $w(s)$.

Later, the concept was generalized to rational curves and surfaces using a dual representation. In Pottmann (1995) it was shown that hypersurfaces with rational offsets correspond exactly to rational hypersurfaces on the so-called *Blaschke hypercylinder*. In other word these are envelopes of system of hyperplanes

$$\Sigma(s) : \mathbf{n}(s) \cdot \mathbf{x} - h(s) = 0, \tag{2.26}$$

where $\mathbf{n}(s)$ is a parameterization of unit sphere $\mathcal{S}^{n-1}$ and $h(s)$ is a rational function. The parameterization of the hypersurface is then obtained as the solutin the system of equations

$$\Sigma(s) = \frac{\partial\,\Sigma(s)}{\partial\, s_1} = \cdots = \frac{\partial\,\Sigma(s)}{\partial\, s_{n-1}} = 0 \tag{2.27}$$

in variables $\mathbf{x}$ with coefficients from $\mathbb{R}(s)$.

The link between convolutions and offsets is obvious – the offset of $\mathcal{X}$ at distance δ is nothing but the convolution of $\mathcal{X}$ with the sphere $\mathcal{S}_\delta^{n-1}$ of radius δ. To see this, let $\mathbf{x} \in \mathcal{X}$ be a regular point and $\mathbf{n}$ the unit normal vector at this point. Then there exist exactly two points on $\mathcal{S}_\delta^{n-1}$ coherent with $\mathbf{x}$, namely the coordinates of these points are $\pm\delta\mathbf{n}$. Thus the convolution $\mathcal{X} \star \mathcal{S}_\delta^{n-1}$ consists of points $\mathbf{x} \pm \delta\mathbf{n}$, which is exactly the offset. Hence any statement proved for convolutions has an immediate consequence for offsets. On the other hand, the offset were studied deeply in recent years and they are well explored. Thus the direction of the investigation may be reversed and convolutions may draw inspiration from offsets. We will see an example in Subsection 4.2.2, where the class of hypersurfaces, behaving like a hypersphere, is studied.

Chapter 3

Algebraic analysis

The summary of methods for dealing with convolutions of hypersurfaces given by implicit, parametric, and dual equation respectively is presented here. The advantages of each of these approaches is discussed. Independently on chosen approach we may prove the so-called fundamental property, which implies that the convolution is an associative operation.
Further the different types of irreducible components of convolution are studied and the singular cases are exactly understood. The upper bound on the number of irreducible components is closely related to the affine invariant of hypersurface – convolution degree. We not only define it but we show how to compute it too. Specially for the curves, the convolution degree formula is presented. Finally the attention to the problems of rationality is devoted.

3.1 Different approaches

The convolution may be viewed as a member of wider class of operations, which, roughly speaking, maps algebraic varieties to an algebraic variety. In the first subsection, we will identify this class precisely. Moreover we will introduce the so-called incidence variety which is valuable tool for proving some theoretical results. The incidence varieties were used e.g. in Arrondo et al. (1997) for study of the offsets, in Sendra and Sendra (2009) for conchoids and in Vršek and Lávička (2010b) this method was applied to the convolutions.

On the other hand, the rational varieties provide the most application potential and thus lot of effort was devoted just to them. This led to the translations of problems to the space of parameters, see Kim and Elber (2000) or Lávička and Bastl (2007) for paper devoted to convolutions. The disadvantage of this approach is obvious. It may be used only for rational varieties with rational convolution.

In addition to described approaches the operation of convolution has considerably simple description in the terms of dual hypersurfaces. This was firstly discovered in Sabin (1974) and in recent years more deeply studied in e.g. Šír et al. (2007); Lávička et al. (2010).

3.1.1 Implicit approach

Let be given two hypersurfaces $\mathcal{V}, \mathcal{W} \subset \mathbb{C}^n$. Then a computation of their convolution consists of the following two steps.

1. Determining all pairs of coherent points, i.e.,

$$\Big\{(\mathbf{v},\mathbf{w}) \in \mathcal{V} \times \mathcal{W} \mid (\mathbf{v},\mathcal{V}) \sim_\star (\mathbf{w},\mathcal{W})\Big\} \subset \mathcal{V} \times \mathcal{W} \tag{3.1}$$

2. The convolution is then obtained as the sum $\mathbf{v}+\mathbf{w}$ of points in (3.1)[1].

This natural and simple decomposition can be found beside a lot of operations in CAGD (see Vršek (2010)) and it is a cornerstone in the formal definition of these operations. In particular, the operation $\diamond$ is given, if for any two admissible[2] varieties $\mathcal{V}$ and $\mathcal{W}$ exists following two objects

1. a variety $\mathcal{I}^\diamond(\mathcal{V},\mathcal{W}) \subset \mathcal{V} \times \mathcal{W}$ such that the natural projections $\pi_\mathcal{V} : \mathcal{I}^\diamond(\mathcal{V},\mathcal{W}) \to \mathcal{V}$ and $\pi_\mathcal{W} : \mathcal{I}^\diamond(\mathcal{V},\mathcal{W}) \to \mathcal{W}$ are dominant and finite,
2. a rational mapping $\sigma_{\mathcal{V},\mathcal{W}} : \mathcal{I}^\diamond(\mathcal{V} \times \mathcal{W}) \to \mathbb{C}^n$,

such that $\mathcal{V} \diamond \mathcal{W} = \mathrm{cl}\left(\sigma_{\mathcal{V},\mathcal{W}}(\mathcal{I}^\diamond(\mathcal{V},\mathcal{W}))\right)$. Thus for convolutions, the set $\mathcal{I}^\star(\mathcal{V},\mathcal{W})$ is given as the algebraic closure of (3.1) and the mapping $\sigma_{\mathcal{V},\mathcal{W}}$ is just the sum $\sigma_{\mathcal{V},\mathcal{W}}(\mathbf{v},\mathbf{w}) = \mathbf{v}+\mathbf{w}$. Since we deal with the convolutions only, we will write $\mathcal{I}(\mathcal{V},\mathcal{W})$ instead of $\mathcal{I}^\star(\mathcal{V},\mathcal{W})$ and σ instead of $\sigma_{\mathcal{V},\mathcal{W}}$ because it does not depend on the varieties $\mathcal{V}$ and $\mathcal{W}$ in our case.

[1] More precisely, one have to take the closure of this set to obtain the convolution variety

[2] For instance for convolutions, these are hypersurfaces with non-degenerated Gauss image.

Definition 3.1. Let be given two hypersurfaces $\mathcal{V}, \mathcal{W}$. By an *incidence variety* we mean

$$\mathcal{I}(\mathcal{V}, \mathcal{W}) = \operatorname{cl}\Big\{(\mathbf{v}, \mathbf{w}) \in \mathcal{V} \times \mathcal{W} \mid (\mathbf{v}, \mathcal{V}) \sim_\star (\mathbf{w}, \mathcal{W})\Big\}. \tag{3.2}$$

It can be checked that all the conditions laid on the mappings $\pi_\mathcal{V}$, $\pi_\mathcal{W}$ and σ are fulfilled.

Lemma 3.2. *If $\mathcal{V}, \mathcal{W}$ are hypersurfaces with the non-degenerated Gauss images, then the projections $\pi_\mathcal{V} : \mathcal{I}(\mathcal{V}, \mathcal{W}) \to \mathcal{V}$ and $\pi_\mathcal{W} : \mathcal{I}(\mathcal{V}, \mathcal{W}) \to \mathcal{W}$ are dominant and finite. Moreover, it holds that $\mathcal{V} \star \mathcal{W} = \operatorname{cl}\big(\sigma(\mathcal{I}(\mathcal{V}, \mathcal{W}))\big)$.*

Proof. Let $\gamma_\mathcal{V} : \mathcal{V} \to \mathrm{Gr}^0(n-1, n)$, $\gamma_\mathcal{W} : \mathcal{W} \to \mathrm{Gr}^0(n-1, n)$ be the Gauss mappings as in Subsection 2.1.2. Then the incidence variety may be identified with the closure of the fibre product $\mathcal{V}_{\mathrm{Reg}} \times_{\mathrm{Gr}^0(n-1,n)} \mathcal{W}_{\mathrm{Reg}}$. Thus for a generic $\mathbf{v}$ on $\mathcal{V}$ we can find

$$\pi_\mathcal{V}^{-1}(\mathbf{v}) = \Big\{\big(\mathbf{v}, \gamma_\mathcal{W}^{-1}(\gamma_\mathcal{V}(\mathbf{v}))\big)\Big\}. \tag{3.3}$$

Since $\mathcal{V}$, $\mathcal{W}$ are hypersurfaces with non-degenerated Gauss images, the associated Gauss mappings are finite and dominant. Hence the preimage of a generic $\mathbf{v}$ under the projection $\pi_\mathcal{V}$ is non-empty and finite. The same argument holds for the projection $\pi_\mathcal{W}$.

First, let us choose $\mathbf{u} \in \sigma(\mathcal{I}(\mathcal{V}, \mathcal{W}))$. Then it follows from the definition of $\mathcal{I}(\mathcal{V}, \mathcal{W})$ that there exists $\mathbf{v} \in \mathcal{V}_{\mathrm{Reg}}$ and $\mathbf{w} \in \mathcal{W}_{\mathrm{Reg}}$ such that $\mathbf{u} = \sigma(\mathbf{v}, \mathbf{w}) = \mathbf{v} + \mathbf{w}$. Moreover $(\mathbf{v}, \mathcal{V}) \sim_\star (\mathbf{w}, \mathcal{W})$ and hence $\mathbf{u} \in \mathcal{V} \star \mathcal{W}$. Conversely, a generic $\mathbf{u} \in \mathcal{V} \star \mathcal{W}$ can be written as $\mathbf{v} + \mathbf{w}$ for coherent $\mathbf{v}$ and $\mathbf{w}$. Then $(\mathbf{v}, \mathbf{w}) \in \mathcal{I}(\mathcal{V}, \mathcal{W})$ and $\mathbf{u} = \sigma(\mathbf{v}, \mathbf{w}) \in \sigma(\mathcal{I}(\mathcal{V}, \mathcal{W}))$. Thus using the fact that both sets are closed, we deduce that they have to be equal. □

The above relations are summarized in the following diagram:

$$\begin{array}{ccccc} \mathcal{V} & \xleftarrow{\;\pi_\mathcal{V}\;} & \mathcal{I}(\mathcal{V}, \mathcal{W}) & \xrightarrow{\;\pi_\mathcal{W}\;} & \mathcal{W} \\ & & \Big\downarrow \sigma & & \\ & & \mathcal{V} \star \mathcal{W} & & \end{array} \tag{3.4}$$

Corollary 3.3. $\dim \mathcal{I}(\mathcal{V}, \mathcal{W}) = n - 1$.

Proof. Since the projection $\pi_\mathcal{V} : \mathcal{I}(\mathcal{V}, \mathcal{W}) \to \mathcal{V}$ is dominant it follows from Harris (1992, p. 138, Thm 11.12) that

$$\dim \mathcal{I}(\mathcal{V}, \mathcal{W}) = \dim \mathcal{V} + \dim \pi_\mathcal{V}^{-1}(\mathbf{v}), \tag{3.5}$$

where $\pi_\mathcal{V}^{-1}(\mathbf{v})$ is a generic fibre. However this is zero-dimensional as it is finite. Hence $\dim \mathcal{I}(\mathcal{V}, \mathcal{W}) = \dim \mathcal{V} = n - 1$. □

Remark 3.4. Originally, in the paper Vršek and Lávička (2010b) the incidence variety was defined as

$$\mathcal{C}(\mathcal{V},\mathcal{W}) := \mathrm{cl}\left\{ (\mathbf{u},\mathbf{v}) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} \mathbf{v} \notin \mathcal{V}_{\mathrm{Sing}}, \\ \mathbf{u}-\mathbf{v} \notin \mathcal{W}_{\mathrm{Sing}}, \\ \mathrm{rank}\begin{bmatrix} \nabla f(\mathbf{v}) \\ \nabla g(\mathbf{u}-\mathbf{v}) \end{bmatrix} < 2 \end{array} \right\}, \tag{3.6}$$

where f and g are defining polynomials of $\mathcal{V}$ and $\mathcal{W}$, respectively. Then for two natural projections

$$\pi_1, \pi_2 : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n, \qquad \pi_1 : (\mathbf{u},\mathbf{v}) \mapsto \mathbf{u} \qquad \text{and} \qquad \pi_2 : (\mathbf{u},\mathbf{v}) \mapsto \mathbf{v}, \tag{3.7}$$

it holds $\mathrm{cl}\,(\pi_1(\mathcal{C}(\mathcal{V},\mathcal{W}))) = \mathcal{V} \star \mathcal{W}$ and $\mathrm{cl}\,(\pi_2(\mathcal{C}(\mathcal{V},\mathcal{W}))) = \mathcal{V}$. This was motivated by the incidence variety introduced in Arrondo et al. (1997) for the study of offsets of algebraic hypersurfaces. We believe that the slightly different definition, which we decided to use in this thesis, makes it more readable.

On the other hand the set $\mathcal{C}(\mathcal{V},\mathcal{W})$ can be usefull when we need to compute the defining polynomial of $\mathcal{V} \star \mathcal{W}$, as the convolution is nothing but the projection from $\mathcal{C}(\mathcal{V},\mathcal{W})$ onto the first n variables. Hence its defining polynomial can be obtained from the defining polynomials of $\mathcal{C}(\mathcal{V},\mathcal{W})$ by eliminating the second n variables. More precisely, consider the ideal generated by

$$f(\mathbf{x}), \quad g(\mathbf{y}-\mathbf{x}), \quad \frac{\partial f(\mathbf{x})}{\partial x_i}\frac{\partial g(\mathbf{y}-\mathbf{x})}{\partial (y_j - x_j)} - \frac{\partial f(\mathbf{x})}{\partial x_j}\frac{\partial g(\mathbf{y}-\mathbf{x})}{\partial (y_i - x_i)}, \tag{3.8}$$

for $1 < j \leq n$ and $i < j$. This is almost the ideal of $\mathcal{C}(\mathcal{V},\mathcal{W})$ but we did not take care about singular locus of input varieties yet. To fix this, we use the standard trick relying on adding two new variables α, β and two polynomials

$$\prod_{i=1}^{n}\left(1 - \alpha\frac{\partial f(\mathbf{x})}{\partial x_i}\right), \quad \prod_{j=1}^{n}\left(1 - \beta\frac{\partial g(\mathbf{y}-\mathbf{x})}{\partial (y_j - x_j)}\right). \tag{3.9}$$

Now the defining polynomial of $\mathcal{V} \star \mathcal{W}$ can be obtained by eliminating variables α, β, y_j from the system (3.8) and (3.9). This can be done e.g. by computing the Gröbner basis of the ideal generated by the system w.r.t. lexicographic order $\alpha > \beta > y_j > x_i$.

It is obvious that the operation of convolution is commutative as the relation of being coherent is symmetric and the addition of complex numbers is commutative. The decision about associativity is a little bit harder. One would use that addition is associative on complex numbers and then to show that the relation of being coherent is transitive and moreover that it holds

$$(\mathbf{v},\mathcal{V}) \sim_\star (\mathbf{w},\mathcal{W}) \quad \text{implies} \quad (\mathbf{v},\mathcal{V}) \sim_\star (\mathbf{v}+\mathbf{w},\mathcal{V} \star \mathcal{W}). \tag{3.10}$$

Unfortunately, the relation $\sim_\star$ is not transitive as the following example shows.

Example 3.5. Let $\mathcal{U}$, $\mathcal{V} \subset \mathbb{C}^3$ be two curves and $\mathcal{W} \subset \mathbb{C}^3$ a surface. If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are points on $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ respectively, such that

$$T_{\mathbf{u}}\mathcal{U} = \langle(1,0,0)\rangle,\ T_{\mathbf{v}}\mathcal{V} = \langle(0,1,0)\rangle \text{ and } T_{\mathbf{w}}\mathcal{W} = \langle(1,0,0),(0,0,1)\rangle, \tag{3.11}$$

then $(\mathbf{u},\mathcal{U}) \sim_\star (\mathbf{v},\mathcal{V})$ and $(\mathbf{u},\mathcal{U}) \sim_\star (\mathbf{w},\mathcal{W})$, but $(\mathbf{v},\mathcal{V}) \not\sim_\star (\mathbf{w},\mathcal{W})$.

In spite of that we will prove the property (3.10) and then it will immediately follow that for hypersurfaces the associativity holds.

Example 3.6. Let us suppose, at this moment, that $\mathcal{V} \subset \mathbb{R}^n$ is a real hypersurface with non-degenerated offset $\mathcal{O}_\delta(\mathcal{V})$, $\mathbf{p} \in \mathcal{V}_{\mathrm{Reg}}$ its regular point and $\mathbf{v}(t) : \mathbb{R}^{n-1} \to \mathcal{V}$ some local parameterization of the neighborhood (in the standard topology) of $\mathbf{p}$ with $\mathbf{v}(0) = \mathbf{p}$. Then the tangent space $T_{\mathbf{p}}\mathcal{V}$ is generated by

$$\left\{\frac{\partial\,\mathbf{v}(0)}{\partial\,t_1}, \frac{\partial\,\mathbf{v}(0)}{\partial\,t_2}, \dots, \frac{\partial\,\mathbf{v}(0)}{\partial\,t_{n-1}}\right\}. \tag{3.12}$$

Now, if $\mathbf{n}(t)$ is the local parameterization of the unit normal vector field associated to $\mathbf{v}$, then two points on the offset $\mathcal{O}_\delta(\mathcal{V})$ generated by $\mathbf{p}$ are $\mathbf{p}_\pm = \mathbf{v}(0) \pm \delta\mathbf{n}(0)$. If these points are regular, the tangent spaces $T_{\mathbf{p}_\pm}\mathcal{O}_\delta(\mathcal{V})$ are determined by

$$\left\{\frac{\partial\,(\mathbf{v}(0) \pm d\mathbf{n}(0))}{\partial\,t_1}, \frac{\partial\,(\mathbf{v}(0) \pm d\mathbf{n}(0))}{\partial\,t_2}, \dots, \frac{\partial\,(\mathbf{v}(0) \pm d\mathbf{n}(0))}{\partial\,t_{n-1}}\right\}. \tag{3.13}$$

However these are exactly same as the subspace $T_{\mathbf{p}}\mathcal{V}$, since each $\partial\,\mathbf{n}/\partial\,t_i$ is a linear combination of vectors $\partial\,\mathbf{v}/\partial\,t_j$. To see this, it suffices to realize that $\mathbf{n}\cdot\mathbf{n} = 1$ implies $\mathbf{n}\cdot\partial\,\mathbf{n}/\partial\,t_i = 0$. Hence, we have showed that generically $T_{\mathbf{p}_\pm}\mathcal{O}_d(\mathcal{V}) = T_{\mathbf{p}}\mathcal{V}$. This identity called in Arrondo et al. (1997) fundamental property of offsets holds for convolutions too.

Lemma 3.7. (FUNDAMENTAL PROPERTY OF CONVOLUTIONS) *For a generic* $\mathbf{v} + \mathbf{w} \in \mathcal{V} \star \mathcal{W}$ *we have* $T_{\mathbf{v}+\mathbf{w}}\mathcal{V} \star \mathcal{W} \subset T_{\mathbf{v}}\mathcal{V} + T_{\mathbf{w}}\mathcal{W}$.

There are more ways to prove this lemma. The analogy to the above example would be a construction of the so-called coherent parameterizations (for the definition of coherent parameterization see Definition 3.11). Moreover if $\mathcal{V} \star \mathcal{W}$ does not contain any degenerated component then the fundamental property can be immediately seen from the dual approach, cf. Subsection 3.1.3. We give here another proof which uses the incidence variety.

Proof. (OF LEMMA 3.7) Let $\mathcal{X}$ be an arbitrary component of $\mathcal{V} \star \mathcal{W}$ and denote $\mathcal{Y} = \sigma^{-1}(\mathcal{X})$ its preimage on $\mathcal{I}(\mathcal{V},\mathcal{W})$. Then by Sard's lemma for varieties (cf. Mumford (1976, p. 42)) there exists a nonempty Zariski open set $X \subset \mathcal{X}$ such that σ is smooth at the points of $Y = \sigma^{-1}(X)\backslash\mathcal{Y}_{\mathrm{Sing}}$, i.e., for each $\mathbf{y} \in$

Y the differential $d\sigma\mid_{\mathbf{y}}: T_{\mathbf{y}}\mathcal{X} \to T_{\sigma(\mathbf{y})}\mathcal{Y}$ is surjective. Since $\mathcal{I}(\mathcal{V},\mathcal{W}) \subset \mathcal{V} \times \mathcal{W}$ we have for each $\mathbf{y} \in Y$ that $T_{\mathbf{y}}\mathcal{I}(\mathcal{V},\mathcal{W}) \subset T_{\mathbf{v}}\mathcal{V} \times \mathcal{T}_{\mathbf{w}}\mathcal{W}$, where $\mathbf{y} = (\mathbf{v},\mathbf{w})$, and hence we may write any tangent vector at $\mathbf{y}$ as $\mathbf{t_y} = (\mathbf{t_v},\mathbf{t_w})$. With this notation the differential $d\sigma$ is given by $(\mathbf{t_v},\mathbf{t_w}) \mapsto \mathbf{t_v} + \mathbf{t_w}$ and hence we obtain that $T_{\mathbf{v}+\mathbf{w}}\mathcal{X} \subset T_{\mathbf{v}}\mathcal{V} + T_{\mathbf{w}}\mathcal{W}$ for generic $\mathbf{v}+\mathbf{w} \in \mathcal{X}$. However the component $\mathcal{X} \subset \mathcal{V} \star \mathcal{W}$ was chosen arbitrarily. This completes the proof. □

Apparently, the property (3.10) is an immediate consequence of the previous lemma. Moreover, let $\mathcal{V},\mathcal{W},\mathcal{U} \subset \mathbb{C}^n$ be hypersurfaces and let $\mathbf{v},\mathbf{w},\mathbf{u}$ be a triple of generic coherent points on them. Then as the consequence of the fundamental property of convolutions, we see that $(\mathbf{v}+\mathbf{w},\mathcal{V}\star\mathcal{W}) \sim_\star (\mathbf{u},\mathcal{U})$ and $(\mathbf{v},\mathcal{V}) \sim_\star (\mathbf{w}+\mathbf{u},\mathcal{W}\star\mathcal{U})$. Since this holds for generic points we arrive at $(\mathcal{U}\star\mathcal{V})\star\mathcal{W} = \mathcal{U}\star(\mathcal{V}\star\mathcal{W})$. Let us formulate it as a corollary.

Corollary 3.8. *For three hypersurfaces $\mathcal{U},\mathcal{V},\mathcal{W}$, we have*

$$(\mathcal{U}\star\mathcal{V})\star\mathcal{W} = \mathcal{U}\star(\mathcal{V}\star\mathcal{W}). \tag{3.14}$$

We will conclude this subsection by introducing another useful concept.

Definition 3.9. A dominant rational mapping $\xi_{\mathcal{U},\mathcal{V}} : \mathcal{U} \to \mathcal{V}$ will be called *coherent* if for a generic $\mathbf{u} \in \mathcal{U}$ it holds $(\mathbf{u},\mathcal{U}) \sim_\star (\xi_{\mathcal{U},\mathcal{V}}(\mathbf{u}),\mathcal{V})$.

In general, for a given hypersurfaces $\mathcal{U},\mathcal{V},\mathcal{W}$ there is no chance that a rational mapping $\varphi : \mathcal{U} \to \mathcal{V}$ could give us some relation between varieties $\mathcal{U}\star\mathcal{W}$ and $\mathcal{V}\star\mathcal{W}$. Contrariwise, if φ is coherent then by the following lemma we may extend it to a rational mapping between corresponding incidence varieties. In this way a coherent mapping relates variety $\mathcal{U}\star\mathcal{W}$ to the variety $\mathcal{V}\star\mathcal{W}$. This will be used later e.g. to derive a genus formula of convolutions with some simple curves.

Lemma 3.10. *For an arbitrary hypersurface $\mathcal{W}$, the coherent mapping $\xi_{\mathcal{U},\mathcal{V}} : \mathcal{U} \to \mathcal{V}$ may be naturally extended to the rational mapping $\zeta : \mathcal{I}(\mathcal{U},\mathcal{W}) \to \mathcal{I}(\mathcal{V},\mathcal{W})$ between incidence varieties. Moreover* $\deg\zeta = \deg\xi_{\mathcal{U},\mathcal{V}}$.

Proof. The mapping ζ, defined by

$$\zeta : (\mathbf{x},\mathbf{y}) \mapsto (\xi_{\mathcal{U},\mathcal{V}}(\mathbf{x}),\mathbf{y}) \tag{3.15}$$

obviously does the job. □

3.1.2 Parametric approach

If the hypersurfaces $\mathcal{V}$ and $\mathcal{W}$ are given by parameterizations $\mathbf{v} : \mathbb{C}^{n-1} \to \mathcal{V}$ and $\mathbf{w} : \mathbb{C}^{n-1} \to \mathcal{W}$, respectively, it is worth studying conditions under which there exists a rational parameterization of (a component of) $\mathcal{V} \star \mathcal{W}$.

Definition 3.11. Two parameterizations $\mathbf{v}(t)$ and $\mathbf{w}(t)$ are said to be *coherent*, written $\mathbf{v}(t) \sim_\star \mathbf{w}(t)$, if for a generic $t_0 \in \mathbb{C}^{n-1}$ it holds $(\mathbf{v}(t_0), \mathcal{V}) \sim_\star (\mathbf{w}(t_0), \mathcal{W})$. The parameterization $\mathbf{w}(t)$ is called *$\mathcal{V}$-coherent*, if there exists a parameterization $\mathbf{v}(t)$ of $\mathcal{V}$ coherent with $\mathbf{w}(t)$.

The following proposition is an immediate consequence of the definitions of convolutions and coherent parameterizations.

Proposition 3.12. *If $\mathbf{v}(t) \sim_\star \mathbf{w}(t)$ then there exists a component of $\mathcal{V} \star \mathcal{W}$ parameterized by $\mathbf{v}(t) + \mathbf{w}(t)$.*

Obviously, two random parameterizations are not coherent and thus the main problem of the parametric approach can be formulated as follows: For given parameterizations $\mathbf{v}(s)$ and $\mathbf{w}(t)$ find rational functions $\varphi, \psi \in \mathbb{C}(u)$ such that $\mathbf{v}(\varphi(u)) \sim_\star \mathbf{w}(\psi(u))$. This can be solved with the help of the so-called parameter variety in the space of variables s and t. This parametric counterpart to the incidence variety was introduced in Kim and Elber (2000).

Definition 3.13. Let $\mathbf{v} : \mathbb{C}^{n-1} \to \mathcal{V}$ and $\mathbf{w} : \mathbb{C}^{n-1} \to \mathcal{W}$ be two parameterization. Then we define the *parameter variety* as

$$\mathcal{P}(\mathbf{v}, \mathbf{w}) = \mathrm{cl}\left\{(s,t) \in \mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \mid (\mathbf{v}(s), \mathcal{V}) \sim_\star (\mathbf{w}(t), \mathcal{W})\right\} \tag{3.16}$$

Lemma 3.14. $\dim \mathcal{P}(\mathbf{v}, \mathbf{w}) = n - 1$.

Proof. Consider the projection $\pi : \mathcal{P}(\mathbf{v}, \mathbf{w}) \to \mathbb{C}^{n-1}$ onto the first $n-1$ coordinates. For a generic $s \in \mathbb{C}^{n-1}$ the fibre $\pi^{-1}(s)$ consists of all (s,t) such that points $\mathbf{v}(s)$ and $\mathbf{w}(t)$ are coherent. Since $\mathcal{W}$ does not have the degenerated Gauss image, we know that there exists only finitely many, say ℓ, points on $\mathcal{W}$ coherent with $\mathbf{v}(s)$. The parameterization $\mathbf{w} : \mathbb{C}^{n-1} \to \mathcal{W}$ is dominant and hence the cardinality of generic fibre $\pi^{-1}(s)$ is $\ell \cdot \deg \mathbf{w}$. Thus the projection π is finite dominant map and we arrive at $\dim \mathcal{P}(\mathbf{v}, \mathbf{w}) = \dim \mathbb{C}^{n-1} = n - 1$. □

Theorem 3.15. *$(\varphi(u), \psi(u))$ is a parameterization of a component of the parameter variety $\mathcal{P}(\mathbf{v}, \mathbf{w})$ if and only if $\mathbf{v}(\varphi(u)) \sim_\star \mathbf{w}(\psi(u))$.*

Proof. The $(\varphi(u), \psi(u))$ parameterizes a component of $\mathcal{P}(\mathbf{v}, \mathbf{w})$ if and only if $(\mathbf{v}(\varphi(u)), \mathcal{V}) \sim_\star (\mathbf{w}(\psi(u)), \mathcal{W})$ for a generic u by the construction of the parameter variety. However this is nothing but the definition of coherent parameterizations. □

Given a parameterization $\mathbf{v}(s) = (v_1(s), \ldots, v_n(s))$ we may compute the normal vector field e.g. using a determinant

$$\hat{\mathbf{n}}_{\mathbf{v}}(s) = \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ \frac{\partial v_1(s)}{\partial s_1} & \frac{\partial v_2(s)}{\partial s_1} & \cdots & \frac{\partial v_n(s)}{\partial s_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial v_1(s)}{\partial s_{n-1}} & \frac{\partial v_2(s)}{\partial s_{n-1}} & \cdots & \frac{\partial v_n(s)}{\partial s_{n-1}} \end{pmatrix}, \tag{3.17}$$

where $\mathbf{e}_i$ are the standard basis vectors. In the same way we may compute the normal vector field $\hat{\mathbf{n}}_{\mathbf{w}}(t)$ associated to the parameterization $\mathbf{w}(t)$. Let us denote

$$\left(\frac{\hat{\mu}_1}{\hat{\mu}_0}, \ldots, \frac{\hat{\mu}_n}{\hat{\mu}_0}\right) = \hat{\mathbf{n}}_{\mathbf{v}}(s) \quad \text{and} \quad \left(\frac{\hat{\nu}_1}{\hat{\nu}_0}, \ldots, \frac{\hat{\nu}_2}{\hat{\nu}_0}\right) = \hat{\mathbf{n}}_{\mathbf{w}}(t). \tag{3.18}$$

Then the polynomials

$$\mu_i := \frac{\hat{\mu}_i}{\gcd(\hat{\mu}_1, \ldots, \hat{\mu}_n)} \quad \text{and} \quad \nu_i := \frac{\hat{\nu}_i}{\gcd(\hat{\nu}_1, \ldots, \hat{\nu}_n)}, \qquad i = 1, \ldots, n, \tag{3.19}$$

lead to the polynomial vector fields with relatively prime coordinates

$$\mathbf{n}_{\mathbf{v}}(s) = (\mu_1(s), \ldots, \mu_n(s)) \quad \text{and} \quad \mathbf{n}_{\mathbf{w}}(t) = (\nu_1(t), \ldots, \nu_n(t)). \tag{3.20}$$

If $\mathbf{v}(s_0)$ and $\mathbf{w}(t_0)$ are regular points on the accordant hypersurfaces then the conditions on points to be coherent may be rewritten such that $\mathbf{n}_{\mathbf{v}}(s_0) = \lambda \mathbf{n}_{\mathbf{w}}(t_0)$ for some nonzero $\lambda \in \mathbb{C}$. This consideration led in Lávička and Bastl (2007) to the so-called *convolution ideal* given by

$$I := \langle \mu_1(s) - \lambda \nu_1(t), \ldots, \mu_n(s) - \lambda \nu_n(t), 1 - w\lambda \rangle, \tag{3.21}$$

where the last polynomial guarantees nonzero λ. The convolution ideal is closely related to the parameter variety, as the ideal of variety $\mathcal{P}(\mathcal{V}, \mathcal{W})$ can be obtained by eliminating variables λ and w from the convolution ideal.

Remark 3.16. If we consider convolution ideal I to be an ideal in the ring $\mathbb{C}(t)[w, s, \lambda]$ then it was shown in Lávička and Bastl (2007) that it is zero-dimensional ideal under the assumption that hypersurface given by parameterization $\mathbf{v}(s)$ does not have degenerated Gauss image. Then it follows that the reduced Gröbner basis of ideal I with respect to the lexicographic order $w > s_1 > \cdots > s_{n-1} > \lambda$ consists of polynomials

$$g_0(w, s_1, \ldots, s_{n-1}, \lambda),\ g_1(s_1, \ldots, s_{n-1}, \lambda),\ g_{n-1}(u_{n-1}, \lambda),\ g_n(\lambda) \tag{3.22}$$

with

$$\mathrm{LT}\,(g_0) = w,\ \mathrm{LT}\,(g_i) = u_i^{r_i}, \quad 1 \le i \le n-1, \quad \mathrm{LT}\,(g_n) = \lambda^{r_n}. \tag{3.23}$$

Remark 3.17. If $\mathcal{V}$ and $\mathcal{W}$ are rational plane curves then $\mathcal{P}(\mathcal{V}, \mathcal{W})$ is a plane curve too and the elimination can be done directly. This leads to the defining equation of the parameter variety

$$p_{\mathbf{v},\mathbf{w}}(s,t) := \mu_1(s)\nu_2(t) - \mu_2(s)\nu_1(t) = 0 \tag{3.24}$$

The defining polynomial of a hypersurface is given uniquely up to a constant multiple and hence the incidence variety (3.6) is unique, too. Contrariwise, there exists a plenty of parameterizations of given (uni)rational variety and the parameter variety depends on chosen parameterizations. Hence we cannot expect its uniqueness. On the other hand for a rational variety there exist proper parameterizations and an arbitrary parameterization of this variety is then just reparameterization of a proper one.

Lemma 3.18. *Let* $\mathbf{v}' = \mathbf{v} \circ \varphi$ *and* $\mathbf{w}' = \mathbf{w} \circ \psi$ *then there exists a rational mapping* $\mathcal{P}(\mathbf{v}', \mathbf{w}') \to \mathcal{P}(\mathbf{v}, \mathbf{w})$ *of the degree* $\deg \varphi \cdot \deg \psi$.

Proof. Let us define $\Phi : (s,t) \mapsto (\varphi(s), \varphi(t))$. It is easy to see that Φ is the desired mapping. Indeed if $(s_0, t_0) \in \mathcal{P}(\mathbf{v}', \mathbf{w}')$ then

$$(\mathbf{v}'(s_0), \mathcal{V}) = (\mathbf{v}(\varphi(s_0)), \mathcal{V}) \sim_\star (\mathbf{w}(\psi(t_0)), \mathcal{W}) = (\mathbf{w}'(t_0), \mathcal{W}), \tag{3.25}$$

and hence $\Phi(s_0, t_0) \in \mathcal{P}(\mathbf{v}, \mathbf{w})$. Next $\Phi^{-1}(\bar{s}, \bar{t}) = \varphi^{-1}(\bar{s}) \times \psi^{-1}(\bar{t})$ which shows that a degree of Φ, i.e. the cardinality generic fibre, is equal to $\deg \varphi \cdot \deg \psi$. □

Since the unirationality of some component of parameter variety corresponds to the existence of coherent reparameterization of input hypersurfaces, the previous lemma implies that in practical computation one should use parameterizations of lowest degree as possible.

Algorithm 2 Parameterization of a simple component of $\mathcal{V} \star \mathcal{W}$

Input: $\mathbf{v}(s) : \mathbb{C} \to \mathcal{V}$ and $\mathbf{w}(t) : \mathbb{C} \to \mathcal{W}$.
Output: A parameterization $\mathbf{x}(u) : \mathbb{C} \to \mathcal{X} \subset \mathcal{V} \star \mathcal{W}$ of a component.
1: If possible, *compute* proper parameterizations $\mathbf{v}'(s)$ and $\mathbf{w}'(t)$.
2: *Find* $\mathcal{P}(\mathbf{v}', \mathbf{w}') \subset \mathbb{C}^{n-1} \times \mathbb{C}^{n-1}$
3: *Decompose* $\mathcal{P}(\mathbf{v}', \mathbf{w}') = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_\ell$.
4: **if** $\mathcal{P}_i$ is unirational **then**
5: *parameterize* $(\varphi_i(u), \psi_i(u)) : \mathbb{C}^{n-1} \to \mathcal{P}_i$
6: **end if**
7: **return** $\mathbf{x}_i(u) = \mathbf{v}'_i(\varphi(u)) + \mathbf{w}'_i(\psi(u))$

It is not obvious at this moment, if each parameterization of some component of $\mathcal{V} \star \mathcal{W}$ can be obtained in this way. We will see later (cf. Remark 3.27) that under the conditions that $\mathcal{V}$ and $\mathcal{W}$ are unirational and $\mathcal{X}$ is a simple unirational component , then each its parameterization can be written as a sum of coherent parameterizations.

Example 3.19. Let be given parameterizations of plane curves, namely

$$\mathbf{v}(r) = \left(\frac{27r^4 - r^2 - 4r - 4}{(2+r)^2}, \frac{9r^2(3r^4 - r^2 - 4r - 4)}{(2+r)^2} \right), \tag{3.26}$$

and

$$\mathbf{w}(t) = \left(\frac{t^8 - 8t^6 + 5t^4 + 1}{t^6 + 1}, \frac{4t^5(t^2 - 2)}{t^6 + 1} \right). \tag{3.27}$$

It may be shown that $\mathbf{w}(t)$ is proper while $\mathbf{v}(r)$ can be written in the form $\mathbf{v} = \hat{\mathbf{v}}(\zeta(s))$ for a proper parameterization $\hat{\mathbf{v}}(s) = (3s^2 - 1, s(s^2 - 3))$ and $\zeta(s) = 3s^2/(s+2)$. Then the associated normal fields admits the parameterizations

$$\mathbf{n}_{\hat{\mathbf{v}}}(s) = (s^2 - 1, -2s) \quad \text{and} \quad \mathbf{n}_{\mathbf{w}}(t) = (2t^3 - 2t, t^4 - t^2 - 2t - 1). \tag{3.28}$$

By Remark 3.17, the parameter variety $\mathcal{P}(\hat{\mathbf{v}}, \mathbf{w})$ is a plane curve given by equation

$$p_{\hat{\mathbf{v}},\mathbf{w}} = \mathbf{n}_{\hat{\mathbf{v}}} \cdot \mathbf{n}_{\mathbf{w}}^{\perp} = (s^2 - 1)(t^4 - t^2 - 2t - 1) + 2s(2t^3 - 2t) = 0 \tag{3.29}$$

The parameter variety is not irreducible, because the polynomial $p_{\hat{\mathbf{v}},\mathbf{w}}$ can be written as a product

$$(1 - s - t - st - t^2 + st^2)(-1 - s - t + st + t^2 + st^2). \tag{3.30}$$

Two factors induce two components of parameter variety $\mathcal{P}(\bar{\mathbf{v}}, \mathbf{w}) = \mathcal{P}_1 \cup \mathcal{P}_2$. Since the variable s is linear in both polynomials it is very easy to find a parameterizations of these components. For instance the first component can be parameterizad by

$$(\varphi(u), \psi(u)) = \left(\frac{u^2 + u - 1}{u^2 - u - 1}, u \right), \tag{3.31}$$

and thus $\hat{\mathbf{h}}[v](\varphi(u)) + \mathbf{w}(\psi(u))$ parameterizes a component of $\mathcal{V} \star \mathcal{W}$.

Although the algorithm looks very simple each step involves quite tough computations already in the surface case, which makes it quite impractical. If it happen – as did in our example, that only one hypersurface need to be reparameterized, then at least the parameterization step is considerably simplified. In Lávička and Bastl (2007) the authors studied conditions under which a given parameterization $\mathbf{v}(s)$ can be reparameterized to be coherent with some other parameterization $\mathbf{w}(t)$. They observed that such a condition is sometimes provided by $\mathbf{v}(s)$ on her own in the form of some algebraic condition which the $\mathbf{n}_{\mathbf{w}}(t)$ has to fulfill. Such a condition is then called *RC-condition*.

In particular, their approach consisted in replacing the normal vector field $\mathbf{n_w}(t)$ in the convolution ideal by the n-tuple of new symbolic variables $(\nu_1, \ldots, \nu_n)$. The RC-conditions then appear when one expresses the variables s from the equations of this modified convolution ideal. Let us illustrate this in the example.

Example 3.20. Let $\mathbf{v}(s) = (3s^2 - 1, s(s^2 - 3))$ be the proper parameterization from the previous example. Thus $\mathbf{n_v}(s) = (s^2 - 1, -2s)$ and instead of $\mathbf{n_w}(t)$ we use (ν_1, ν_2). With these data, the Gröbner basis w.r.t. $w > s > \lambda$ of the convolution ideal has the form

$$g_0 = 1 - w\lambda, \quad g_1 = 2s + \lambda\nu_2, \quad g_2 = \lambda^2\nu_2^2 - 4\lambda\nu_1 - 4. \tag{3.32}$$

The computation of the reparameterization of $\mathbf{v}(s)$ is now reduced to the expressing s from (3.32) as the function of ν_1 and ν_2. Therefore we arrive at

$$s = -\frac{\nu_1 \pm \sqrt{\nu_1^2 + \nu_2^2}}{\nu_2}. \tag{3.33}$$

which implies that $s = s(t)$ is rational if and only if normal field $\mathbf{n_w}(t) = (\nu_1(t), \nu_2(t))$ fulfils

$$\nu_1^2(t) + \nu_2^2(t) = \sigma^2(t), \tag{3.34}$$

for some polynomial $\sigma(t)$.

3.1.3 Dual approach

Let $\mathbf{v} \in \mathcal{V}$ and $\mathbf{w} \in \mathcal{W}$ be two generic coherent points on hypersurfaces in $\mathbb{C}^n$. And let us suppose that an affine tangent hyperplane $T_{\mathbf{v}}^A\mathcal{V}$ is given by

$$\mathbf{n} \cdot \mathbf{x} = n_1x_1 + \cdots + n_nx_n = h_{\mathbf{v}}. \tag{3.35}$$

Then the $T_{\mathbf{w}}^A\mathcal{W}$ is described by $\mathbf{n} \cdot \mathbf{x} = h_{\mathbf{w}}$ and by Lemma 3.7 the affine tangent hyperplane at the point $\mathbf{v} + \mathbf{w} \in \mathcal{V} \star \mathcal{W}$ has the equation

$$T_{\mathbf{v}+\mathbf{w}}^A(\mathcal{V} \star \mathcal{W}) : \ \mathbf{n} \cdot \mathbf{x} = h_{\mathbf{v}} + h_{\mathbf{w}}, \tag{3.36}$$

i.e., it is obtained by summing $h_{\mathbf{v}}$ and $h_{\mathbf{w}}$.

It follows that the description of convolutions in dual space is considerably very simple. In fact, the original definition Sabin (1974) is based on this dual description. Later, these ideas were developed in more detail e.g. in the papers Šír et al. (2007); Gravesen et al. (2008) with the help of the so-called *support function representation*, which is tool well known from the convex geometry, cf. Gruber and Wills (1993)

By the *support function* it is meant an arbitrary rational function $h : \mathcal{S}^{n-1} \to \mathbb{C}$. This encodes a dual representation of the hypersurface $\mathcal{X}$ as the envelope of the hyperplanes

$$\mathbf{n} \cdot \mathbf{x} = h(\mathbf{n}), \tag{3.37}$$

where $\mathbf{n}$ runs over the unit sphere. If we restrict ourself to the field of real numbers, then the support function may be apprehended as the function measuring the oriented distance of the hyperplanes (3.37) from the origin.

As we saw above, the convolution of two hypersurfaces is then reduced to the sum of their support functions. Unfortunately, not every hypersurface admits support functions representation. It is immediately seen that such a hypersurface $\mathcal{X}$ has to be rational and it can have at most two tangent planes with a given direction – otherwise the function $\mathcal{S}^{n-1} \to \mathbb{C}$ would be one-to-many. Moreover this condition is necessary but not sufficient as the following example shows.

Example 3.21. Let $\mathcal{X} \subset \mathbb{C}^2$ be an ellipse given by $x^2/a^2 + y^2/b^2 = 1$, where $a, b \in \mathbb{R}\backslash\{0\}$. Then its dual equation can be expressed as $a^2n_1^2 + b^2n_2^2 = h^2$. Under the assumption $n_1^2 + n_2^2 = 1$ this may be rearranged into

$$h = \pm\sqrt{(a^2 - b^2)n_1^2 + b^2}, \tag{3.38}$$

which is seen to be a rational function only if $a^2 = b^2$, i.e., if $\mathcal{X}$ is a circle.

This led authors of Lávička et al. (2010) to define the so-called *implicit support function* which removes these drawbacks. It turns out that it is exactly the dual equation of a hypersurface as defined in Subsection 2.1.2. In the same paper the relation between convolutions and dual equation of hypersurface was established and we formulate it as a proposition here.

Proposition 3.22. *Let $F^\vee(\mathbf{n}, h_1)$ and $G^\vee(\mathbf{n}, h_2)$ be the defining polynomials of $\mathcal{V}^\vee$ and $\mathcal{W}^\vee$, respectively. Then the defining equation of $(\mathcal{V} \star \mathcal{W})^\vee$ is obtained by eliminating variables h_1 and h_2 from the system*

$$F^\vee(\mathbf{n}, h_1), \quad G^\vee(\mathbf{n}, h_2), \quad h - h_1 - h_2. \tag{3.39}$$

This can be again translated into the language of incidence varieties as follows. Let ν denotes the projection $P^n\mathbb{C} \to P^{n-1}\mathbb{C}$ given by

$$(n_1 : \cdots : n_n : h) \mapsto (n_1 : \cdots : n_n), \tag{3.40}$$

which is well defined outside the point $\mathbf{s} = (0 : \cdots : 0 : 1)$. Then the points $\mathbf{v} \in \mathcal{V}^\vee$ and $\mathbf{w} \in \mathcal{W}^\vee$ such that $\nu(\mathbf{v}) = \nu(\mathbf{w})$ represent tangent hyperplanes at the coherent points on the hypersurfaces $\mathcal{V}$ and $\mathcal{W}$. Thus it is natural to define *dual incidence variety* $\mathcal{I}^\vee(\mathcal{V}^\vee, \mathcal{W}^\vee)$ as the closure of the set of pairs $(\mathbf{v}, \mathbf{w})$ such that $\nu(\mathbf{v}) = \nu(\mathbf{w})$, i.e., it is the closure of the fibre product

$$\mathcal{I}^\vee(\mathcal{V}^\vee, \mathcal{W}^\vee) = \mathrm{cl}\left(\mathcal{V}^\vee\backslash\{\mathbf{s}\} \times_{P^{n-1}\mathbb{C}} \mathcal{W}^\vee\backslash\{\mathbf{s}\}\right). \tag{3.41}$$

Then the "sum of support functions" is performed by the rational mapping $\sigma^\vee : \mathcal{I}^\vee(\mathcal{V}^\vee, \mathcal{W}^\vee) \to P^n\mathbb{C}$ defined by

$$\sigma^\vee\big((\mathbf{n} : h_1), (\mathbf{n} : h_2)\big) \mapsto (\mathbf{n} : h_1 + h_2). \tag{3.42}$$

The above considerations lead to the lemma

Lemma 3.23. $(\mathcal{V} \star \mathcal{W})^\vee = \sigma^\vee(\mathcal{I}^\vee(\mathcal{V}^\vee, \mathcal{W}^\vee))$.

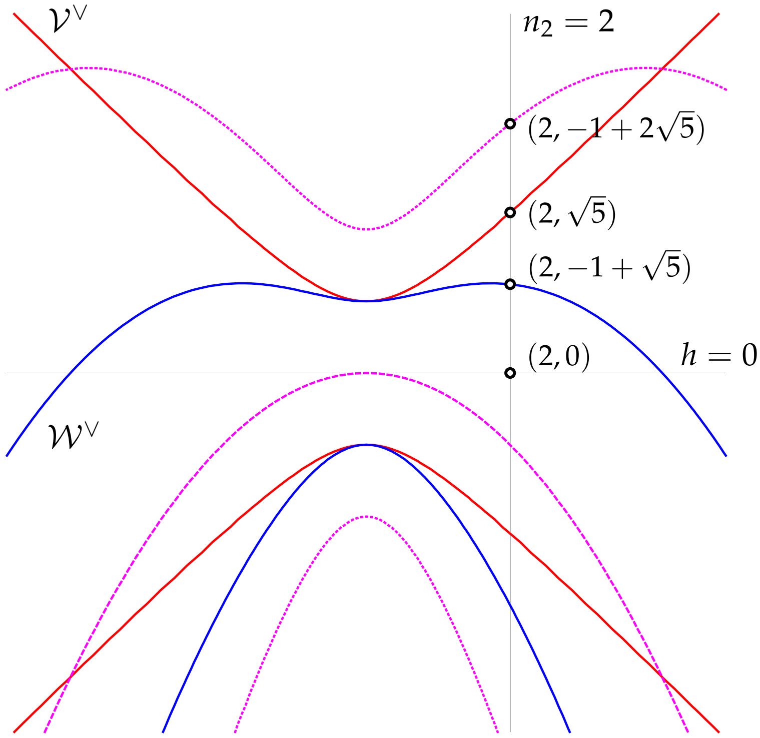


Figure 3.1: The construction of the convolution curve in the dual plane (the dual image is dehomogenized by setting $n_1 = 1$) – the red and blue curves are the irreducible input curves, their reducible convolution is in purple color.

3.2 Properties of convolutions

In this section we are going to analyze algebraic properties of convolutions. Using an affine invariant of curve called the convolution degree, we will be able to give a bound to the number of components of a convolution. Next, we will give a complete characterization of special and degenerated components. As a consequence of this analysis some conditions on rationality of convolutions components will be formulated. The most of results introduced in this section is based on Vršek and Lávička (2010b).

3.2.1 Types of convolution components and their characterization

Definition 2.9 introduced three different types of convolution components. Using the set $\mathcal{I}(\mathcal{V},\mathcal{W})$ and the associated mappings, the particular type corresponds to the degree of the mapping σ. It turns out that the most interesting cases are simple components since then the mapping σ restricted on this simple component is a birational mapping.

Lemma 3.24. *An irreducible component $\mathcal{X}$ of $\mathcal{V}\star\mathcal{W}$ is* simple, special, *or* degenerated *if and only if* $\deg\sigma|_{\sigma^{-1}(\mathcal{X})}$ *is equal to* 1, k *for* $1<k<\infty$, *or* ∞, *respectively.*

Proof. For a generic $\mathbf{u}\in\mathcal{X}\subset\mathcal{V}\star\mathcal{W}$ the fibre $\sigma^{-1}(\mathbf{u})$ consists of $\{\mathbf{v}_\alpha,\mathbf{w}_\alpha\}_{\alpha\in A}$ such that $\mathbf{u}=\mathbf{v}_\alpha+\mathbf{w}_\alpha\in\mathcal{W}$ and $(\mathbf{v}_\alpha,\mathcal{V})\sim_\star(\mathbf{w}_\alpha,\mathcal{W})$. Hence, the cardinality of a generic fibre is equal to the number of pairs $(\mathbf{v}_\alpha,\mathbf{w}_\alpha)$ generating $\mathbf{u}$. By the definition of the degree of a rational mapping it is nothing but the degree of σ. □

Moreover, it is seen that for any special component we have one fixed number k such that a generic fibre has the cardinality k. Hence in what follows, we will use the name *k-special* component.

For degenerated components we immediately obtain:

Corollary 3.25. *Let $\mathcal{X}\subset\mathcal{V}\star\mathcal{W}$ be a degenerated component. Then* $\dim\mathcal{X}<n-1$.

Proof. Since the fibres of σ are infinite for a generic point $\mathbf{x}$ on the degenerated component $\mathcal{X}$, it follows that $\dim\mathcal{X}<\dim\mathcal{I}(\mathcal{V},\mathcal{W})=n-1$, cf. the proof of Corollary 3.3. □

Corollary 3.26. *Let $\mathcal{X}$ be an irreducible simple component of $\mathcal{V}\star\mathcal{W}$, then there exist the rational mappings $\mathcal{X}\to\mathcal{V}$ and $\mathcal{X}\to\mathcal{W}$. In particular $\mathcal{X}$ cannot be unirational if $\mathcal{V}$ and $\mathcal{W}$ are not.*

Proof. Since $\mathcal{X}$ is simple, we have $\deg\sigma|_{\sigma^{-1}(\mathcal{X})}=1$ and thus there exists its inverse. The wanted mappings are $\sigma^{-1}\circ\pi_\mathcal{V}$ and $\sigma^{-1}\circ\pi_\mathcal{W}$.

If $\mathcal{X}$ is unirational, then there exists its parameterization $\mathbf{x}:\mathbb{C}^{n-1}\to\mathcal{X}$. Then the composed mapping $\pi_\mathcal{V}\circ\sigma^{-1}\circ\mathbf{x}:\mathbb{C}^{n-1}\to\mathcal{V}$ is dominant as it is the composition of dominant mappings and thus a parameterization of $\mathcal{V}$. The unirationality of $\mathcal{W}$ follows analogously. □

Remark 3.27. As a special consequence of Corollary 3.26 we see that a rational parameterization $\mathbf{x}(s)$ of a simple component $\mathcal{X}$ of $\mathcal{V} \star \mathcal{W}$ can by pushed forward a rational parameterizations $\mathbf{v}(s)$ and $\mathbf{w}(s)$ of $\mathcal{V}$ and $\mathcal{W}$, respectively. Moreover from the definition of incidence variety it follows that $\mathbf{v}(s) \sim_\star \mathbf{w}(s)$ and $\mathbf{x}(s) = \mathbf{v}(s) + \mathbf{w}(s)$. Consequently every parameterization of $\mathcal{X}$ is the sum of coherent parameterizations of input curves, which justifies Algorithm 2.

The rationality problem is the most simple in the curve case because it depends only on vanishing the genus of the curve. Moreover we may formulate stronger result which relates the genera of input curves and the genus of a simple component of their convolution for curves.

Corollary 3.28. *Let $\mathcal{V}, \mathcal{W} \subset \mathbb{C}^2$ be curves and $\mathcal{X}$ a simple component of $\mathcal{V} \star \mathcal{W}$, then* $\mathrm{g}(\mathcal{X}) \geq \max\{\mathrm{g}(\mathcal{V}), \mathrm{g}(\mathcal{W})\}$.

Proof. We will prove that $\mathrm{g}(\mathcal{X}) \geq \mathrm{g}(\mathcal{V})$, the second inequality would be proved analogously. If $\mathrm{g}(\mathcal{V}) = 0$ then there is nothing to prove. Next, we assume $\mathrm{g}(\mathcal{V}) > 0$. Since $\mathcal{X}$ is simple, there exists a rational mapping $\pi_{\mathcal{V}} \circ \sigma^{-1} : \mathcal{X} \to \mathcal{V}$, by Corollary 3.26. Let $\varphi : \mathcal{X}^P \to \mathcal{V}^P$ be its projective extension. Hironaka's theorem ensures that there exist smooth curves $\widetilde{\mathcal{X}}^P$ and $\widetilde{\mathcal{V}}^P$ which are birational to $\mathcal{X}^P$ and $\mathcal{V}^P$, respectively. Moreover, the mapping φ is resolved to the rational mapping $\widetilde{\varphi} : \widetilde{\mathcal{X}}^P \to \widetilde{\mathcal{V}}^P$. Invoking Riemann-Hurwitz formula (Theorem 2.4) and after some simple calculations we arrive at

$$\mathrm{g}\left(\widetilde{\mathcal{X}}^P\right) = \mathrm{g}\left(\widetilde{\mathcal{V}}^P\right) + 2\left(\deg\widetilde{\varphi} - 1\right)\left(\mathrm{g}\left(\widetilde{\mathcal{V}}^P\right) - 1\right) + \frac{1}{2}\deg D_R, \tag{3.43}$$

where all the terms on the right side of (3.43) are nonnegative. Thus using $\mathrm{g}(\mathcal{X}) = \mathrm{g}\left(\widetilde{\mathcal{X}}^P\right)$ and $\mathrm{g}(\mathcal{V}) = \mathrm{g}\left(\widetilde{\mathcal{V}}^P\right)$ we get $\mathrm{g}(\mathcal{X}) \geq \mathrm{g}(\mathcal{V})$ and the statement is proved. □

Let us emphasize that Corollary 3.28 does not hold for special components as one can see in the following example.

Example 3.29. Given two curves

$$\mathcal{V} = \mathbb{V}\left(x_1^8 - 12x_1^6 - 2x_2^4x_1^4 + 48x_1^4 - 20x_2^4x_1^2 - 64x_1^2 + x_2^8 + 4x_2^4\right) \tag{3.44}$$

and

$$\mathcal{W} = \mathbb{V}\left(x_1^2 + x_2^2 - 1\right). \tag{3.45}$$

The curve $\mathcal{V}$ has genus equal to 1 and thus it is not rational, the curve $\mathcal{W}$ is the unit circle which genus equals zero. The convolution $\mathcal{V} \star \mathcal{W}$ factorizes into two components

$$\begin{aligned} \mathcal{X}_1 = \mathbb{V}\,(&x_1^8 - 30x_1^6 - 2x_2^4x_1^4 - 30x_2^2x_1^4 + 309x_1^4 - 50x_2^4x_1^2 + 330x_2^2x_1^2 - \\ &-1180x_1^2 + x_2^8 - 18x_2^6 + 133x_2^4 - 516x_2^2 + 900) \end{aligned} \tag{3.46}$$

and

$$\mathcal{X}_2 = \mathbb{V}\left(x_1^2 - x_2^2 - 1\right). \qquad (3.47)$$

The component $\mathcal{X}_1$ is simple and $\mathrm{g}(\mathcal{X}_1) = 1 \geq 1 = \max\{\mathrm{g}(\mathcal{V}), \mathrm{g}(\mathcal{W})\}$. On the other hand the component $\mathcal{X}_2$ is 2-special with $\mathrm{g}(\mathcal{X}_2) = 0 < 1$.

3.2.2 Convolution degree

In this subsection, we define the so-called convolution degree which reflects a complexity of a given hypersurface with respect to the operation of convolution. Let us recall that for a $(n-1)$-space $H \subset \mathbb{C}^n$, the $\mathcal{X}_H$ stands for the set of points $\mathbf{p} \in \mathcal{X}$ such that $T_{\mathbf{p}}\mathcal{X} = H$.

Definition 3.30. The *convolution degree* $\kappa_{\mathcal{X}}$ of a hypersurface $\mathcal{X}$ is equal to the cardinality of $\mathcal{X}_H$ for a generic $(n-1)$-space H.

It is not obvious if $\sharp\mathcal{X}_H$ is constant for generic $(n-1)$-spaces H and henceforth if the definition makes sense. Let $\gamma_{\mathcal{X}} : \mathcal{X} \to \mathrm{Gr}^0(n-1,n)$ be the Gauss mapping defined in Subsection 2.1.2 and let $\mathbf{n}$ be the representant of H in $\mathrm{Gr}^0(n-1,n)$. Then, since H is generic, the fibre $\gamma_{\mathcal{X}}^{-1}(\mathbf{n})$ consists exactly of points $\mathbf{p}$ on $\mathcal{X}$ such that $T_{\mathbf{p}}\mathcal{X} = H$. Hence the convolution degree is nothing but the cardinality of the fibre $\gamma_{\mathcal{X}}^{-1}(\mathbf{n})$ for a generic $\mathbf{n}$. This leads to the proposition, which justifies Definition 3.30, as well.

Proposition 3.31. *Let be given an algebraic hypersurface $\mathcal{V}$ with the Gauss mapping $\gamma_{\mathcal{V}}$. Then $\kappa_{\mathcal{V}} = \deg \gamma_{\mathcal{V}}$.*

Remark 3.32. From the geometric interpretation it is obvious that the convolution degree of a hypersurface is an affine invariant, i.e., it is not affected by applying any affine transformation.

Example 3.33. Let $\mathcal{S}^{n-1} \subset \mathbb{C}^n$ be the unit sphere, and $\mathbf{a} \in \mathbb{C}^n$ a vector such that $\mathbf{a} \cdot \mathbf{a} = a_1^2 + \cdots + a_n^2 \neq 0$. If we take the $(n-1)$-space $A : \mathbf{a} \cdot \mathbf{x} = 0$ then it is not hard to find out that the set $\mathcal{S}_A^{n-1}$ consists exactly of two points $\mathbf{a}/\sqrt{\mathbf{a} \cdot \mathbf{a}}$. Since $\mathbf{a}$ was chosen generically, we obtain $\kappa_{\mathcal{S}}^{n-1} = 2$.

As the convolution degree is closely related to the complexity of a hypersurface with respect to the operation of convolution, it is important to be able to compute it even for more complicated hypersurfaces then are the spheres. We have described three different approaches used for studying convolutions and each of them provides a natural way how the convolution degree can be computed.

Theorem 3.34. *Let $\mathcal{V}$ be an algebraic hypersurface, $F^{\vee}(\mathbf{n},h)=0$ its dual equation and let $\mathcal{W}$ be an arbitrary hypersurface. Moreover, if $\mathcal{V}$ is a unirational hypersurface given by the parameterization $\mathbf{v}(s)$ then denote by $g_0,\dots g_n$ the Gröbner basis as in Remark 3.16. Under these assumptions it holds*

(i) $\kappa_{\mathcal{V}}=\deg \pi_{\mathcal{W}}$,

(ii) $\kappa_{\mathcal{V}}=\frac{r_1\cdot r_2\cdots r_n}{\deg \mathbf{v}}$,

(iii) $\kappa_{\mathcal{V}}=\deg_h F^{\vee}(\mathbf{n},h)$.

Proof. (i) Follows immediately from the definition of the incidence variety $\mathcal{I}(\mathcal{V},\mathcal{W})$.

(ii) See Lávička and Bastl (2007).

(iii) See Lávička et al. (2010). □

Remark 3.35. The convolution degree appeared firstly in Lávička and Bastl (2007). More precisely the integer $\delta=r_1\cdot r_2\cdots r_n$, called *degree of the construction* of convolution hypersurface, was introduced in mentioned paper. It is seen that it is a convolution degree of hypersurface multiplied by the degree of a given parameterization and hence it is more related to the parameterization then to the hypersurface itself.

Remark 3.36. If $\mathcal{X}$ is a rational curve then $\kappa_{\mathcal{X}}$ can be computed immediately from its normal vector field instead of computing Gröbner basis. Namely, if $\mathbf{x}(s)$ is a proper parameterization and $\mathbf{n_x}(s)=(n_1(s),n_2(s))$ the associated normal vector field, then we have

$$\kappa_{\mathcal{X}}=\max\{\deg n_1,\deg n_2\}. \tag{3.48}$$

Let $\mathcal{X}$ be a hypersurface given by the irreducible polynomial $f\in\mathbb{C}[\mathbf{x}]$. For some fixed nonzero $\mathbf{a}\in\mathbb{C}^n$ we set $\{\mathbf{a}_1,\dots,\mathbf{a}_{n-1}\}$ as a basis of a $(n-1)$-space $\mathbf{a}\cdot\mathbf{x}=0$. Then for a point $\mathbf{p}\in\mathcal{X}$ it holds $T_{\mathbf{p}}\mathcal{X}=\{\mathbf{a}\cdot\mathbf{x}=0\}$ if and only if $\mathbf{p}$ is the solution of the system of equations

$$f(\mathbf{x})=0,\qquad \nabla f(\mathbf{x})\cdot\mathbf{a}_i=0,\quad i=1,\dots,n-1, \tag{3.49}$$

and it is not a singular point on $\mathcal{X}$. Thus generically we have one equation of the degree $\deg f=\deg\mathcal{X}$ and $n-1=\dim\mathcal{X}$ equations of the degrees $\deg\mathcal{X}-1$. Thus, using Bézout theorem, we arrive at the estimation.

Proposition 3.37. *For an arbitrary hypersurface $\mathcal{X}$ with non-degenerated Gauss image it holds*

$$\kappa_{\mathcal{X}}\le\deg\mathcal{X}(\deg\mathcal{X}-1)^{\dim\mathcal{X}}. \tag{3.50}$$

Since each singular point belongs to the solution of system (3.49) however it is not counted into the convolution degree it is obvious that singular points decreases the convolution degree of a variety. Similarly, proper use of Bézout theorem would require the projective extension of (3.49). However the ideal points of $\mathcal{X}$ do not contribute to the convolution degree, too.

Although a generic hypersurface is smooth and its intersection with the ideal hyperplane does not affect the convolution degree, we are mostly interested in rational varieties, which are usually singular. In what follows, we will refine the computations in order to obtain a compact formula expressing the convolution degree of a given algebraic curve. For this purpose, it is necessary to introduce another affine invariant of curves describing their relation to the ideal line ω.

Definition 3.38. Let $\mathcal{X}$ be an affine curve and $\mathcal{X}^P$ its projective closure. We define the so-called *ω-correction* of $\mathcal{X}$ as the integer

$$\Omega_{\mathcal{X}} := \sum_{\mathbf{x}\in \operatorname{Reg}\mathcal{X}^P} \left(I_{\mathbf{x}}\left(\mathcal{X}^P, \omega\right) - 1\right). \tag{3.51}$$

Remark 3.39. Since $\mathcal{X}^P$ intersects ω only in a finite number of points, the ω-correction is finite, too. Moreover, it can be easily computed in the following way. Let $\mathcal{X}^P = \mathbb{V}(F)$ where $F = \sum_{i+j+k=n} a_{jk} x_0^i x_1^j x_2^k$. Then the intersections of $\mathcal{X}^P$ with ω are the solutions of the equation $f_{(n)} = \sum_{j+k=n} a_{jk} x_1^j x_2^k = 0$. This equation is homogeneous and can be rewritten as

$$f_{(n)}(x_1, x_2) = \sum_{i+j=n} a_{ij} x_1^i x_2^j = \prod_{i=1}^{m} (\alpha_i x_1 + \beta_i x_2)^{k_i} = 0. \tag{3.52}$$

Hence, the points in $\mathcal{X}^P \cap \omega$ are $\mathbf{x}_i = (0 : -\beta_i : \alpha_i)$. In addition, the intersection multiplicity $I_{\mathbf{x}_i}(\mathcal{X}^P, \omega)$ is equal to k_i.

If $f_{(n)} = x_1^n$ then $\Omega_{\mathcal{X}}$ equals $n-1$ for $(0 : 0 : 1)$ being a regular point of $\mathcal{X}^P$, and 0 otherwise. The case $f_{(n)} = x_2^n$ can be handled analogously. Thus, we can exclude these special cases from further considerations. Next, we have

$$\operatorname{GCD}\left(f_{(n)}, \frac{\partial f_{(n)}}{\partial x_1}, \frac{\partial f_{(n)}}{\partial x_2} \right) = \prod_{i=1}^{m} (\alpha_i x_1 + \beta_i x_2)^{k_i - 1}. \tag{3.53}$$

Of course, some factors of the polynomial (3.53) may correspond to the singular points of $\mathcal{X}^P \cap \omega$. We may identify them by computing a square-free part $s(x_1, x_2)$ of the polynomial

$$\operatorname{GCD}\left(f_{(n-1)}, \frac{\partial f_{(n)}}{\partial x_1}, \frac{\partial f_{(n)}}{\partial x_2} \right). \tag{3.54}$$

After repeated division of (3.53) by the square-free polynomial $s(x_1, x_2)$, we arrive at the polynomial $g(x_1, x_2) = \prod_{i\in A} (\alpha_i x_1 + \beta_i x_2)^{k_i - 1} = 0$, where $A \subset$

$\{1,\ldots,m\}$. From the considerations mentioned above, it follows $\Omega_{\mathcal{X}} = \sum_{i\in A}(k_i - 1)$. Therefore, we obtain

$$\Omega_{\mathcal{X}} = \deg g. \tag{3.55}$$

In addition, this immediately gives a bound for the ω-correction

$$0 \leq \Omega_{\mathcal{X}} \leq \deg \mathcal{X} - 1. \tag{3.56}$$

Theorem 3.40. (Convolution degree formula) *Let $\mathcal{X}$ be an affine curve, $\mathcal{X}^P$ its projective closure and denote $\Delta_{\mathbf{x}} := m_{\mathbf{x}}(\mathcal{X}^P) - r_{\mathbf{x}}\left(\mathcal{X}^P\right)$ for any $\mathbf{x} \in \mathcal{X}^P_{\text{Sing}}$. If no tangent at a singular point of $\mathcal{X}^P$ coincides with ω then the convolution degree of $\mathcal{X}$ is equal to*

$$\kappa_{\mathcal{X}} = 2\left(\deg(\mathcal{X}) + \mathrm{g}\left(\mathcal{X}\right) - 1\right) - \sum_{\mathbf{x}\in \operatorname{Sing}\mathcal{X}^P} \Delta_{\mathbf{x}} - \Omega_{\mathcal{X}}. \tag{3.57}$$

Before proving the theorem, we recall one necessary lemma whose proof can be found e.g. in Tutaj (1993).

Lemma 3.41. (Weak version of Teissier's lemma) *Assume that we have an irreducible curve $\mathcal{X} = \mathbb{V}(f)$, where $f \in \mathbb{C}[x_1,x_2]$ such that $\mathbf{o} = (0,0) \in \mathcal{X}$ and $f(0,x_2) \not\equiv 0$. Then*

$$\mu_{\mathbf{o}}(\mathcal{X}) = I_{\mathbf{o}}\left(\mathcal{X}, \mathbb{V}\left(\frac{\partial f}{\partial x_2}\right)\right) - I_{\mathbf{o}}\left(\mathcal{X}, \mathbb{V}(x_1)\right) + 1. \tag{3.58}$$

Proof. (of Theorem 3.40) Let $\mathbf{p} = (p_0 : p_1 : p_2) \in P^2\mathbb{C}$ be a generic point and let us denote $\mathcal{P}_{\mathbf{p}}\mathcal{X}^P$ the curve with the defining equation

$$\frac{\partial F}{\partial x_0}p_0 + \frac{\partial F}{\partial x_1}p_1 + \frac{\partial F}{\partial x_2}p_2 = 0, \tag{3.59}$$

where $F(x_0,x_1,x_2)$ is the homogenization of the defining polynomial of $\mathcal{X}$. We will refer to $\mathcal{P}_{\mathbf{p}}\mathcal{X}^P$ as the polar curve of $\mathcal{X}$ at $\mathbf{p}$. If we choose generic $\mathbf{p} \in \omega$, then the convolution degree is exactly the number of regular points in $\mathcal{X}^P \cap \mathcal{P}_{\mathbf{p}}\mathcal{X}^P$ which do not lie at infinity.

First we will find the number of regular points in the intersection and then subtract some correction for the intersection points on ω. We denote $n = \deg(\mathcal{X}^P)$ and thus for any $\mathcal{X}$ we have generically $\deg \mathcal{P}_{\mathbf{p}}\mathcal{X}^P = n - 1$. Then by Bézout theorem one obtains

$$n(n-1) = \sum_{\mathbf{x}\in \operatorname{Reg}\mathcal{X}^P} I_{\mathbf{x}}\left(\mathcal{X}^{\mathcal{P}}, \mathcal{P}_{\mathbf{p}}\mathcal{X}^P\right) + \sum_{\mathbf{x}\in \operatorname{Sing}\mathcal{X}^P} I_{\mathbf{x}}\left(\mathcal{X}^{\mathcal{P}}, \mathcal{P}_{\mathbf{p}}\mathcal{X}^P\right). \tag{3.60}$$

Now, to determine the number of regular points it is enough to compute the intersection multiplicities $I_{\mathbf{x}}(\mathcal{X}^{\mathcal{P}}, \mathcal{P}_{\mathbf{p}}\mathcal{X}^P)$ for both the regular and singular points.

We may assume w.l.o.g. that $\mathbf{x} = (1:0:0)$ and $\mathbf{p} = (0:0:1) \in \omega$, i.e., we can work with the affine curve $\mathcal{X}$ and the polar given by $\mathcal{P}_{(0,1)}\mathcal{X} = \mathbb{V}(\partial f/\partial x_2)$. By Teissier's lemma we know that the Milnor number of $\mathbf{x}$ is

$$\mu_{\mathbf{x}}(\mathcal{X}) = I_{\mathbf{x}}\left(\mathcal{X}, \mathcal{P}_{\mathbf{p}}\mathcal{X}\right) - I_{\mathbf{x}}\left(\mathcal{X}, \mathbb{V}(x_1)\right) + 1. \tag{3.61}$$

There is only a finite number of lines in a tangent cone at any singular point $\mathbf{x}$ and we can assume that none of them passes through $\mathbf{p}$ (the only exception occurs when $\mathbf{x}$ lies on ω and ω is tangent to $\mathcal{X}^P$ at $\mathbf{x}$ – however, this configuration was excluded by the assumptions of the theorem). Since $\mathbb{V}(x_1)$ is not a component of the tangent cone at $\mathbf{x}$, the number $I_{\mathbf{x}}(\mathcal{X}, \mathbb{V}(x_1))$ is exactly equal to the multiplicity $m_{\mathbf{x}}(\mathcal{X})$. Using (3.61) and the Milnor identity (2.7) we arrive at

$$I_{\mathbf{x}}\left(\mathcal{X}^P, \mathcal{P}_{\mathbf{p}}\mathcal{X}^P\right) = I_{\mathbf{x}}\left(\mathcal{X}, \mathcal{P}_{\mathbf{p}}\mathcal{X}\right) = 2\delta_{\mathbf{x}} + \Delta_{\mathbf{x}}. \tag{3.62}$$

If $\mathbf{x} \in \mathcal{X} \cap \mathcal{P}_{\mathbf{p}}\mathcal{X}$ is a regular point of $\mathcal{X}$ then $\mathbb{V}(x_1)$ is the tangent line to $\mathcal{X}$ at $\mathbf{x}$. Hence $I_{\mathbf{x}}\left(\mathcal{X}, \mathbb{V}(x_1)\right) \geq 2$ and the inequality is sharp if and only if $\mathbf{x}$ is flex of the curve. However any curve, distinct from the line, cannot have more than $3n(n-2)$ flexes. Thus from (3.61) for a generic direction $\mathbf{p}$ and any point $\mathbf{x} \in \mathcal{X} \cap \mathcal{P}_{\mathbf{p}}\mathcal{X}$ regular on $\mathcal{X}$, we get

$$I_{\mathbf{x}}\left(\mathcal{X}, \mathcal{P}_{\mathbf{p}}\mathcal{X}\right) = I_{\mathbf{x}}\left(\mathcal{X}, \mathbb{V}(x_1)\right) - \mu_{\mathbf{x}}(\mathcal{X}) - 1 = 2 - 0 - 1 = 1. \tag{3.63}$$

Summarizing the previous calculations, we can see that the number of regular points m in the intersection $\mathcal{X}^p \cap \mathcal{P}_{\mathbf{p}}\mathcal{X}^P$ for a generic direction $\mathbf{p}$ is equal to

$$\begin{aligned} m &= \sum_{\mathbf{x}\in \operatorname{Reg}\mathcal{X}^P} I_{\mathbf{x}}(\mathcal{X}^{\mathcal{P}}, \mathcal{P}_{\mathbf{p}}\mathcal{X}^P) = n(n-1) - \sum_{\mathbf{x}\in \operatorname{Sing}\mathcal{X}^P} I_{\mathbf{x}}(\mathcal{X}^{\mathcal{P}}, \mathcal{P}_{\mathbf{p}}\mathcal{X}^P) = \\ & n(n-1) - \sum_{\mathbf{x}\in \operatorname{Sing}\bar{\mathcal{X}}} (2\delta_{\mathbf{x}} + \Delta_{\mathbf{x}}). \end{aligned} \tag{3.64}$$

Using Max Noether's formula (Theorem 2.3) we can substitute for the genus and obtain

$$m = 2(\deg(\mathcal{X}) + \mathrm{g}(\mathcal{X}) - 1) - \sum_{\mathbf{x}\in \operatorname{Sing}\bar{\mathcal{X}}} \Delta_{\mathbf{x}}. \tag{3.65}$$

Finally, any regular point $\mathbf{x}$ on $\mathcal{X}^P \cap \omega$ such that $\omega = T_{\mathbf{x}}\mathcal{X}^P$ lies also on the polar $\mathcal{P}_{\mathbf{p}}\mathcal{X}^P$ for all $\mathbf{p} \in \omega$. Hence, to get the convolution degree we have to subtract from m the intersection multiplicities $I_{\mathbf{x}}\left(\mathcal{X}^P, \mathcal{P}_{\mathbf{p}}\mathcal{X}^P\right)$ for these points. However, this number is just the ω-correction defined above. Thus, we arrive at

$$\kappa_{\mathcal{X}} = m - \Omega_{\mathcal{X}} = 2\left(\deg(\mathcal{X}) + \mathrm{g}(\mathcal{X}) - 1\right) - \sum_{\mathbf{x}\in \operatorname{Sing}\mathcal{X}^P} \Delta_{\mathbf{x}} - \Omega_{\mathcal{X}}, \tag{3.66}$$

which completes the proof. □

Since the multiplicity of an ordinary singularity equals to the number of branches through it, we have an immediate corollary for ordinary curves.

Corollary 3.42. *Let all assumptions of Theorem 3.40 hold and let each singularity of $\mathcal{X}^P$ be ordinary. Then the convolution degree can be computed as*

$$\kappa_{\mathcal{X}} = 2\left(\deg(\mathcal{X}) + \mathrm{g}\left(\mathcal{X}\right) - 1\right) - \Omega_{\mathcal{X}}. \tag{3.67}$$

Example 3.43. Let us compute the convolution degree of all regular conic sections. It is an affine invariant by Remark 3.32 and thus it is enough to compute it only for the parabola, the ellipse and the hyperbola in canonical positions.

All these curves are non-singular, rational and of algebraic degree 2. Hence, their convolution degree is equal to $\kappa_{\mathcal{X}} = 2(2+0-1) - 0 - \Omega_{\mathcal{X}} = 2 - \Omega_{\mathcal{X}}$. Now, consider the parabola $\mathcal{X} = \mathbb{V}(x_2 - x_1^2)$. The homogenization of its defining polynomial is $x_0x_2 - x_1^2$ and thus, by Remark 3.39, we have $\Omega_{\mathcal{X}} = 1$. Hence the convolution degree of the parabola is equal to one. Since both the hyperbola and the ellipse intersect ω in two distinct points, their ω-corrections are zero and thus we get for them that their convolution degree is two.

Moreover, this example shows that the convolution degree cannot be a projective invariant – parabolas and ellipses are projectively equivalent, however of different convolution degrees. Indeed looking at formula (3.57), one can see that all the terms are projectively invariant except the ω-correction. This is caused by the fact that projectivities do not preserve the ideal line from whose position the ω-correction is derived.

Now, let us return back to the hypersurfaces. For a better understanding of convolution hypersurfaces and their properties we will study the projections $\pi_{\mathcal{V}}$ and $\pi_{\mathcal{W}}$ with respect to components of the convolution. Unlike the convolution degree, this new characteristic obviously depends on both input hypersurfaces.

Definition 3.44. Let $\mathcal{X} \subset \mathcal{V} \star \mathcal{W}$ be an irreducible component. Then the integers

$$i_{\mathcal{X}}^{\mathcal{V}} := \deg \pi_{\mathcal{V}}|_{\sigma^{-1}(\mathcal{X})} \quad \text{and} \quad i_{\mathcal{X}}^{\mathcal{W}} := \deg \pi_{\mathcal{W}}|_{\sigma^{-1}(\mathcal{X})} \tag{3.68}$$

are called *indices of the component* $\mathcal{X}$ with respect to the hypersurface $\mathcal{V}$ and $\mathcal{W}$, respectively.

As an immediate corollary of the definitions of the index and the convolution degree we obtain

$$\sum_{j=1}^{\ell} i_{\mathcal{X}_j}^{\mathcal{V}} = \kappa_{\mathcal{W}} \quad \text{and} \quad \sum_{j=1}^{\ell} i_{\mathcal{X}_j}^{\mathcal{W}} = \kappa_{\mathcal{V}}, \tag{3.69}$$

where $\mathcal{V} \star \mathcal{W} = \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_\ell$ is the irreducible decomposition.

Since $\mathcal{V} \star \mathcal{W}$ is generated by the pairs of coherent points on the hypersurfaces $\mathcal{V}$ and $\mathcal{W}$, the statement $\mathcal{X} \subset \mathcal{V} \star \mathcal{W}$ has indices $i_{\mathcal{X}}^{\mathcal{V}}$ and $i_{\mathcal{X}}^{\mathcal{W}}$ means that one has to trace $i^{\mathcal{V}}_{\mathcal{X}}$ times the input hypersurface $\mathcal{V}$, and $i^{\mathcal{W}}_{\mathcal{X}}$ times the input hypersurface $\mathcal{W}$ to obtain all points of the irreducible component $\mathcal{X}$. This obvious fact and the fundamental property of convolutions (cf. Lemma 3.7) immediately imply the following lemma which is given without proof.

Lemma 3.45. *Let $\mathcal{X} \subset \mathcal{V} \star \mathcal{W}$ be a k-special component, where by 1-special we mean a simple component. Then*

$$\kappa_{\mathcal{X}} = \frac{i^{\mathcal{V}}_{\mathcal{X}} \cdot \kappa_{\mathcal{V}}}{k} = \frac{i^{\mathcal{W}}_{\mathcal{X}} \cdot \kappa_{\mathcal{W}}}{k}. \tag{3.70}$$

Using this lemma we can show that the convolution degrees of the input hypersurface give us an upper bound for the number of components of the convolution.

Theorem 3.46. *$\mathcal{V} \star \mathcal{W}$ has at most* $\mathrm{GCD}(\kappa_{\mathcal{V}}, \kappa_{\mathcal{W}})$ *irreducible components.*

Proof. We prove only the case when $\mathcal{V} \star \mathcal{W}$ does not have a degenerated component. Otherwise, Theorem 3.58 can be applied.

Let $\mathcal{V} \star \mathcal{W} = \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_\ell$ be the irreducible decomposition into ℓ components, where $\mathcal{X}_j$ is k_j-special. By Lemma 3.45, we may write

$$\sum_{j=1}^{\ell} k_j \cdot \kappa_{\mathcal{X}_j} = \kappa_{\mathcal{V}} \cdot \sum_{j=1}^{\ell} i^{\mathcal{V}}_{\mathcal{X}_j} = \kappa_{\mathcal{V}} \cdot \kappa_{\mathcal{W}}. \tag{3.71}$$

Furthermore, by the same lemma we have $\kappa_{\mathcal{V}} | k_j \cdot \kappa_{\mathcal{X}_j}$ and $\kappa_{\mathcal{W}} | k_j \cdot \kappa_{\mathcal{X}_j}$ which implies $k_j \cdot \kappa_{\mathcal{X}_j} = \mu_j \cdot \mathrm{LCM}(\kappa_{\mathcal{V}}, \kappa_{\mathcal{W}})$, where μ_j is a non-zero natural number. Next, (3.71) can be rewritten

$$\sum_{j=1}^{\ell} \mu_j \cdot \mathrm{LCM}(\kappa_{\mathcal{V}}, \kappa_{\mathcal{W}}) = \kappa_{\mathcal{V}} \cdot \kappa_{\mathcal{W}} = \mathrm{GCD}(\kappa_{\mathcal{V}}, \kappa_{\mathcal{W}}) \cdot \mathrm{LCM}(\kappa_{\mathcal{V}}, \kappa_{\mathcal{W}}) \tag{3.72}$$

and we arrive at $\ell \le \sum_{j=1}^{\ell} \mu_j = \mathrm{GCD}(\kappa_{\mathcal{V}}, \kappa_{\mathcal{W}})$, which completes the proof. □

Corollary 3.47. *Let be given a hypersurface $\mathcal{V}$. Then for any hypersurface $\mathcal{W}$, the convolution $\mathcal{V} \star \mathcal{W}$ cannot have more than $\kappa_{\mathcal{V}}$ components. Moreover, if the number of components is equal to $\kappa_{\mathcal{V}}$ then every simple component is birationally equivalent to $\mathcal{W}$.*

Proof. The first part of the proof is obvious. To prove the second part, we consider the irreducible decomposition $\mathcal{V} \star \mathcal{W} = \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_{\kappa_{\mathcal{V}}}$ and denote $\mathcal{Y}_j := \sigma^{-1}(\mathcal{X}_j)$ the preimages of the components in the mapping σ. Since $\sum_{j=1}^{\kappa_{\mathcal{V}}} i^{\mathcal{W}}_{\mathcal{X}_j} = \kappa_{\mathcal{V}}$, we see that $i^{\mathcal{W}}_{\mathcal{X}_j} = 1$ for any j. Hence the mappings $\varphi_j = \pi_{\mathcal{W}}|_{\mathcal{Y}_j} : \mathcal{Y}_j \to \mathcal{W}$ are birational. If the component $\mathcal{X}_j$ is simple then, by Lemma 3.24, the mapping $\psi_j = \sigma|_{\mathcal{Y}_j} : \mathcal{Y}_j \to \mathcal{X}_j$ is birational, too. Hence the composed mapping $\varphi_j^{-1} \circ \psi_j$ is birational mapping from $\mathcal{W}$ to $\mathcal{X}_j$. □

Corollary 3.48. *If* $\mathrm{GCD}(\kappa_{\mathcal{V}}, \kappa_{\mathcal{W}}) = 1$ *then* $\mathcal{V} \star \mathcal{W}$ *is irreducible.*

Remark 3.49. Let us emphasize that the converse statement of Corollary 3.48 does not hold. For instance, we can consider a nodal cubic $\mathcal{W}$ in Weierstrass's form, i.e., a curve given by the equation

$$x_2^2 - a x_1^2 (x_1 - b) = 0. \tag{3.73}$$

Any such curve with $a, b \neq 0$ is a rational curve of degree 3 with one ordinary double node at the origin. Next, one can compute $\Omega_{\mathcal{X}} = 2$ an hence we obtain $\kappa_{\mathcal{X}} = 2(3 + 0 - 1) - 0 - 2 = 2$. However, the convolution with the unit circle is reducible only for $a = 1/9$ and $b = -3$, which is the only cubic curve, up to a similarity, (called Tschirhausen cubic) admitting a polynomial PH parameterization, cf. Farouki and Sakkalis (1990).

Let us consider the set of nodal cubics in the Weierstrass form parameterized by assigning to each $(a, b) \in \mathbb{C}\backslash\{0\} \times \backslash\mathbb{C}\{0\}$ the curve (3.73). Then by previous remark the set of points (a, b) such that the convolution of the associated cubic with circle is closed – because it is finite. Therefore we may rephrase it such that the convolution of cubic in Weierstrass form with a circle is generically irreducible. This observation can be generalized. Our proof is motivated by the proof of an analogous result on conchoids of algebraic curves, cf. Albano and Roggero (2010), where one of Bertini's theorems was used, see e.g. Lazarsfeld (2004, Theorem 3.3.1, p. 207).

Theorem 3.50. (Bertini's theorem for generic linear sections) *Let* $\mathcal{X}$ *be an irreducible variety and* $f : \mathcal{X} \to P^r\mathbb{C}$ *a morphism. Fix an integer* $d < \dim \mathrm{cl} f(\mathcal{X})$. *If* $\mathcal{L} \subset P^\ell\mathbb{C}$ *is a generic* $(r - d)$*-plane, then* $f^{-1}(\mathcal{L})$ *is irreducible.*

Theorem 3.51. *For generic hypersurfaces* $\mathcal{V}, \mathcal{W} \subset \mathbb{C}^n$ *the convolution* $\mathcal{V} \star \mathcal{W}$ *is irreducible.*

Proof. Since $\mathcal{V} \subset \mathbb{C}^n$ is generic we may assume w.l.o.g that the dual hypersurface $\mathcal{V}^\vee \subset P^n\mathbb{C}$ is irreducible and it does not pass through the point $\mathbf{s} = (0 : \cdots : 0 : 1)$. Then, following the notation introduced in Subsection 3.1.3, let $\nu : P^n\mathbb{C} \to P^{n-1}\mathbb{C}$ be the projection onto the first n coordinates. Consider the set

$$\mathcal{I}_{\mathcal{V}}^\vee = \mathrm{cl}\left(\mathcal{V}^\vee \times_{P^{n-1}\mathbb{C}} P^n\mathbb{C}\backslash\{\mathbf{s}\}\right) \tag{3.74}$$

and let us denote by $\pi_1 : \mathcal{I}_{\mathcal{V}}^\vee \to \mathcal{V}^\vee$ and $\pi_2 : \mathcal{I}_{\mathcal{V}}^\vee \to P^n\mathbb{C}$ two natural projections. If we look at the fibres of projection π_2, we find out that they are finite – more precisely $\sharp\{\pi_2^{-1}(\mathbf{p})\} = \kappa_{\mathcal{V}}$ for a generic $\mathbf{p}$. Thus π_2 is dominant, finite mapping and we arrive at $\dim \mathcal{I}_{\mathcal{V}}^\vee = \dim P^n\mathbb{C} = n$. The fibres of π_1 are not finite. Any point $\mathbf{v} \in \mathcal{V}^\vee$ can be written in the form $\mathbf{v} = (n_1 : \cdots : n_n : 1)$. Then the fibre $\pi_1^{-1}(\mathbf{v})$ is the line parameterized by

$$\alpha(n_1 : \cdots : n_n : 0) + \beta(0 : \cdots : 0 : 1), \tag{3.75}$$

where not both $\alpha, \beta \in \mathbb{C}$ equal to zero, and thus the fibres are irreducible. Therefore using the irreducibility of $\mathcal{V}^\vee$ we deduce that $\mathcal{I}_\mathcal{V}^\vee$ must be irreducible, too.

$\mathcal{I}_\mathcal{V}^\vee$ may be seen as a kind of a dual incidence variety with the unknown hypersurface $\mathcal{W}^\vee$. Indeed, comparing (3.74) with (3.41) it is easy to see that

$$\mathcal{I}^\vee(\mathcal{V}^\vee, \mathcal{W}^\vee) = \pi_2^{-1}(\mathcal{W}^\vee), \tag{3.76}$$

for a hypersurface $\mathcal{W}^\vee$. Since $\mathcal{V} \star \mathcal{W}$ is irreducible if and only if $(\mathcal{V} \star \mathcal{W})^\vee$ is, we deduce from (3.76) and Lemma 3.23 that the irreducibility of $\pi_2^{-1}(\mathcal{W}^\vee)$ implies the irreducibility of $\mathcal{V} \star \mathcal{W}$. (This is because the image of an irreducible variety under a rational mapping cannot be reducible.)

If m stays for the degree of $\mathcal{W}^\vee$, then we denote by $\phi_m : P^n\mathbb{C} \to P^\ell\mathbb{C}$ the m-tuple Veronesse embedding. Now for a hypersurface $\mathcal{W}^\vee$ of degree m, the image of $\pi_2^{-1}(\mathcal{W}^\vee)$ under

$$\phi_m \circ \pi_2 : \mathcal{I}_\mathcal{V}^\vee \to P^\ell\mathbb{C} \tag{3.77}$$

is the section of $\phi_m \circ \pi_2(\mathcal{I}_\mathcal{V}^\vee)$ by some $(r-1)$-plane $\mathcal{L}$. Since $1 < 2 \leq \dim \mathrm{cl}\left(\phi_m \circ \pi_2(\mathcal{I}_\mathcal{V}^\vee)\right)$ we can use Theorem 3.50 to obtain that $\pi_2^{-1}(\mathcal{W}^\vee)$ is generically irreducible. This completes the proof. □

To conclude this subsection we establish the relation between convolution degrees, indices of components and coherent mappings.

Lemma 3.52. *Let $\xi_{\mathcal{U},\mathcal{V}} : \mathcal{U} \to \mathcal{V}$ be a coherent mapping. Then* $\deg \xi_{\mathcal{U},\mathcal{V}} = \kappa_\mathcal{U}/\kappa_\mathcal{V}$.

Proof. Let $H \subset \mathbb{C}^n$ be a generic $(n-1)$-space. Then the preimage of $\mathcal{V}_H$ under $\xi_{\mathcal{U},\mathcal{V}}$ is the set $\mathcal{U}_H$. But $\sharp\{\mathcal{V}_H\} = \kappa_\mathcal{V}$, $\sharp\{\mathcal{U}_H\} = \kappa_\mathcal{U}$, and hence the degree of $\xi_{\mathcal{U},\mathcal{V}}$ has to be $\kappa_\mathcal{U}/\kappa_\mathcal{V}$. □

Lemma 3.53. *There exists a correspondence between coherent mappings $\xi_{\mathcal{V},\mathcal{W}} : \mathcal{V} \to \mathcal{W}$ and components $\mathcal{X} \subset \mathcal{V} \star \mathcal{W}$ such that $i_\mathcal{X}^\mathcal{W} = 1$.*

Proof. For a coherent mapping $\xi_{\mathcal{V},\mathcal{W}}$, define $\phi : \mathbf{v} \mapsto (\mathbf{v}, \xi_{\mathcal{V},\mathcal{W}}(\mathbf{v}))$. Obviously the closure of the image of $\mathcal{V}$ under ϕ is a component of $\mathcal{I}(\mathcal{V}, \mathcal{W})$, say $\mathcal{Y}$. Then $\mathcal{X} = \sigma(\mathcal{Y})$ is the component of the convolution whose index with respect to $\mathcal{V}$ is equal to one.

Contrary if $\mathcal{X} \subset \mathcal{V} \star \mathcal{W}$ is a component such that $i_\mathcal{X}^\mathcal{V}$, then the projection $\pi_\mathcal{V}|_{\sigma^{-1}(\mathcal{X})}$ from incidence variety is birational and thus there exists its inverse. Then, by the definition of incidence variety, it is obvious that the composed mapping $\pi_\mathcal{W} \circ (\pi_\mathcal{V}|_{\sigma^{-1}(\mathcal{X})})^{-1} : \mathcal{V} \to \mathcal{W}$ is coherent. □

Despite Theorem 3.51, the previous lemma immediately implies that for almost every hypersurface there exists another hypersurface such that their convolution is reducible.

Corollary 3.54. *If $\kappa_{\mathcal{V}} > 1$ then $\mathcal{V} \star \mathcal{V}$ is reducible.*

Proof. The identity mapping $\xi_{\mathcal{V},\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ is coherent and thus by the previous lemma there exists a component $\mathcal{X}$ of index one. On the other hand if $\kappa_{\mathcal{V}} > 1$ then so is the sum of indices of the component and $\mathcal{V} \star \mathcal{V}$ has to be reducible. □

3.2.3 Convolutions containing special and degenerated components

Although we are mainly interested in simple components (as for them the mapping σ is birational), at least a short analysis of special and degenerated components is necessary to have a better insight into the properties of convolution hypersurfaces.

The degenerated and special components of offsets to algebraic curves were introduced and studied in Sendra and Sendra (2000). In particular it was shown that $\mathcal{X}$ is always a component of $\mathcal{O}_\delta(\mathcal{O}_\delta(\mathcal{X}))$ and the conditions under which $\mathcal{X}$ is degenerated or special component of an offset to $\mathcal{O}_\delta(\mathcal{X})$ were identified. Moreover it was proved that all such components arise in this way.

If $\mathcal{X}$ is a degenerated component then by Corollary 3.25 $\dim \mathcal{X} < n - 1$. In this case we understand under its offset $\mathcal{O}_\delta(\mathcal{X})$ just the convolution with a sphere, i.e., $\mathcal{X} \star \mathcal{S}_\delta^{n-1}$. It is not hard to see that this is an irreducible hypersurface. For example in the surface case, it is a sphere or a pipe surface in dependence on the dimension of $\mathcal{X}$, cf. Fig 3.2.

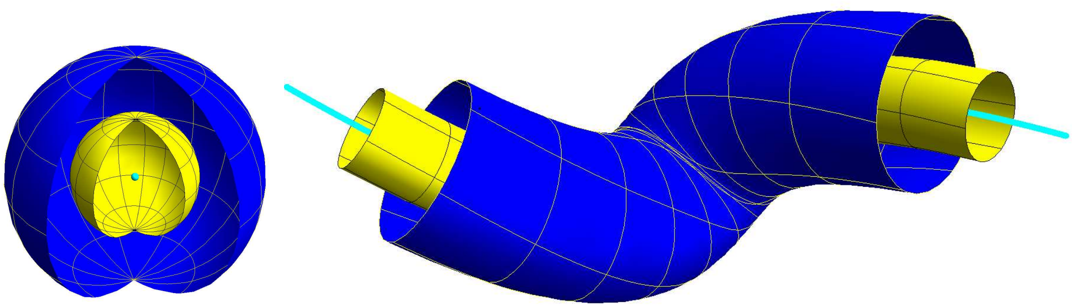


Figure 3.2: Degenerated components of a convolution with sphere, i.e., offset. Left: A component of the offset to a sphere degenerates to a point. Right: A component of an offset to a pipe surface degenerates to its spine curve.

The extension of the results of Sendra and Sendra (2000) for convolutions is straightforward except for one difference. If $\mathcal{V}$ is a more complicated hypersurface then the sphere and $\mathcal{X}$ is a variety of greater codimension, then the convolution $\mathcal{X} \star \mathcal{V}$ need not to be an irreducible hypersurface, as the following example shows.

Example 3.55. Let $\mathcal{V} \subset \mathbb{C}^3$ be a torus given by

$$f(x,y,z) = (x^2 + y^2 + z^2 + R^2 - r^2)^2 - 4R^2(x^2 + y^2) = 0 \tag{3.78}$$

and $\mathcal{X}$ be a curve lying in the plane $z = 0$. If $\mathbf{p} \in \mathcal{X}_{\text{Reg}}$, then the tangent line $T_{\mathbf{p}}\mathcal{X}$ is determined by some vector $\mathbf{a} = (a_1, a_2, 0)$. The set of points on the torus $\mathcal{V}$ coherent with $\mathbf{p}$ is given as the solution of the system $f = 0$ and $\mathbf{a} \cdot \nabla f = 0$ After expressing the second equation

$$4(a_1 x + a_2 y)(x^2 + y^2 + z^2 - R^2 - r^2), \tag{3.79}$$

we figure out that the solution consists of four components. The first two are circles $\mathcal{C}_1$, $\mathcal{C}_2$ obtained as the intersection of the torus with the hyperplane $a_1 x + a_2 y = 0$, while the remaining two components are circles $\mathcal{D}_1$ and $\mathcal{D}_2$ in the intersections of $\mathcal{V}$ with hyperplanes $z = \pm r$ and as such they do not depend on the vector $\mathbf{a}$, cf. Fig. 3.3.

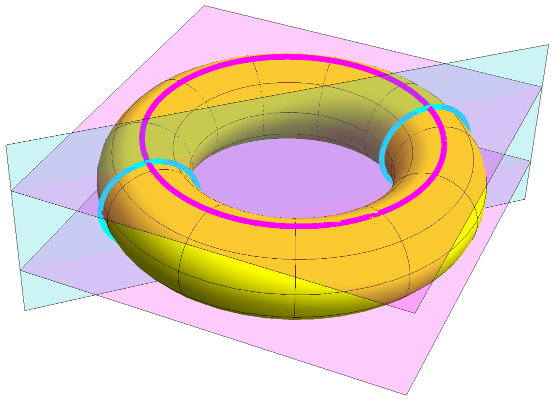


Figure 3.3: The points on the torus which tangent planes contain the fixed vector $(a_1, a_2, 0)$. The circles $\mathcal{C}_1$, $\mathcal{C}_2$ are colored by cyan, while the intersection of $\mathcal{V}$ with the planes $z = \pm r$ are in magenta (there is another magenta circle on the bottom of the torus).

Thus the planes $z = \pm r$ are obviously two components of the $\mathcal{X} \star \mathcal{V}$, as they are obtained by sweeping $\mathcal{D}_1$ and $\mathcal{D}_2$ along the curve $\mathcal{X}$. Moreover, under some conditions on $\mathcal{X}$, it may be shown that the rest of the convolution has two components (roughly speaking these correspond to the two circles $\mathcal{C}_i$).

The components $z = \pm r$ from the previous example are somehow specific for convolutions in higher codimensions. Let $\mathcal{V}$ be a hypersurface and $\mathcal{X}$ a variety. If it happens that the set

$$\mathcal{V}_{\mathcal{X}} := \{\mathbf{v} \in \mathcal{V} \mid \forall \mathbf{x} \in \mathcal{X}_{\mathrm{Reg}} : (\mathbf{v}, \mathcal{V}) \sim_{\star} (\mathbf{x}, \mathcal{X})\} \tag{3.80}$$

is nonempty then the set $\mathcal{V}_{\mathcal{X}} + \mathcal{X} = \{\mathbf{v} + \mathbf{x}\}$ for all $\mathbf{v} \in \mathcal{V}_{\mathcal{X}}$ and $\mathbf{x} \in \mathcal{X}$ is a component of $\mathcal{V} \star \mathcal{X}$. Let us call such a component *non-ordinary*. Obviously if $\mathcal{X}$ was a hypersurface, then the convolution cannot have a such a kind of component.

Definition 3.56. Let $X \subset \mathbb{C}^n$ be an arbitrary set. Then we denote by X^- the set centrally symmetric with X, i.e.,

$$X^- := \{\mathbf{x} \in \mathbb{C}^n \mid -\mathbf{x} \in X\}. \tag{3.81}$$

Now we are ready to prove the analogy of the statement that a variety is a component of the δ-offset to its δ-offset.

Lemma 3.57. *If $\mathcal{U} \subset \mathcal{V} \star \mathcal{W}$ is not a non-ordinary component then $\mathcal{W} \subset \mathcal{U} \star \mathcal{V}^-$ is an ordinary component, too.*

Proof. A generic point $\mathbf{p} \in \mathcal{U}$ can be written in the form $\mathbf{v} + \mathbf{w}$ for coherent points $\mathbf{v} \in \mathcal{V}$, $\mathbf{w} \in \mathcal{W}$. Contrary, under the assumption that $\mathcal{U}$ is not non-ordinary, for generic $\mathbf{w} \in \mathcal{W}$ there exists a coherent $\mathbf{v} \in \mathcal{V}$ such that $\mathbf{v} + \mathbf{w} \in \mathcal{U}$.

Now, let us start with a generic point $\mathbf{w} \in \mathcal{W}$. By the above consideration we may find $\mathbf{u} = \mathbf{v} + \mathbf{w} \in \mathcal{U}$. Invoking Lemma 3.7, we arrive at $(\mathbf{u}, \mathcal{U}) \sim_{\star} (\mathbf{v}, \mathcal{V})$. It follows that $\mathbf{u}$ is also coherent to $-\mathbf{v} \in \mathcal{V}^-$, because $T_{\mathbf{v}}\mathcal{V} = T_{-\mathbf{v}}\mathcal{V}^-$. Hence $\mathbf{w} = \mathbf{u} + (-\mathbf{v}) \in \mathcal{U} \star \mathcal{V}^-$ and it follows immediately that $\mathcal{W}$ is not non-ordinary. □

Now, we are ready to formulate the conditions under which a component of convolution is degenerated or special.

Theorem 3.58. *Let $\mathcal{V}$ and $\mathcal{W}$ be two hypersurfaces. Then $\mathcal{V} \star \mathcal{W}$ contains a degenerated component $\mathcal{X}$ if and only if $\mathcal{W} \subset \mathcal{X} \star \mathcal{V}^-$ is not non-ordinary.*

Proof. It is an immediate consequence of the preceding lemma. □

Theorem 3.59. *Let $\mathcal{V}$, $\mathcal{W} \subset \mathbb{C}^n$ be algebraic hypersurfaces. Then $\mathcal{X} \subset \mathcal{V} \star \mathcal{W}$ is k-special if and only if $\mathcal{V} \subset \mathcal{X} \star \mathcal{W}^-$ and $i^{\mathcal{X}}_{\mathcal{V}} = k$.*

Proof. To prove the theorem, we have to show that $\mathcal{X}$ is k-special if and only if the degree of the projection $\pi_{\mathcal{X}}|_{\mathcal{U}}$, cf. the following diagram, is k.

$$\begin{array}{ccccc} \mathcal{W}^- & \xleftarrow{\pi_{\mathcal{W}^-}} & \pi_1^{-1}(\mathcal{V}) = \mathcal{U} \subset \mathcal{I}(\mathcal{X}, \mathcal{W}^-) & \xrightarrow{\pi_{\mathcal{X}}} & \mathcal{X} \\ & & \Big\downarrow \sigma & & \\ & & \mathcal{V} \subset \mathcal{X} \star \mathcal{W}^- & & \end{array} \tag{3.82}$$

The component $\mathcal{X}$ is k-special if and only if for generic $\mathbf{x} \in \mathcal{X}$ it holds $\mathbf{x} = \mathbf{v}_i + \mathbf{w}_i$ for $\{(\mathbf{v}_1, \mathbf{w}_i)\}_{i=1}^k \subset \mathcal{I}(\mathcal{V}, \mathcal{W})$. Since $(\mathbf{x}, \mathcal{X}) \sim_\star (\mathbf{w}_i, \mathcal{W})$, by Lemma 3.7, we can write equivalently $(\mathbf{x}, -\mathbf{w}_i) \in \mathcal{I}(\mathcal{X}, \mathcal{W}^-)$ and moreover all these points lies on $\sigma^{-1}(\mathcal{V})$ as $\sigma(\mathbf{x}, -\mathbf{w}_i) = \mathbf{v}_i \in \mathcal{V}$. The generic choice of $\mathbf{x}$ implies that the degree $\deg \pi_{\mathcal{X}}$ is equal to the $\sharp\{\pi_{\mathcal{X}}^{-1}(\mathbf{x})\} = \sharp\{(\mathbf{x}, -\mathbf{w_i})\}_{i=1}^k = k$. □

To conclude the part devoted to special and degenerated component, we prove that the convolution cannot consists of special componets only. Let us note that analogical statement does not hold for degenerated components. For example if $\mathcal{V}$ is a hypersurface with $\kappa_{\mathcal{V}} = 1$ then $\mathcal{V} \star \mathcal{V}^-$ is irreducible by Theorem 3.46 and thus Theorem 3.58 says that its only component is degenerate.

Theorem 3.60. *For any two hypersurfaces $\mathcal{V}, \mathcal{W}$, the convolution $\mathcal{V} \star \mathcal{W}$ has at least one non-special component.*

Proof. Let be given a generic $\mathbf{n} \in \mathbb{C}^n \backslash \{\mathbf{o}\}$. Then there exist $\kappa_{\mathcal{V}}$ distinct points on $\mathcal{V}^\vee$ with coordinates $(\mathbf{n} : h_1), \ldots, (\mathbf{n} : h_{\kappa_{\mathcal{V}}})$ for some $h_i \in \mathbb{C}$ and similarly on $\mathcal{W}^\vee$ lie $\kappa_{\mathcal{W}}$ distinct points $(\mathbf{n} : g_1), \ldots, (\mathbf{n} : g_{\kappa_{\mathcal{W}}})$. From the dual interpretation of convolution we see that

$$\{(\mathbf{n} : h_i + g_j)\}_{i,j=1}^{\kappa_{\mathcal{V}}, \kappa_{\mathcal{W}}} \tag{3.83}$$

are points on $(\mathcal{V} \star \mathcal{W})^\vee$. Moreover $\mathcal{V} \star \mathcal{W}$ has only special components if and only if for each i, j there exist $i' \neq i$ and $j' \neq j$ such that $h_i + g_j = h_{i'} + g_{j'}$. To see that this is not possible, we may w.l.o.g. assume that h_i's are ordered such that $\Re(h_1) \geq \cdots \geq \Re(h_{\kappa_{\mathcal{V}}})$ and analogously $\Re(g_1) \geq \cdots \geq \Re g_(\kappa_{\mathcal{W}})$, where $\Re(a)$ stands for the real part of complex number a. If $\Re(h_1) > \Re(h_2)$ and $\Re(g_1) > \Re(g_2)$ we arrive at $\Re(h_1 + g_1) > \Re(h_i + g_j)$ for all $i, j \neq 1$ and hence $h_1 + g_1 \neq h_i + g_j$. In the case when the inequalities are not strict we may choose another basis of $\mathbb{C}$ (considered as two-dimensional vector space $\mathbb{R}^2$) to ensure this.

□

Chapter 4

Hypersurfaces of low convolution degree

We begin with the study of hypersurfaces of convolution degree one. Despite showing that they are rational, we will identify them with the well known class of LN hypersurfaces. Their properties with respect to convolutions are listed and the decomposition of LN curves into the simple ones is shown. We conclude by a brief review of the methods of approximations of convolutions with the help of LN curves.
The second part is devoted to the study of hypersurfaces of convolution degree two. After summarizing their properties with respect to convolution, the interesting class – the so-called QN hypersurfaces will be identified. Afterwards the detailed analysis of convolutions with QN hypersurfaces is accomplished and for the curve case the genus formula is presented. We conclude by providing a decomposition of QN curves into simple ones.

4.1 Hypersurfaces of convolution degree one

Clearly, the most simple hypersurfaces with respect to the operation of convolution are the hypersurfaces with convolution degree one. In this section we specialize the general results from the previous chapter to provide a full algebraic analysis of corresponding convolution hypersurfaces with any arbitrary hypersurface (convolution degree, number and types of components). These results for the curve case can be found in Vršek and Lávička (2010a).

Moreover we show that all hypersurfaces with convolution degree one are rational and in addition they coincide exactly with the well-known class of LN hypersurfaces. This observation relates our work with the results e.g. from Jüttler (1998); Sampoli et al. (2006); Lávička and Bastl (2007).

We conclude this section by providing a decomposition of curves with convolution degree one into the convolution of a finite number of suitable fundamental curves and by a short discussion of methods used in the approximation of convolutions.

4.1.1 Introduction and elementary properties

Assumption 4.1. Throughout this section we will assume that $\mathcal{V}$ has the convolution degree one.

Example 4.2. By a hyperparaboloid, we mean a variety in $\mathbb{C}^n$ given by the equation

$$\mathcal{P}^{n-1} : f(\mathbf{x}) = x_1^2 + \cdots + x_{n-1}^2 - x_n = 0. \tag{4.1}$$

Let $H \subset \mathbb{C}^n$ be a generic $(n-1)$-space identified with $\mathbf{a} \cdot \mathbf{x} = 0$ for some $\mathbf{a} = (a_1, \ldots, a_n)$. Then, by Definition 3.30, $\kappa_{\mathcal{P}^{n-1}}$ equals $\sharp\{\mathcal{P}_H^{n-1}\}$. A point $\mathbf{p}$ is in $\mathcal{P}_H^{n-1}$ if and only if it is a solution of the equation

$$\nabla f(\mathbf{x}) = (2x_1, \ldots, 2x_{n-1}, -1) = \lambda(a_1, \ldots, a_n). \tag{4.2}$$

where λ is assumed to be non-zero. Obviously for $a_n \neq 0$ we obtain exactly one solution while for $a_n = 0$ there is no solution. Since $a_n \neq 0$ is Zariski open subset of the set of all vectors $\mathbf{a}$, there is a one solution for a generic $\mathbf{a} \in \mathbb{C}^n$ and hence the convolution degree of $\mathcal{P}^{n-1}$ is equal to one.

In what follows, we would like to show that hypersurfaces with convolution degree one are always rational, and moreover they possess the well known LN parameterizations. Let us start with the following lemma.

Lemma 4.3. *Let $\mathcal{V}$ be a hypersurface with $\kappa_{\mathcal{V}} = 1$ and $\mathcal{U}$ an arbitrary hypersurface. Then there exists a coherent mapping $\xi_{\mathcal{U},\mathcal{V}} : \mathcal{U} \to \mathcal{V}$.*

Proof. Consider the Gauss mappings $\gamma_{\mathcal{V}} : \mathcal{V} \to \mathrm{Gr}^0(n-1, n)$ and $\gamma_{\mathcal{U}} : \mathcal{U} \to \mathrm{Gr}^0(n-1, n)$. Since $\kappa_{\mathcal{V}} = 1$, we see by Proposition 3.31 that $\deg \gamma_{\mathcal{V}} = 1$ and hence it is birational. Thus we obtain the rational mapping $\xi_{\mathcal{U},\mathcal{V}} = \gamma_{\mathcal{V}}^{-1} \circ \gamma_{\mathcal{U}} : \mathcal{U} \to \mathcal{V}$. From the construction of this mapping it follows that $\xi_{\mathcal{U},\mathcal{V}}(\mathbf{u}) = \{\mathbf{v} \in \mathcal{V} \mid T_{\mathbf{v}}\mathcal{V} = T_{\mathbf{u}}\mathcal{U}\}$, which means that points $\mathbf{u} \in \mathcal{U}$ and $\xi_{\mathcal{U},\mathcal{V}}(\mathbf{u}) \in \mathcal{V}$ are coherent and so is the mapping. □

Corollary 4.4. *An arbitrary hypersurface with convolution degree equal to one is rational.*

Proof. Let $\mathcal{V}$ be such a hypersurface. Then by the previous lemma there exists a coherent mapping $\zeta_{\mathcal{P}^{n-1},\mathcal{V}} : \mathcal{P}^{n-1} \to \mathcal{V}$. Since for the hyperparaboloid $\mathcal{P}^{n-1}$ it holds $\kappa_{\mathcal{P}^{n-1}} = 1$ too, the coherent mapping is birational by Lemma 3.52. Moreover, $\mathcal{P}^{n-1}$ is rational hypersurface, as it admits a proper parameterization

$$\mathbf{p}(s) = (s_1, \dots, s_{n-1}, s_1^2 + \cdots s_{n-1}^2), \tag{4.3}$$

and thus $\mathcal{V}$ is rational, too. □

Consider the normal vector field associated to the parameterization (4.3) from the previous proof. It can be expressed, with the help of (3.17), as

$$\mathbf{n}_{\mathbf{p}}(s) = \det \left(\begin{array}{cccc|c} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_{n-1} & \mathbf{e}_n \\ \hline 1 & 0 & \dots & 0 & 2s_1 \\ 0 & 1 & \dots & 0 & 2s_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 2s_{n-1} \end{array} \right) = (-1)^n \big(2s_1, \dots, 2s_{n-1}, -1\big). \tag{4.4}$$

Therefore hyperparaboloids admit parameterizations with a remarkable normal vector field – expressed by linear functions. Curves and surfaces with this property was deeply studied in recent years because of their nice geometric properties. The original definition is due to Jüttler (1998).

Definition 4.5. Let $\mathcal{V}$ be a hypersurface and $\mathbf{v}(s) : \mathbb{C}^{n-1} \to \mathcal{V}$ its parameterization. We say that $\mathbf{v}(s)$ has LN property (where LN stands for *linear normals*) or it is an LN parameterization if

$$\mathbf{n}_{\mathbf{v}}(s) = \mathbf{p}_0 + \sum_{i=1}^{n-1} \mathbf{p}_i s_i, \tag{4.5}$$

where $\mathbf{p}_0, \dots \mathbf{p}_{n-1}$ are linearly independent vectors in $\mathbb{C}^n$. A hypersurface is called an *LN hypersurface* if it admits an LN parameterization.

Remark 4.6. In some papers, the vectors $\mathbf{p}_0, \dots \mathbf{p}_{n-1}$ are not assumed to be linearly independent. Although such a hypersurface admits a linear normal vector field, its Gauss image is degenerated, and hence we omit them from our further considerations.

Obviously any LN hypersurface has its convolution degree one. Indeed a generic $\mathbf{a} \in \mathbb{C}^n$ is not a linear combination of vectors $\mathbf{p}_i$, $i = 1, \dots, n-1$ and thus the system of linear equations $\lambda \mathbf{a} = \mathbf{n}_{\mathbf{v}}(s)$ has the unique solution.

Contrariwise, if $\mathcal{V}$ is a curve and $(n_1(s), n_2(s))$ the normal vector field associated to a proper parameterization, then by Remark 3.36 it holds $\kappa_{\mathcal{V}} = \max\{\deg n_1(s), \deg n_2(s)\} = 1$. Thus, each curve with the convolution degree one is immediately an LN curve.

The situation becomes slightly more complicated whenever one deals with varieties of dimension greater then one. To illustrate this, consider a surface $\mathcal{Q} \subset \mathbb{C}^3$ parameterized by

$$\mathbf{q}(s) = (s_1^2, s_2^2, s_1^2 + s_2^2). \tag{4.6}$$

The computation reveals that $\mathbf{n_q}(s) = (-s_1, -s_2, 2s_1s_2)$ and thus even though $\kappa_{\mathcal{Q}} = 1$ and $\mathbf{q}(s)$ is proper, the normal vector field is quadratic. On the other hand it was shown in Peternell and Odehnal (2008) that after reparameterization $t_1 = -1/2s_1$, and $t_2 = -1/2s_2$ one obtain a normal field $(t_1, t_2, 1)$.

This observation can be generalized for an arbitrary hypersurface of convolution degree one. Although the proof which we give here is not the simplest possible one, its advantage is, that it is constructive – i.e., it gives the method how to reparameterize hypersurfaces with convolution degree one to obtain their LN parameterizations.

Theorem 4.7. *$\mathcal{V}$ is an LN hypersurface if and only if $\kappa_{\mathcal{V}} = 1$.*

Proof. As mentioned above the if part is obvious. Contrary, let $\mathcal{V}$ be an LN hypersurface, then it is by Corollary 4.4 rational, and thus there exists a proper parameterization $\mathbf{u}(t) : \mathbb{C}^{n-1} \to \mathcal{V}$. Let $\mathbf{n_u}(t)$ be the associated normal vector field and let us denote $h(t) = \mathbf{n}(t) \cdot \mathbf{u}(t)$, i.e., $\mathcal{V}$ is an envelope of the family of hyperplanes parameterized by $\mathbf{n_u}(t) \cdot \mathbf{x} = h(t)$. Now, fix linearly independent vectors $\mathbf{p}_0, \ldots \mathbf{p}_{n-1}$ and set

$$\mathbf{n_v}(s) = \mathbf{p}_0 + \sum_{i=1}^{n-1} \mathbf{p}_i s_i. \tag{4.7}$$

Since $\mathbf{u}(t)$ is the proper parameterization of LN hypersurface, the equation

$$\mathbf{n_u}(t) = \lambda \mathbf{n_v}(s_0) \tag{4.8}$$

has exactly one solution for a generic $s_0 \in \mathbb{C}^{n-1}$, and thus we may express t depending rationally on s, such that $\mathbf{n_u}(t(s)) = \lambda(s)\, \mathbf{n_v}(s)$. Then the family of hyperplanes

$$\lambda(s)\, \mathbf{n_v}(s) \cdot \mathbf{x} = \lambda(s)\, h(t(s)) \tag{4.9}$$

describes $\mathcal{V}$ as their envelope. Applying (2.12) on the system (4.9) we arrive at the parameterization $\mathbf{v}(s)$ of the hypersurface $\mathcal{V}$ with the associated normal field $\mathbf{n_v}(s)$. □

It is seen from the above proof that a hyperplane parameterized by the normal vector field $\mathbf{n_v}(s)$ can by chosen almost arbitrarily – however it must not pass through the origin. Hence, any LN surface admits a parameterization $\mathbf{v}(s)$ such that

$$\mathbf{n_v}(s) = (s_1, \ldots, s_{n-1}, 1). \tag{4.10}$$

It follows that a parameterization of an arbitrary LN hypersurface $\mathcal{V}$ can be obtained just by choosing $h \in \mathbb{C}(s)$ and the subsequent computing of the envelope of the system $\mathbf{n_v}(s) \cdot \mathbf{x} = h(s)$. This leads to the parameterizations of the form (see Sampoli et al. (2006) for surfaces)

$$\mathbf{v}(s) = \left(\frac{\partial h}{\partial s_1}, \ldots \frac{\partial h}{\partial s_{n-1}}, \sum_{i=1}^{n-1} \frac{\partial h}{\partial s_i} s_i - h \right). \tag{4.11}$$

Let us return back to parameterization (4.6). The quadratic polynomial parameterizations are well explored, see e.g. Jörg and Reif (1998). Recently, these surfaces were related with convolutions in Lávička and Bastl (2007), where their LN property was discovered. The proof relies on the computation of convolution ideal for each of affine classes of quadratic parameterizations. Another proof based on the Cremona transformations was later published in Peternell and Odehnal (2008). In spite of the effort devoted to this problem, we venture to present another proof here. Moreover our proof, in addition to being simple, does not depend on the dimension of the parameterized hypersurface.

Corollary 4.8. *Let $\mathbf{q}(s) : \mathbb{C}^{n-1} \to \mathcal{Q}$ be a quadratic polynomial parameterization of a hypersurface with non-degenerated Gauss image. Then $\mathcal{Q}$ is an LN hypersurface.*

Proof. Let $H : \mathbf{a} \cdot \mathbf{x} = 0$ be a generic $(n-1)$-space, then $\kappa_{\mathcal{Q}} = \sharp\{\mathcal{Q}_H\}$. Tangent hyperplanes to $\mathcal{Q}$ are generated by vectors $\partial \mathbf{q} / \partial s_i$ $(i = 1, \ldots, n-1)$ and thus

$$\sharp\{\mathcal{Q}_H\} = \frac{\sharp\{\text{solutions of the system } \frac{\partial \mathbf{q}}{\partial s_i} \cdot \mathbf{a} = 0\}}{\deg \mathbf{q}}, \tag{4.12}$$

where $\deg \mathbf{q}$ is the degree of a mapping in the usual sense. Since $\mathbf{q}$ is given by quadratic polynomials, the equations in the numerator of (4.12) are linear in variables $s_1, \ldots s_{n-1}$. Now, using that $\mathcal{Q}$ has non-degenerated Gauss image, $\sharp\{\mathcal{Q}_H\}$ is non-zero finite integer and we conclude that the numerator of (4.12) must be equal to one. This immediately implies $\kappa_{\mathcal{Q}} = 1$ and by Theorem 4.7 the hypersurface $\mathcal{Q}$ fulfills the LN property. □

The interest in LN hypersurfaces is not caused by its rationality and quite simple parameterizations only. It is e.g. well known that a convolution of LN curve/surface with any rational curve/surface is again rational. In the following theorem we summarize important properties of convolutions with LN hypersurfaces which follow from the general results in Chapter 3.

Theorem 4.9. *Let $\mathcal{V}$ be an LN hypersurface and $\mathcal{W}$ an arbitrary hypersurface. Then*

(i) $\mathcal{V} \star \mathcal{W}$ is irreducible,

(ii) $\mathcal{V} \star \mathcal{W}$ cannot be special,

(iii) if $\mathcal{V} \star \mathcal{W}$ is not degenerated then it is birationally equivalent to $\mathcal{W}$.

Proof. (i) By Theorem 3.46 , the number of irreducible components of $\mathcal{V} \star \mathcal{W}$ is less or equal to GCD $(1, \kappa_{\mathcal{W}}) = 1$.

(ii) If $\mathcal{V} \star \mathcal{W}$ is special, then by Theorem 3.59 it has to be a component of $(\mathcal{V} \star \mathcal{W}) \star \mathcal{V}^-$ whose index w.r.t. to $\mathcal{V} \star \mathcal{W}$ is strictly greater then one. However this index is less or equal to $\kappa_{\mathcal{V}} = 1$ by definition.

(iii) If $\mathcal{V} \star \mathcal{W}$ is non-degenerated, then by (ii) it is simple, and hence $\sigma : \mathcal{I}(\mathcal{V}, \mathcal{W}) \to \mathcal{V} \star \mathcal{W}$ is birational. Moreover $\deg \pi_{\mathcal{W}} = \kappa_{\mathcal{V}} = 1$ and the composition of these mappings is the desired birational mapping. □

The result on rationality of convolutions with LN hypersurface follows immediately from the last item in the previous theorem. More precisely, it says that for (uni)rational $\mathcal{W}$ the convolution $\mathcal{V} \star \mathcal{W}$ is again (uni)rational, under the condition that the convolution does not degenerate. If $\mathcal{V} \star \mathcal{W}$ is degenerated then one can still construct a rational mapping $\mathcal{W} \to \mathcal{V} \star \mathcal{W}$ (of course it is far away from a birational one) which pushes forward a parameterization of $\mathcal{W}$ to the dominant rational mapping $\mathbb{C}^{n-1} \to \mathcal{V} \star \mathcal{W}$.

The parameterization of $\mathcal{V} \star \mathcal{W}$ can be found by methods described in Subsection 3.1.1, i.e., for given $\mathbf{v}(s) : \mathbb{C}^{n-1} \to \mathcal{V}$ and $\mathbf{w}(t) : \mathbb{C}^{n-1} \to \mathcal{W}$ the parametrization $(\varphi(u), \psi(u))$ of some component of $\mathcal{P}(\mathbf{v}, \mathbf{w})$ leads to the parameterization of $\mathcal{V} \star \mathcal{W}$ in the form $\mathbf{v}(\varphi(u)) + \mathbf{w}(\psi(u))$. The great advantage of LN hypersurfaces is that the tough computation involved in this method becomes very simple whenever $\kappa_{\mathcal{V}} = 1$. This was shown in Sampoli et al. (2006) and Lávička and Bastl (2007) for surfaces. Let us reformulate this result for hypersurfaces here.

Let $\mathbf{v}(s)$ and $\mathbf{w}(t)$ be arbitrary parameterizations of $\mathcal{V}$ and $\mathcal{W}$, respectively. If $\mathbf{v}(s)$ happens not to be LN then apply Theorem 4.7 to obtain LN parameterization such that

$$\mathbf{n_v}(s) = \mathbf{p}_0 + \sum_{i=1}^{n-1} \mathbf{p}_i s_i, \tag{4.13}$$

for $\mathbf{p}_0, \ldots, \mathbf{p}_n$ linearly independent vectors. Then the condition on $\mathbf{v}(s)$ and $\mathbf{w}(t)$ to be coherent is stated as

$$\mathbf{p}_0 + \sum_{i=1}^{n-1} \mathbf{p}_i s_i = \lambda \mathbf{n_w}(t), \tag{4.14}$$

which is a system of linear equations in variables $s_1, \ldots, s_{n-1}, \lambda$ with coefficients in $\mathbb{C}(t)$. The matrix $(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{n}_\mathbf{w}(t))$ of this system is generically regular. To see this, realize that rank $(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}) = n-1$ and thus the matrix is singular if and only if $\mathbf{n}_\mathbf{w}(t)$ lies in the subspace generated by vectors $\mathbf{p}_i$. However this may happen only for almost none parameter t, otherwise the hypersurface $\mathcal{W}$ would have degenerated Gauss image. Finally, by Cramer's rule, the reparameterization can be written in the form

$$s_i = \frac{\det(\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_0, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_{n-1}, \mathbf{n}_\mathbf{w}(t))}{\det(\mathbf{p}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{n}_\mathbf{w}(t))}. \tag{4.15}$$

Let us conclude by an example, we have promised in the preliminary section, showing that the boundary of Minkowski sum need not to be necessarily a subset of the convolution of the associated boundaries.

Example 4.10. Let A be e.g. a domain in plane $\mathbb{R}^2$ bounded by the real part of an LN curve. For an arbitrary $\mathbf{p} \in \mathbb{R}^2$ we set $B := \{-\mathbf{x} + \mathbf{p} \mid \mathbf{x} \in A\}$. Then $\partial B = (\partial A)^- \star \{\mathbf{p}\}$ and by Theorem 4.9 it holds $\partial A \star \partial B = \mathbf{p}$. However the boundary of Minkowski sum $A \oplus B$ is a curve in $\mathbb{R}^2$, see Fig. 4.1. Hence

$$\partial(A \oplus B) \not\subseteq \partial A \star \partial B. \tag{4.16}$$

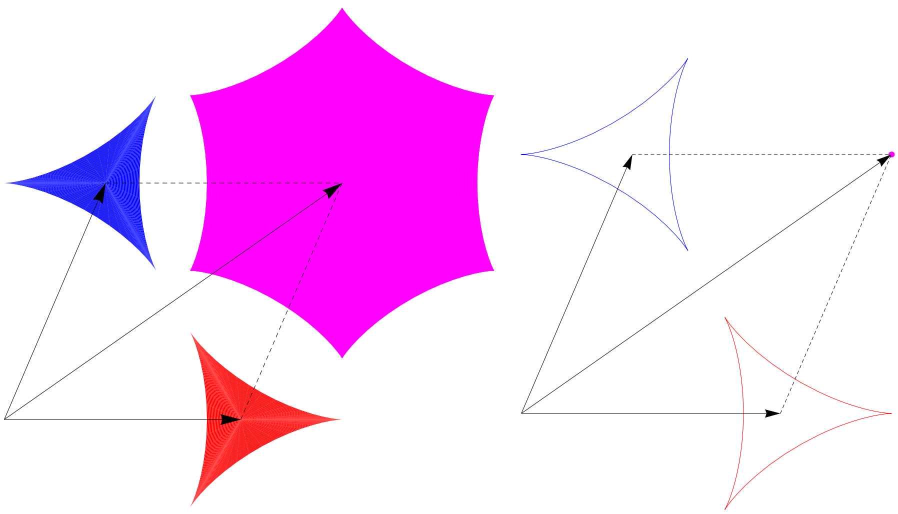


Figure 4.1: Minkowski sum of domains bounded by deltoids (left) and degeneracy of convolution of these LN curves (right).

4.1.2 Decomposition of LN curves

By Theorem 3.34 the dual equation of an arbitrary LN hypersurface $\mathcal{V} \subset \mathbb{C}^n$ has the form

$$f_{m-1}(\mathbf{n})h + f_m(\mathbf{n}) = 0, \tag{4.17}$$

where f_{m-1} and f_m are homogeneous polynomials of degrees $m-1$ and m, respectively. Thus there is a close relation between LN hypersurfaces and rational functions in $n-1$ variables. To see this write h as $-f_m/f_{m-1}$ and then dehomogenize polynomials f_i e.g. by setting $x_1 = n_i/n_n$ for $i = 1, \ldots, n-1$ (see also Gravesen et al. (2008).

If $\mathcal{W}$ is another LN hypersurface, represented dually as $g_{n-1}(\mathbf{n})h + g_n(\mathbf{n}) = 0$, then it follows from Subsection 3.1.3 that the dual equation of convolution $\mathcal{V} \star \mathcal{W}$ can be expressed as

$$\Big(f_{m-1}(\mathbf{n})g_{n-1}(\mathbf{n})\Big)h + \Big(f_m(\mathbf{n})g_{n-1}(\mathbf{n}) + f_{m-1}(\mathbf{n})g_n(\mathbf{n})\Big) = 0. \tag{4.18}$$

In this way the convolution of two LN hypersurfaces corresponds to the sum of two rational functions. Moreover rational functions ϕ and its non-zero multiple represent the same dual hypersurface and thus writing $\phi = \sum_{i=1}^{k} \alpha_i \phi_i$ is equivalent to writing the LN hypersurface corresponding to ϕ as the convolution of k LN hypersurfaces each of them corresponding to ϕ_i. Specially in the curve case, it is possible to identify functions ϕ_i for each ϕ, such that the resulting LN curves are very simple.

Theorem 4.11. *Any LN curve can be obtained as the convolution of a finite number of affine images of the canonical curves* $x_1^k - x_2^{k+1} = 0$.

Proof. Let the dual equation of an LN curve be written in the form

$$F^{\vee}(\mathbf{n}, h) = f_{m-1}(\mathbf{n})h + f_m(\mathbf{n}), \tag{4.19}$$

where $\mathbf{n} = (n_1, n_2)$. Write

$$h = -\frac{f_m(\mathbf{n})}{f_{m-1}(\mathbf{n})} \tag{4.20}$$

and assume for the sake of simplicity, we assume that n_1 divides neither $f_{m-1}(\mathbf{n})$, nor $f_m(\mathbf{n})$. Thus after dehomogenization $n = n_2/n_1$ we may write the decomposition of (4.20) into the partial fractions

$$h = h_0 + \cdots + h_\ell, \quad \text{where} \quad h_i = \frac{\alpha_i n_1^{k+1}}{(\beta_i n_1 + \gamma_i n_2)^k} \tag{4.21}$$

for some $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ and $k \in \mathbb{N}$. Then the transformation $n_1' = \alpha_i n_1$, $n_2' = \beta_i n_1 + \gamma_i n_2$ and $h' = h$ induces an affine transformation which maps a curve described by h_i to the desired canonical form. □

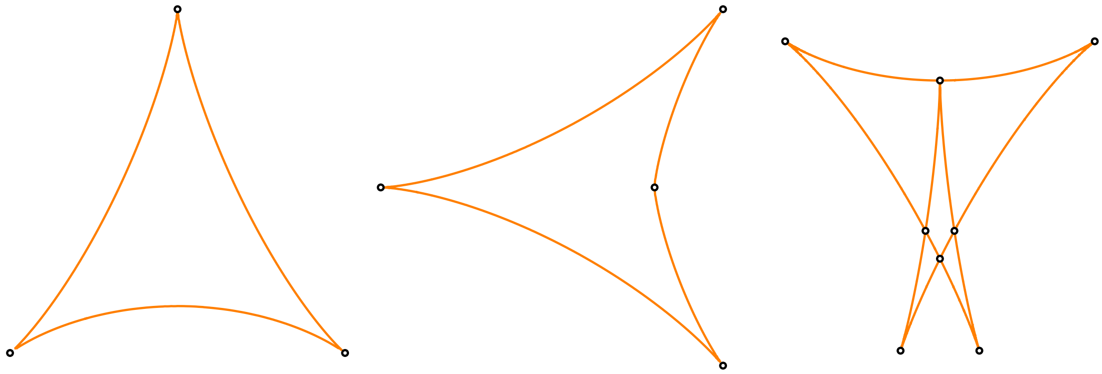


Figure 4.2: Fundamental LN curves from Remark 4.12; $k = 2$, $i = 0, 1, 2$ (the black dots denote real singularities).

Remark 4.12. Let us emphasize that despite starting with a real LN curve, it may happen that some of the factors (4.21) possess imaginary coefficients and hence the induced transformation does, too. So, if we prefer to work over reals we have to introduce new real fundamental curves corresponding to irreducible factors. In particular, these LN curves are given by the dual equations

$$(n_1^2 + n_2^2)^k h - n_1^i n_2^{2k+1-i} = 0, \tag{4.22}$$

where $k \geq 1$ and $0 \leq i \leq k$. These fundamental LN curves for $k = 2$ are shown in Fig. 4.2; the first curve is the well-known hypocycloid called deltoid (or tricuspoid).

4.1.3 Approximation with LN curves

Despite being far away from the topic of the thesis, it is apt to make a short note on approximations of convolutions at this moment. As observed earlier, the rationality of $\mathcal{V}$ and $\mathcal{W}$ does not guarantee the rationality of $\mathcal{V} \star \mathcal{W}$. Even worse, it seems that the rationality is rather rarely preserved. Thus one needs to apply some approximation technique. Let us restrict ourselves to the best explored case – curves.

The approximate methods for the resulting convolution $\mathcal{V} \star \mathcal{W}$ are often used since they provide a universal solution of many problems appearing in technical practice – but usually at the expense of great computational effort because any small modification of an input curve leads to a new approximation of the gained convolution curve. Therefore, it is worthwhile to investigate also techniques when not the convolution curve but one or both input rational curve(s) is/are suitably approximated.

If only one input curve is to be approximated by $\mathbf{v}^a(s) : \mathbb{C} \to \mathcal{V}^a$, then for $\mathbf{w}(t) : \mathbb{C} \to \mathcal{W}$ the approximation of convolution is parameterized by

$$\mathbf{v}^a(s(u)) + \mathbf{w}(t(u)) : \mathbb{C} \to \mathcal{V} \star^a \mathcal{W}. \tag{4.23}$$

Of course, a natural question is how to approximate a given curve and which primitives are for this procedure suitable. The LN curves analyzed in this section seems to serve as acceptable primitives, as their convolution with an arbitrary rational curve is rational and thus (4.23) makes sense. Let us recall that approximating convolutions with the help of LN curves has its history – a particular example is a quadratic curve approximation (QAC) method designed and studied in Lee et al. (1998). Moreover, as shown in Lávička and Bastl (2007), there exist one affine class of planar LN cubics and thus an analogous approach may be formulated also for a cubic Bézier curve with linear normals.

A simultaneous approximation of both curves is beneficial especially when a polynomial description of convolution curves is demanded. After a construction of the so-called compatible subdivision of given curves (producing segments on both curves with one-to-one correspondence between coherent points, cf. Lee et al. (1998)), the approximation by a suitable arcs is applied. The advantage of this approach is that the class of primitives can be significantly extended. For instance by involving curves of convolution degree two studied in the next section (e.g. we can also use other conic sections then parabola) whereas the computational effort is still feasible.

4.2 Hypersurfaces with convolution degree two

Despite a great application potential of LN hypersurfaces, the most of hypersurfaces do not belong to this class. For instance a majority of regular quadrics have convolution degree two. Fortunately the complexity of convolutions with hypersurfaces with convolution degree two is not so large.

In this section, we provide an algebraic analysis of convolutions of these hypersurfaces with an arbitrary hypersurface. Further we define the so-called QN hypersurfaces and prove some results on the rationality of convolutions with them (for the curve case see Vršek and Lávička (2010a)). We conclude this section with the decomposition of QN curves into the convolution of suitable fundamental curves.

4.2.1 Elementary properties of hypersurfaces with convolution degree two

Assumption 4.13. Throughout this section, we consider by $\mathcal{V}$ a hypersurface with convolution degree two.

In contrast to LN hypersurfaces, the rationality of hypersurfaces of convolution degree two is not guaranteed. By Theorem 3.34, the dual equation of such a hypersurface is of the form

$$F^{\vee}(\mathbf{n}, h) = f_{m-2}(\mathbf{n})h^2 + f_{m-1}(\mathbf{n})h + f_m(\mathbf{n}). \tag{4.24}$$

If we consider an affine part of $\mathcal{V}^{\vee}$ given by $n_n \neq 0$, then we see that $\mathcal{V}$ is the envelope of a system of the hyperplanes

$$\sum_{i=1}^{n-1} s_i x_i + x_n = -\frac{f_{m-1}(s) + \sqrt{f_{m-1}^2(s) - 4f_{m-2}(s)f_m(s)}}{2f_{m-2}(s)}, \tag{4.25}$$

where $f_j(s) = f_j(s_1, \ldots, s_{n-1}, 1)$. Therefore the parameterization of $\mathcal{V}$ obtained from (4.25) with use of (2.12) is not rational in general. However, it is parameterizable in terms of s and $\sqrt{P(s)}$, where $P(s)$ is a polynomial in s. Such a hypersurface is then called *square-root parameterizable*. In particular, it is known that the only curves which admits square-root parameterization are rational, elliptic or hyper-elliptic. Hence we obtain

Proposition 4.14. *Any curve with convolution degree 2 is rational, elliptic, or hyper-elliptic.*

Owning to the greater convolution degree, the resulting convolution hypersurfaces are generally more complicated than the convolutions with LN hypersurfaces. Nevertheless, the properties can be still summarized relatively simply.

Theorem 4.15. *For an arbitrary hypersurface $\mathcal{W}$ it holds:*

(i) $\mathcal{V} \star \mathcal{W}$ has at most two components.

(ii) If $\mathcal{X}$ is special component of $\mathcal{V} \star \mathcal{W}$, then $\mathcal{X}$ is 2-special.

(iii) If $\mathcal{V} \star \mathcal{W}$ is reducible, then each simple component is birationally equivalent to $\mathcal{W}$.

Proof. (i) By Theorem 3.46 the number of components is less or equal to $\mathrm{GCD}(2, \kappa_{\mathcal{W}}) \leq 2$.

(ii) If $\mathcal{X} \subset \mathcal{V} \star \mathcal{W}$ then by Theorem 3.59 we have $\mathcal{W} = \mathcal{X} \star \mathcal{V}^{-}$. Therefore $\mathcal{W}$ has index 2 w.r.t. $\mathcal{X}$ and by the same theorem $\mathcal{X}$ is 2-special.

(iii) It is an immediate consequence of Corollary 3.47. □

Remark 4.16. Since for two coherent points $(\mathbf{v}, \mathcal{V}) \sim_\star (\mathbf{w}, \mathcal{W})$ the point $\mathbf{v} + \mathbf{w} \in \mathcal{V} \star \mathcal{W}$ is generically coherent to both points $\mathbf{v}$ and $\mathbf{w}$, cf. Lemma 3.7, we can easily compute the convolution degree of the components of $\mathcal{V} \star \mathcal{W}$. In particular, if $\mathcal{V} \star \mathcal{W}$ is irreducible we have $\kappa_{\mathcal{V}\star\mathcal{W}} = 2\kappa_{\mathcal{W}}$ and in the case of reducible convolution we arrive at $\kappa_{\mathcal{X}} = \kappa_{\mathcal{W}}$ for the simple component $\mathcal{X}$ and $\kappa_{\mathcal{X}} = \frac{1}{2}\kappa_{\mathcal{W}}$ for the 2-special component $\mathcal{X}$.

A main disadvantage of hypersurfaces of the convolution degree two, despite being (uni)rational, is that the convolution $\mathcal{V} \star \mathcal{W}$ is not (uni)rational in general. However from Theorem 4.15-(iii) it follows that each simple component of $\mathcal{V} \star \mathcal{W}$ is (uni)rational, whenever $\mathcal{W}$ is and $\mathcal{V} \star \mathcal{W}$ posses two components. On the other hand, Theorem 3.51 implies that $\mathcal{V} \star \mathcal{W}$ tends to be irreducible. Thus it would be worth having some condition on input hypersurfaces whose ensure the reducibility of resulting convolution. For instance, we see from Corollary 3.48, that the convolution can be reducible only if $\kappa_{\mathcal{W}}$ is even. However this necessary condition is far away of being sufficient.

Lemma 4.17. *The following statements are equivalent:*

1. *There exists at least one coherent mapping* $\xi_{\mathcal{W},\mathcal{V}} : \mathcal{W} \to \mathcal{V}$.
2. *There exist exactly two coherent mappings* $\xi^{\pm}_{\mathcal{W},\mathcal{V}} : \mathcal{W} \to \mathcal{V}$.
3. $\mathcal{V} \star \mathcal{W}$ *is reducible.*

Proof. If $\mathcal{V} \star \mathcal{W}$ is reducible then it has by (i) two components – say $\mathcal{X}$ and $\mathcal{Y}$. By (3.69) we have $i^{\mathcal{W}}_{\mathcal{X}} + i^{\mathcal{W}}_{\mathcal{Y}} = \kappa_{\mathcal{V}} = 2$. Hence the indices of both components w.r.t. $\mathcal{W}$ equal one. The proof then follows from the correspondence between coherent mappings and components of index one established in Lemma 3.53. □

It is not easy to decide in general whether there exists a coherent mapping between two varieties or not. The partial answer will be given in the next subsection. In particular, we will identify an interesting class of hypersurfaces with convolution degree two. For any member of this class and any rational hypersurface we will, besides other, give an effective method to decide whether there exist coherent mappings between them.

4.2.2 QN hypersurfaces and their convolutions

The main aim of this subsection is to identify a direct analogy to LN parameterizations for hypersurfaces $\mathcal{V}$ of the convolution degree two. Afterwards we will analyze the associated conditions on the rational hypersurface $\mathcal{W}$ to have reducible and/or rational convolution $\mathcal{V} \star \mathcal{W}$.

The condition on $\mathbf{v}(s)$ to be LN can be stated in two ways, either it is assumed that the normal vector field $\mathbf{n}_{\mathbf{v}}(s)$ has the form (4.5), or equivalently it is a proper polynomial parameterization of a hyperplane not passing through the origin.

To generalize the first formulation, one could assume that the coordinates of $\mathbf{n}_{\mathbf{v}}$ are quadratic polynomials. This does not lead to well defined generalization because such a normal vector field can belong to a hypersurface of the higher convolution degree, as one can easily see. Thus we use the second formulation to define the natural generalization of the LN property.

Definition 4.18. A parametrization $\mathbf{v}(s)$ of a given hypersurface $\mathcal{V}$ is said to have a *QN property* (where QN stands for *quadratic normals*), or $\mathbf{v}(s)$ is a *QN* parameterization if $\mathbf{n}_{\mathbf{v}}(s)$ is a proper parameterization of some quadric not passing through the origin. The hypersurface $\mathcal{V}$ admitting such a parameterization is then called a *QN hypersurface*.

Lemma 4.19. *Each QN hypersurface has the convolution degree two. On contrary, if $\mathbf{v}(s)$ is a proper parameterization of $\mathcal{V}$ ($\kappa_{\mathcal{V}} = 2$) such that $\mathbf{n}_{\mathbf{v}}(s)$ is proper and without base points,*[1] *then $\mathbf{v}(s)$ is a QN parameterization.*

Proof. If $\mathbf{v}(s)$ is a QN parameterization and $\mathbf{a} \in \mathbb{C}^n$ a generic vector, then $\kappa_{\mathcal{V}}$ can be identified with the integer

$$\sharp\{s \in \mathbb{C}^{n-1} \mid \mathbf{n}_{\mathbf{v}}(s) = \lambda \mathbf{a}, \text{ and } \lambda \neq 0\}, \tag{4.26}$$

Let us denote by $\mathcal{A}$ the line parameterized by $\lambda \mathbf{a}$ and by $\mathcal{Q}_{\mathbf{v}}$ the quadric with the parameterization $\mathbf{n}_{\mathbf{v}}(s)$. Since $\mathcal{Q}_{\mathbf{v}}$ does not pass through the origin, (4.26) can be rewritten as

$$\kappa_{\mathcal{V}} = \sharp\{\mathcal{Q}_{\mathbf{v}} \cap \mathcal{A}\}, \tag{4.27}$$

Since a generic line $\mathcal{L}$ will intersect $\mathcal{Q}_{\mathbf{v}}$ transversally, we arrive at $\kappa_{\mathcal{V}} = \deg \mathcal{Q}_{\mathbf{v}} \cdot \deg \mathcal{L} = 2$.

The contrary statement follows from the same arguments. ☐

Let $A \in \mathbf{GL}_n(\mathbb{C})$ then the transformation $\mathbf{v}(s) \mapsto A \cdot \mathbf{v}(s)$ induces the transformation on the corresponding normal vector fields by

$$\mathbf{n}_{\mathbf{v}}(s) \mapsto \mathbf{n}_{A\cdot\mathbf{v}}(s) = A^{-T} \cdot \mathbf{n}_{\mathbf{v}}(s), \tag{4.28}$$

where A^{-T} stands for $(A^{-1})^T$. For the sake of clarity, we will write $A(\mathcal{X})$ for the set $\{A \cdot \mathbf{x} \mid \mathbf{x} \in \mathcal{X}\}$.

Using simultaneously this notation and (4.28), we can find a regular matrix A such that $A^{-T}(\mathcal{Q}_{\mathbf{v}})$ is in a standard position. In particular, using the assumptions

[1] Under a *base point* we mean the point where all the coordinates of normal vector field vanish simultaneously.

that $\mathcal{Q}_{\mathbf{v}}$ does not pass through the origin and it is parameterizable by polynomials (and hence irreducible), we deduce that the equation of $A^{-T}(\mathcal{Q}_{\mathbf{v}})$ can be written in the form, cf. Berger (1987)

$$\sum_{i=1}^{r} x_i^2 + 2\sum_{i=1}^{r} a_i x_i - 2x_n + b = 0, \tag{4.29}$$

where $b \neq 0$ and $1 \leq r < n$.

Example 4.20. Consider a surface $\mathcal{V}$ parameterized by

$$\mathbf{v}(s) = \left(s_1(1 - \frac{s_1^2}{3} + s_2^2), -s_2(1 - \frac{s_1^2}{3} + s_2^2), \frac{1}{\sqrt{2}}(s_1^2 - s_2^2) \right)^T. \tag{4.30}$$

This is a proper parameterization with the associated normal field given by

$$\mathbf{n}_{\mathbf{v}}(s) = \left(\sqrt{2}s_1, \sqrt{2}s_2, s_1^2 + s_2^2 - 1 \right)^T. \tag{4.31}$$

It is not hard to show that $\mathbf{n}_{\mathbf{v}}(s)$ is also proper and moreover it has no base point. (If it has a base point, then it follows $s_1 = s_2 = 0$ from the first two coordinates. However substituting into the third coordinate we get $0^2 + 0^2 - 1 \neq 0$). Direct computation reveals that the variety parameterized by $\mathbf{n}_{\mathbf{v}}(s)$ is the paraboloid

$$\mathcal{Q}_{\mathbf{v}}: \quad x_1^2 + x_2^2 - 2x_3 - 2 = 0. \tag{4.32}$$

Therefore $\mathbf{v}(s)$ is a QN parameterization and $\kappa_{\mathcal{V}} = 2$.

Remark 4.21. Let us emphasize that working over the field of real number – as it is usual in CAGD, it is not always possible to find a matrix $A \in \mathbf{GL}_n(\mathbb{R})$ such that a transformed quadric has the form (4.29). However, it can be always transformed into the form

$$\sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2 + 2\sum_{i=1}^{p} a_i x_i - 2\sum_{i=p+1}^{p+q} a_i x_i - 2x_n + b = 0, \tag{4.33}$$

where again $b \neq 0$ and $1 \leq p + q = r < n$.

Now, for a later use (see the proof of Theorem 4.25) we will consider a polynomial associated to (4.29), given by

$$-b\sum_{i=1}^{r} x_i^2 + \left(\sum_{i=1}^{r} a_i x_i - x_n \right)^2. \tag{4.34}$$

This is a homogeneous polynomial of the degree two and thus it can be written as $\mathbf{x}^T \cdot \Delta^{r,n} \cdot \mathbf{x}$, where $\Delta^{r,n}$ is a symmetric $n \times n$ matrix with the coefficients depending on $a_1, \ldots, a_r$ and b.

Definition 4.22. Let $\mathcal{V}$ be a QN hypersurface admitting a QN parameterization $\mathbf{v}(s)$ and let $\mathbf{n}_{\mathbf{v}}(s)$ parameterizes a quadric $\mathcal{Q}_{\mathbf{v}}$. If $A \in \mathbf{GL}_n(\mathbb{C})$ such that $\mathcal{Q}_{A\cdot\mathbf{v}}$ has the equation (4.29), then by a *coherent form* we mean

$$D_{\mathcal{V}}(\mathbf{x}) = \mathbf{x}^T \cdot (A^{-1} \cdot \Delta^{r,n} \cdot A^{-T}) \cdot \mathbf{x}. \tag{4.35}$$

Example 4.23. Let $\mathbf{v}(s)$ be a parameterization from Example 4.20. As we have seen this is a QN parameterization and the quadric $\mathcal{Q}_{\mathbf{v}}$ is given by (4.32). Since it is already in the standard form (4.29), we immediately deduce that the coherent form is given by

$$D_{\mathcal{V}}(\mathbf{x}) = 2x_1^2 + 2x_2^2 + x_3^2. \tag{4.36}$$

Let us recall that under the rank of quadratic form we understand the rank of its matrix.

Lemma 4.24. *Let $D_{\mathcal{V}}(\mathbf{x})$ be the coherent form given by* (4.35)*. Then its rank is equal to $r+1$.*

Proof. Since $(A^{-1} \cdot \Delta^{r,n} \cdot A^{-T})$ and $\Delta^{r,n}$ are similar, they have the same rank. Now, the computation of the rank of $\Delta^{r,n}$ is only a technical application of Gauss elimination method and it results to the rank $\Delta^{r,n} = r+1$. □

It is probably not obvious at this point what the coherent form is and why it was chosen in the form (4.35). However its significance will be justified by the following theorem.

Theorem 4.25. *Let $\mathcal{V}$ be a QN hypersurface and $\mathcal{W}$ an arbitrary unirational hypersurface with a parameterization $\mathbf{w}(t)$. Then $\mathbf{w}(t)$ is $\mathcal{V}$-coherent if and only if there exists $\sigma \in \mathbb{C}[t]$ such that*

$$D_{\mathcal{V}}(\mathbf{n}_{\mathbf{w}}(t)) = \sigma^2(t). \tag{4.37}$$

Proof. The theorem will be proved in two steps. First let us assume that $\mathcal{V}$ admits a QN parameterization $\mathbf{v}(s)$ such that $\mathbf{n}_{\mathbf{v}}(s)$ parameterizes a quadric $\mathcal{Q}_{\mathbf{v}}$ given by (4.29). Since both $\mathbf{v}(s)$ and $\mathbf{n}_{\mathbf{v}}(s)$ are proper, we may assume (after a suitable reparameterization) that

$$\mathbf{n}_{\mathbf{v}}(s) = \left(s_1 - a_1, \ldots, s_r - a_r, s_{r+1}, \ldots, s_{n-1}, \frac{1}{2}\left(\sum_{i=1}^{r}(s_i^2 - a_i^2) + b\right)\right)^T. \tag{4.38}$$

The parameterization $\mathbf{w}(t)$ is by definition $\mathcal{V}$-coherent if and only if there exists a rational reparameterization $\phi(t) : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ such that $\mathbf{v}(\phi(t)) \sim_\star \mathbf{w}(t)$. Rewritten into the language of normals, this relation becomes

$$\mathbf{n}_{\mathbf{v}}(\phi(t)) = \lambda(t)\mathbf{n}_{\mathbf{w}}(t). \tag{4.39}$$

Now we invoke the trick made by the authors of Lávička and Bastl (2007) (cf. Subsection 3.1.1). They replaced the normal vector field $\mathbf{n_w}(t)$ by a vector of symbolic variables $(\nu_1, \dots, \nu_n)$. Afterwards the necessary and sufficient condition for the existence of rational reparameterization $\phi(t)$ could be extracted from the Gröbner basis of the associated convolution ideal. This can be applied immediately also to our problem. The Gröbner basis of the ideal

$$\langle \mathbf{n_v}(s) - (\nu_1, \dots, \nu_n), 1 - \lambda w \rangle \tag{4.40}$$

w.r.t. lexicographic order $w > s_1 > \dots > s_{n-1} > \lambda$ is

$$\begin{aligned} g_0 &= bw + \left(\sum_{i=1}^r \nu_i^2\right)\lambda + 2\sum_{i=1}^r a_i\nu_i - 2\nu_n, && (4.41)\\ g_i &= s_i - a_i - \lambda\nu_i, \quad i = 1, \dots, r && (4.42)\\ g_j &= s_j - \lambda\nu_j, \quad j = r+1, \dots, n-1 && (4.43)\\ g_n &= \left(\sum_{i=1}^r \nu_i^2\right)\lambda^2 + 2\left(\sum_{i=1}^r a_i\nu_i - 2\nu_n\right)\lambda + b. && (4.44) \end{aligned}$$

All these polynomials are linear in variables $w, s_1, \dots, s_{n,-1}, \lambda$ except of the last one, which is quadratic in λ. Expressing λ from g_n yields the formula

$$\lambda = 2\frac{\nu_n - \sum_{i=1}^r a_i\nu_i \pm \sqrt{-b\sum_{i=1}^r \nu_i^2 + \left(\sum_{i=1}^r a_i\nu_i - \nu_n\right)^2}}{\sum_{i=1}^r \nu_i^2}. \tag{4.45}$$

Comparing the polynomial under the square root with (4.34) we see that it is nothing but a coherent form $D_{\mathcal{V}}(\nu_1, \dots, \nu_n)$. Thus substituting $\mathbf{n_w}(t)$ back for the $(\nu_1, \dots, \nu_n)$, we arrive at rational $\lambda(t)$ (and therefore rational $s = \phi(t)$) if and only if $D_{\mathcal{V}}(\mathbf{n_w}(t)) = \sigma^2(t)$ for some rational $\sigma(t)$.

Now, let $\mathcal{V}$ be a hypersurface with a QN parametrization $\mathbf{v}(s)$ such that the quadric $Q_{A\cdot\mathbf{v}}$ is in the form (4.29) for some $A \in \mathbf{GL}_n(\mathbb{C})$. Obviously $A(\mathcal{V}) \star A(\mathcal{W}) = A(\mathcal{V} \star \mathcal{W})$, and therefore $\mathbf{w}(t)$ is $\mathcal{V}$-coherent if and only if $A \cdot \mathbf{w}(t)$ is $A(\mathcal{V})$-coherent. Writing the coherent form $D_{A(\mathcal{V})}(\mathbf{x})$ as $\mathbf{x}^T \cdot \Delta^{r,n} \cdot \mathbf{x}$ we conclude that $\mathbf{w}(t)$ is $\mathcal{V}$-coherent if and only if

$$\begin{aligned} D_{A(\mathcal{V})}(\mathbf{n}_{A\cdot\mathbf{w}}) &= \mathbf{n}_{A\cdot\mathbf{w}}^T \cdot \Delta^{r,n} \cdot \mathbf{n}_{A\cdot\mathbf{w}} = \\ &\left(A^{-T} \cdot \mathbf{n_w}\right)^T \cdot \Delta^{r,n} \cdot \left(A^{-T} \cdot \mathbf{n_w}\right) = \\ &\mathbf{n_w}^T \cdot \left(A^{-1} \cdot \Delta^{r,n} \cdot A^{-T}\right) \cdot \mathbf{n_w} = \\ &D_{\mathcal{V}}(\mathbf{n_w}) = \sigma^2, \end{aligned} \tag{4.46}$$

which completes the proof. □

Remark 4.26. Let us recall, that conditions guaranteeing the rationality of the convolution $\mathcal{V} \star \mathcal{W}$ are called RC-conditions in Lávička and Bastl (2007). Thus, $D_{\mathcal{V}}(\mathbf{n_w}^T(t)) = \sigma^2(t)$ is the RC condition of a QN hypersurface $\mathcal{V}$.

Remark 4.27. If $\mathcal{V}$ is a QN curve, then the computation of coherent form becomes considerably simpler. Let $\mathbf{v}(s)$ be a QN parameterization. Then

$$\mathbf{n_v}(s) = (\mu_1(s), \mu_2(s))^T \tag{4.47}$$

is a proper polynomial parameterization of a parabola. Therefore

$$\mu_1(s)y - \mu_2(s)x = 0 \tag{4.48}$$

is a quadratic equation in variable s and coefficients in $\mathbb{C}[x, y]$. It can be shown that the coherent form is nothing but the discriminant of this equation.

Now we are ready to answer the question from the end of the previous subsection, i.e., we will provide a criterion enabling to decide whether there exists a coherent mapping between QN hypersurface $\mathcal{V}$ and an arbitrary rational hypersurface $\mathcal{W}$. Let us suppose that $\mathbf{w}(t)$ is a proper $\mathcal{V}$-coherent parameterization of $\mathcal{W}$. Thus there exists $\mathbf{v}(t) : \mathbb{C}^{n-1} \to \mathcal{V}$ such that $\mathbf{v}(t) \sim_\star \mathbf{w}(t)$. Then it is easy to see that the mapping $\xi : \mathcal{W} \to \mathcal{V}$ defined by $\xi = \mathbf{v} \circ \mathbf{w}^{-1}$ is coherent. Conversely, if there exists a coherent mapping $\xi_{\mathcal{W},\mathcal{V}} : \mathcal{W} \to \mathcal{V}$, then for an arbitrary parametrization $\mathbf{w}(t)$ of $\mathcal{W}$ the $\xi_{\mathcal{W},\mathcal{V}} \circ \mathbf{w}$ is a parameterization of $\mathcal{V}$ coherent with $\mathbf{w}$. Thus $\mathbf{w}(t)$ is $\mathcal{V}$-coherent. Hence combining with Theorem 4.25 we arrive at the proposition.

Proposition 4.28. *Let $\mathcal{V}$ be a QN hypersurface and $\mathcal{W}$ a rational hypersurface. Then there exists a coherent mapping $\xi_{\mathcal{W},\mathcal{V}} : \mathcal{W} \to \mathcal{V}$ if and only if for some proper (and thus for all) parameterization $\mathbf{w}(t)$ of $\mathcal{W}$ it holds $D_{\mathcal{V}}(\mathbf{n_w}(t)) = \sigma^2(t)$.*

It turns out that QN hypersurfaces with the same coherent forms (up to multiplication by scalar) behaves similarly with respect to the operation of convolution. Let us denote by $\mathfrak{Q}_\Delta$ the set of QN hypersurfaces $\mathcal{V}$ such that $D_{\mathcal{V}}(\mathbf{x}) = \alpha(\mathbf{x}^T \cdot \Delta \cdot \mathbf{x})$ for a symmetric $n \times n$ matrix Δ and a nonzero number α. Moreover the common coherent form will be often denoted by $D_\Delta(\mathbf{x})$. If Δ is a regular matrix (it happens whenever $r = n - 1$, cf. Lemma 4.24), then the following proposition reveals a simple representant of $\mathfrak{Q}_\Delta$. As its proof has a purely technical character, we omit it here.

Proposition 4.29. *Let Δ be regular. Then the quadric given by the equation*

$$\mathcal{S}^\Delta : \ \mathbf{x}^T \cdot \Delta^{-1} \cdot \mathbf{x} = 1 \tag{4.49}$$

lies in $\mathfrak{Q}_\Delta$.

Example 4.30. Let $\mathcal{T}$ be the Tschirnhausen cubic given parameterically by

$$\mathbf{t}(s) = \Big(3(s^2-3), s(s^2-3)\Big)^T. \tag{4.50}$$

Then $\mu_1(s) = 6s$, $\mu_2(s) = 3(s^2-1)$ and the coherent form is by Remark 4.27 the discriminant of the quadratic equation $3(s^2-3)x - s(s^2-3)y = 0$, i.e., $36(x^2 + y^2)$. Since the multiplication by nonzero scalar does not play any role, we can write $D_{\mathcal{T}}(\mathbf{x}) = \mathbf{x}^T \cdot \Delta \cdot \mathbf{x}^T$, where Δ is unit 2×2 matrix. Therefore $\Delta^{-1} = \Delta$ and the set of QN curves containing the Tschirnhausen cubic can be represented by the conic section with the equation $x^2 + y^2 = 1$, i.e., by the unit circle. In addition the RC condition of Tschirnhausen cubic is exactly the well-known PH condition.

It follows from the definition of coherent form, that for two QN hypersurfaces $\mathcal{V}, \mathcal{W} \subset \mathbb{C}^n$ with the coherent forms of the same rank there exists some $A \in \mathbf{GL}_n(\mathbb{C})$ such that $A(\mathcal{V})$ and $\mathcal{W}$ lie in the same set $\mathfrak{Q}_\Delta$, i.e., $\mathbf{GL}_n(\mathbb{C})$ acts transitively on the set of all classes of QN hypersurfaces with equally ranked forms.

Specially, all the QN curves can be transformed such that their coherent form has the form $x_1^2 + x_2^2$ (or equivalently their RC condition is exactly PH condition). To see this, use Lemma 4.24 to obtain that each QN curve has fully ranked coherent form. However, let us emphasize that when working with real curves only, it is not generally ensured that for a real curve $\mathcal{V}$ the curve $A(\mathcal{V})$ is real, too. In particular, any real QN curve can be transformed either to a real curve with PH condition, or to a real curve with MPH condition (the class of MPH curves is determined by the conic section $x^2 - y^2 = 1$, cf. Kosinka and Jüttler (2006, 2007); Moon (1999)).

As shown in Peternell and Pottmann (1998), a parameterized curve/surface fulfills the PH/PN property if and only if it can be represented dually as a rational curve/surface on the so-called Blaschke cylinder $\mathcal{B}(\mathbf{n}, h) : \sum_i^k n_i^2 = 1$, where k equals 2 or 3, respectively. It can be expected that an analogous condition will be fulfilled for all classes of QN hypersurfaces, too.

Definition 4.31. Let us consider $D_\Delta(\mathbf{n})$ as a polynomial in $\mathbb{C}[\mathbf{n}, h]$. Then the hypersurface

$$\mathcal{B}_\Delta : D_\Delta(\mathbf{n}) - 1 = 0 \tag{4.51}$$

is called a *generalized Blaschke hypercylinder*.

Corollary 4.32. *If $\mathcal{V} \in \mathfrak{Q}_\Delta$ then there is a correspondence between $\mathcal{V}$-coherent parameterizations and parameterizations of varieties of codimension one on $\mathcal{B}_\Delta$.*

Proof. Let $\mathbf{w}(t) : \mathbb{C}^{n-1} \to \mathcal{W}$ be a $\mathcal{V}$-coherent parameterization. Then by Theorem 4.25 there exists $\sigma \in \mathbb{C}[t]$ such that $D_\Delta(\mathbf{n}_\mathbf{w}(t)) = \sigma^2(t)$. We set

$h(t) = -\mathbf{w}(t) \cdot \mathbf{n}_{\mathbf{w}}(t)$. The image of the rational mapping

$$t \mapsto \frac{1}{\sigma(t)} \left(\mathbf{n}_{\mathbf{w}}(t), h(t)\right)^T \tag{4.52}$$

lies on $\mathcal{B}_\Delta$. Moreover, since $\mathcal{W}$ has non-degenerated Gauss image, its dimension is $n-1$ and therefore it has codimension one on the generalized Blaschke hypercylinder.

Conversely let $(\mathbf{n}(t), h(t))^T$ be an arbitrary rational parameterization of a variety of codimension one on $\mathcal{B}_\Delta$. Then, we can easily skip to the representation of a hypersurface as the envelope of $(n-1)$-parameter family of hyperplanes $\Sigma(t)$: $\mathbf{n}(t) \cdot \mathbf{x} + h(t) = 0$, where $D_\Delta(\mathbf{n}(t)) = 1$. Solving the system

$$\Sigma(t) = 0, \ \frac{\partial \Sigma(t)}{\partial t_1} = 0, \dots, \frac{\partial \Sigma(t)}{\partial t_{n-1}} = 0 \tag{4.53}$$

we arrive at the corresponding $\mathcal{V}$-coherent parameterization. □

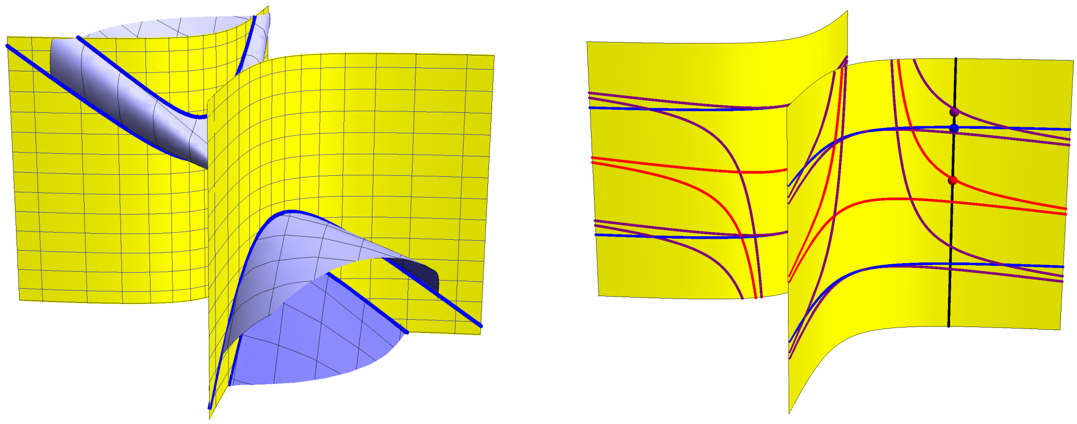


Figure 4.3: Left: the generalized Blaschke cylinder $n_1^2 - n_2^2 - 1 = 0$ (yellow) with the image of a QN-curve (blue); Right: the construction of the convolution curve (purple) of two QN-curves (red and blue) solved on the associated Blaschke cylinder.

Remark 4.33. The above mentioned correspondence is not bijective but one-to-two. More precisely, both rational parameterizations $\mathbf{x}(t) : \mathbb{C}^{n-1} \to \mathcal{B}_\Delta$ and $-\mathbf{x}(t) : \mathbb{C}^{n-1} \to \mathcal{B}_\Delta$ have the same image in the set of $\mathcal{V}$-coherent parameterizations, cf. Fig. 4.3.

Having a rational parameterization of $\mathcal{V}$, Corollary 4.32 gives us a straightforward method for expressing the $\mathcal{V}$-coherent parameterizations. For the sake of simplicity, we will summarize the main steps for the curve case only. The generalization to the higher dimensions is straightforward. In addition we formulate the algorithm for real curves to emphasize the difference between real and complex case.

Algorithm 3 Universal formula for $\mathcal{V}$-coherent parameterizations of curves

Input: A QN parameterization $\mathbf{v}(s) : \mathbb{R} \to \mathcal{V}$.
Output: A $\mathcal{V}$-coherent parameterization $\mathbf{w}(t) : \mathbb{R} \to \mathcal{W}$.
1: $D_{\mathcal{V}}(x,y) := (x,y) \cdot \Delta \cdot (x,y)^T$ (see Remark 4.27).
2: *Find* $A \in \mathbf{GL}_2(\mathbb{R})$: $A \cdot \Delta \cdot A^T = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}$, where $\tau = \pm 1$.
3: *Set*

$$(n_1(t), n_2(t), h(t)) := (2a(t)b(t), a^2(t) - \tau b^2(t), (a^2(t) + \tau b^2(t))c(t)),$$

where $a(t), b(t) \in \mathbb{R}[t]$ and $c(t) \in \mathbb{R}(t)$.
4: *Find* the solution $x = x(t)$ and $y = y(t)$ of the following system:

$$\begin{aligned} n_1(t)x + n_2(t)y &= h(t), \\ n_1'(t)x + n_2'(t)y &= h'(t). \end{aligned}$$

5: **return** $\mathbf{w}(t) := A^{-1} \cdot (x(t), y(t))^T$.

Obviously, if $\mathcal{V} = \mathcal{S}^1$ then the output of Algorithm 3 are all rational PH parameterizations, presented in Pottmann (1995).

We mentioned above that two hypersurfaces with the same coherent forms behave similarly with respect to the operation of convolution. Now, we are going to establish this analogy more precisely. In particular, we will show that from the birational point of view the convolutions $\mathcal{V} \star \mathcal{W}$ and $\mathcal{U} \star \mathcal{V}$ (where $\mathcal{V}, \mathcal{U}$ are QN hypersurfaces with the same coherent form) can differ only in non-simple components.

Lemma 4.34. *If $\mathcal{U}, \mathcal{V} \in \mathfrak{Q}_\Delta$ then there exists a birational coherent mapping $\xi_{\mathcal{U},\mathcal{V}} : \mathcal{U} \to \mathcal{V}$.*

Proof. Let $\mathbf{u}(s)$ be a proper parameterization of $\mathcal{U}$. This is trivially $\mathcal{U}$-coherent. Since $\mathcal{U}$ and $\mathcal{V}$ have the same coherent forms, it is by Theorem 4.25 also $\mathcal{V}$-coherent. Therefore there exists a parameterization $\mathbf{v}(s)$ of $\mathcal{V}$ such that $\mathbf{u}(s) \sim_\star \mathbf{v}(s)$. Since $\mathbf{u}(t)$ is proper it has a rational inverse and the mapping

$$\xi_{\mathcal{U},\mathcal{V}} = \mathbf{v} \circ \mathbf{u}^{-1} \tag{4.54}$$

is obviously coherent. Finally using Lemma 3.52 we obtain $\deg \xi_{\mathcal{U},\mathcal{V}} = \kappa_{\mathcal{U}} / \kappa_{\mathcal{V}} = 1$ and thus it is birational. □

Corollary 4.35. *Let $\mathcal{U}, \mathcal{V} \in \mathfrak{Q}_\Delta$ and let $\mathcal{W}$ be an arbitrary hypersurface. Then it holds*

(i) $\mathcal{U} \star \mathcal{W}$ is irreducible if and only if $\mathcal{V} \star \mathcal{W}$ is,

(ii) each simple component of $\mathcal{V} \star \mathcal{W}$ is birationally equivalent to each simple component of $\mathcal{U} \star \mathcal{W}$.

Proof. (i) Let us suppose that that $\mathcal{V} \star \mathcal{W}$ is irreducible while $\mathcal{U} \star \mathcal{W}$ has two components. Then by Lemma 4.17 there exists a coherent mapping $\xi_{W,V} : \mathcal{W} \to \mathcal{U}$, whereas there exists no such a mapping between $\mathcal{W}$ and $\mathcal{V}$. However by Lemma 4.34 we have coherent mapping $\xi_{\mathcal{U},\mathcal{V}} :\to \mathcal{U} \to \mathcal{V}$ and the composition of coherent mappings i again coherent. Thus we obtain $\xi_{\mathcal{U},\mathcal{W}} = \xi_{\mathcal{V},\mathcal{W}} \circ \xi_{\mathcal{U},\mathcal{V}}$ the coherent mapping from $\mathcal{W}$ to $\mathcal{U}$, which is a contradiction.

(ii) If $\mathcal{U} \star \mathcal{W}$ and $\mathcal{V} \star \mathcal{W}$ are reducible, then there is nothing to prove because by Theorem 4.15 each simple component is birationally equivalent to $\mathcal{W}$. If the convolutions are irreducible and simple then they are birationally equivalent to the corresponding incidence varieties. Then invoking Lemma 3.10 we can extend coherent mapping $\xi_{\mathcal{U},\mathcal{V}}$ to the birational mapping $\xi : \mathcal{I}(\mathcal{U},\mathcal{V}) \to \mathcal{I}(\mathcal{V},\mathcal{W})$, which completes the proof. □

The importance of the above corollary consists of possibility of replacing generally complicated QN hypersurface by a simple one, anytime we study e.g. the rationality of resulting convolution hypersurface. For instance, when the QN hypersurface has a fully ranked coherent form, it can be replaced by a quadric, cf. Proposition 4.29. To illustrate this, we conclude this subsection by giving a formula for computing genera of convolutions of QN curves and a wide class of algebraic curves. This formula uses and generalizes the result presented in Arrondo et al. (1999) for the offsets (i.e., for convolutions with circles).

Let us suppose now, that $\mathcal{V}$ and $\mathcal{W}$ are curves. If $\mathcal{V} \star \mathcal{W}$ is irreducible then it is simple[2], and hence combining Corollary 4.35 with the genus formula for classical offsets from Arrondo et al. (1999), allows us to write down the analogous genus formula for a wide class of curves $\mathcal{W}$ fulfilling that $\mathcal{V} \star \mathcal{W}$ is irreducible. (The reducible case is solved by Theorem 4.15.) Let a given QN curve $\mathcal{V}$ lie in some class $\mathfrak{Q}_\Delta$ represented be the conic section $\mathcal{S}^\Delta$. Let $A \in \mathbf{GL}_2(\mathbb{C})$ be a linear transformation such that $A(\mathcal{S}^\Delta)$ is the unit circle $\mathcal{S}^1$ and $\mathcal{W}^P$ be the projective closure of $\mathcal{W}$. If all singularities of $A(\mathcal{W}^P)$ are affine and ordinary, no tangent line at the inflection or singular point of $A(\mathcal{W}^\mathcal{P})$ passes through the circular point $(0 : 1 : \pm \mathrm{i})$ and the ideal line $\omega : x_0 = 0$ is not tangent to $A(\mathcal{W}^P)$ (all of these conditions are taken from Arrondo et al. (1999) – see there for more details), then we may apply the mentioned genus formula to compute the genus of the offset $\mathcal{S}^1 \star A(\mathcal{W})$ and hence of $\mathcal{S}^\Delta \star \mathcal{W} = A^{-1}(\mathcal{S}^1 \star A(\mathcal{W}))$, too. Combining this with Lemma 4.35 we can formulate the following theorem.

Theorem 4.36. (GENUS FORMULA). *Let $\mathcal{V} \in \mathfrak{Q}_\Delta$ and let $\mathcal{W}$ be a curve such that $\mathcal{V} \star \mathcal{W}$ is irreducible and all of the above mentioned conditions are fulfilled. Then the*

[2]It is non-special by Theorem 3.60. To see that it is not degenerated suppose contrary, i.e., $\mathcal{V} \star \mathcal{W} = \mathbf{p}$ for some $\mathbf{p} \in \mathbb{C}^2$ Then by Theorem 3.58 $\mathcal{W} = \mathcal{V}^- \star \{\mathbf{p}\}$. Now if $H \subset \mathbb{C}^2$ is generic subspace of dimension one, we can write $\mathcal{V}_H = \{\mathbf{v}_1, \mathbf{v}_2\}$. With this notation we have $\mathcal{W}_H = \{\mathbf{p} - \mathbf{v}_1, \mathbf{p} - \mathbf{v}_2\}$. Thus $\mathcal{V} \star \mathcal{W}$ is generated by points of the form $\mathbf{p}$, $\mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{v}_2 - \mathbf{v}_1$. Therefore $\mathbf{p} \subsetneq \mathcal{V} \star \mathcal{W}$ which is a contradiction.

genus of $\mathcal{V} \star \mathcal{W}$ can be computed by the Arrondo-Sendra-Sendra formula*:*

$$\mathrm{g}\left(\mathcal{V} \star \mathcal{W}\right) = 4\mathrm{g}\left(\mathcal{W}\right) + 2\deg \mathcal{W} - 3. \tag{4.55}$$

Analogously to Arrondo et al. (1999) it is obvious that the convolution of two regular conic sections is an elliptic curve assuming it is irreducible. On contrary, if we consider quadrics in three-dimensional space we obtain the following surprising result.

Theorem 4.37. *The convolution of any two regular quadrics in three-dimensional space is rational.*

Proof. If the convolution is reducible, then there is nothing to prove. Let us suppose that $\mathcal{X}$ and $\mathcal{Y}$ are regular quadrics admitting the irreducible convolution $\mathcal{X} \star \mathcal{Y}$. Since $\mathcal{X}$ is regular there exists a transformation $A \in \mathbf{GL}_3(\mathbb{C})$ such that $A(\mathcal{X})$ is the unit sphere $\mathcal{S}^1$. Invoking Peternell and Pottmann (1998, Theorem 3.2) we conclude that

$$A(\mathcal{X} \star \mathcal{Y}) = A(\mathcal{X}) \star A(\mathcal{Y}) = \mathcal{S}^1 \star A(\mathcal{Y}) = \mathcal{O}_1(\mathcal{Y}) \tag{4.56}$$

is rational. Hence we have proved the rationality of the convolution $\mathcal{X} \star \mathcal{Y}$. □

4.2.3 Decomposition of QN curves

Similarly to the case of LN curves we would like to give a description of the set of QN curves. Since any QN curve lies in a certain class $\mathfrak{Q}_\Delta$ for some conic section $\mathcal{S}^\Delta$, we will work with a fixed class $\mathfrak{Q}_\Delta$. Let us denote by $\mathfrak{L}$ the set of LN curves and we set $\mathfrak{D}_\Delta = \mathbb{C}^2 \cup \mathfrak{L} \cup \mathfrak{Q}_\Delta$. Then, the following statement is obvious:

Lemma 4.38. *If $\mathcal{V}, \mathcal{W} \in \mathfrak{D}_\Delta$ then $\mathcal{V} \star \mathcal{W} \subset \mathfrak{D}_\Delta$.*

Hence, the set $\mathfrak{D}_\Delta$ is closed under the operation of convolution[3] and our goal is to find its generators, as in the case of LN curves. More precisely, we are going to find a family of fundamental curves $\{\mathcal{G}_\lambda\}_{\lambda\in\Lambda}$, where Λ is an (infinite) index set, such that an arbitrary curve $\mathcal{V} \in \mathfrak{Q}_\Delta$ is a component of

$$\left(\bigstar_{\lambda\in\Gamma} \mathcal{G}_\lambda\right) \star \mathcal{L} \star \mathbf{p}, \tag{4.57}$$

where $\mathcal{L} \in \mathfrak{L}$, $\mathbf{p} \in \mathbb{C}^2$ and $\Gamma \subset \Lambda$ is finite.

[3]For curves $\mathcal{V}, \mathcal{W} \in \mathfrak{Q}_\Delta \subset \mathfrak{D}_\Delta$ the convolution can be reducible and hence we have to write $\subset$ instead of $\in$ in Lemma 4.38.

As known, the dual equation of a general curve with the convolution degree 2 has the form

$$F^{\vee}(\mathbf{n}, h) = f_{m-2}(\mathbf{n})h^2 + f_{m-1}(\mathbf{n})h + f_m(\mathbf{n}) = 0, \tag{4.58}$$

where $f_i(\mathbf{n})$ are homogeneous polynomials of degree i. It is not difficult to realize that the set of all curves in $\mathfrak{Q}_\Delta$ such that $f_{m-1} = 0$ is closed under the operation of convolution. From the geometric point of view, the curves with this special dual representation are centrally symmetric.

Now, let $\mathcal{V}, \mathcal{W} \in \mathfrak{Q}_\Delta$ be two curves with the centers of symmetry $\mathbf{c}_1$ and $\mathbf{c}_2$, respectively. Then $\mathcal{V} \star \mathcal{W}$ is decomposed into two curves with the common center $\mathbf{c}_1 + \mathbf{c}_2$. As these curves play an important role in our further considerations, we will introduce for them a notation $\mathfrak{Q}^0_\Delta$.

Lemma 4.39. *Any $\mathcal{V} \in \mathfrak{Q}_\Delta$ can be written uniquely (up to a translation) as $\mathcal{V} = \mathcal{L} \star \mathcal{Q}$, where $\mathcal{L} \in \mathfrak{L}$ and $\mathcal{Q} \in \mathfrak{Q}^0_\Delta$.*

Proof. Let $\mathcal{V}$ has the dual equation $F^{\vee}(\mathbf{n}, h) = f_{m-2}(\mathbf{n})h^2 + f_{m-1}(\mathbf{n})h + f_m(\mathbf{n}) = 0$ and $\mathcal{L}'$ be an LN curve with the dual equation $g_{n-1}(\mathbf{n})h + g_n(\mathbf{n}) = 0$. For the sake of brevity, we will write f_i and g_i instead of $f_i(\mathbf{n})$ and $g_i(\mathbf{n})$. Using Proposition 3.22, one can compute the dual equation of $\mathcal{V} \star \mathcal{L}'$

$$\begin{aligned} f_{m-2}g_{n-1}^2h^2 + &(2f_{m-2}g_n + f_{m-1}g_{n-1})g_{n-1}h + \\ &+ f_{m-2}g_n^2 + f_{m-1}g_{n-1}g_n + f_m g_{n-1}^2 = 0. \end{aligned} \tag{4.59}$$

Hence, $\mathcal{V} \star \mathcal{L}' \in \mathfrak{Q}^0_\Delta$ if and only if $2f_{m-2}g_n + f_{m-1}g_{n-1} \equiv 0$. Let us write $f_{m-i} = f \cdot \tilde{f}_{m-i}$, for $i = 1, 2$, where $f = \mathrm{GCD}(f_{m-2}, f_{m-1})$ and set $g_{n-1} := -2\tilde{f}_{m-2}$ and $g_n := \tilde{f}_{m-1}$. Then $\mathcal{L}'$ is an LN curve such that $\mathcal{Q} = \mathcal{V} \star \mathcal{L}' \in \mathfrak{Q}^0_\Delta$ and denoting $\mathcal{L} = (\mathcal{L}')^-$ we arrive at $\mathcal{V} = \mathcal{Q} \star \mathcal{L}$.

Next, let $\mathcal{V} = \mathcal{Q} \star \mathcal{L} = \hat{\mathcal{Q}} \star \hat{\mathcal{L}}$. If $\hat{\mathcal{L}} \neq \mathcal{L}$ then $\mathcal{L} \star \hat{\mathcal{L}}^-$ is a LN curve such that

$$\mathcal{Q} = \hat{\mathcal{Q}} \star \left(\mathcal{L} \star \hat{\mathcal{L}}^-\right). \tag{4.60}$$

However, it is easy to see that the convolution of a curve in $\mathfrak{Q}^0_\Delta$ with an LN curve cannot lie in $\mathfrak{Q}^0_\Delta$, which is a contradiction. □

Remark 4.40. As the curves with rational explicit support function are exactly the curves in the set $\mathfrak{D}_\Delta$ for the conic section $\mathcal{S}^\Delta$ being the unit circle, the previous lemma may be understood as an implicit reformulation of the decomposition of support functions into odd and even part in Gravesen et al. (2008).

By Lemma 4.39 it is seen that solving the problem stated by (4.57) can be reduced to finding generators of the set $\mathfrak{Q}^0_\Delta$. The following lemma shows that the dual equations of curves from $\mathfrak{Q}^0_\Delta$ possess a special form.

Lemma 4.41. *$\mathcal{V}$ lies in $\mathfrak{Q}^0_\Delta$ if and only if the defining polynomial of $\mathcal{V}^\vee$ can be written in the form*

$$F^\vee(\mathbf{n},h) = d_1(\mathbf{n})f^2(\mathbf{n})h^2 + d_2(\mathbf{n})g^2(\mathbf{n}), \tag{4.61}$$

where $d_1 \cdot d_2 = D_\Delta$.

Proof. $\mathcal{V} \in \mathfrak{Q}^0_\Delta$ if and only if it is a rational curve defined dually as $\mathcal{V}^\vee : f_{n-2}(\mathbf{n})h^2 + f_n = 0$ and its convolution with the corresponding conic section $\mathcal{S}^\Delta$ is reducible.

First, let us assume that $\mathcal{V} \star \mathcal{S}^\Delta$ has a degenerated component. Then by Theorem 4.15 $\mathcal{V} = \mathcal{S}^{\Delta^-} = \mathcal{S}^\Delta$ and hence $F^\vee(\mathbf{n},h) = h^2 + D_\Delta(\mathbf{n},h)$.

Second, we will show that $\mathcal{V} \star \mathcal{S}^\Delta$ cannot have a special component. Recalling again Theorem 4.15, we get $\mathcal{V} = \mathcal{S}^\Delta \star \mathcal{L}$, where $\mathcal{L}$ has to be an LN curve since it holds $2 = \kappa_\mathcal{V} = \kappa_{\mathcal{S}^\Delta} \cdot \kappa_\mathcal{L} = 2\kappa_\mathcal{L}$. However as mentioned at the end of the proof of Lemma 4.39, the convolution of an LN curve and a curve from $\mathfrak{Q}^0_\Delta$ is not in $\mathfrak{Q}^0_\Delta$.

Thus, we may assume that both components of $\mathcal{V} \star \mathcal{S}^\Delta$ are simple and after computing the dual equation of the convolution we arrive at

$$\begin{aligned}(f_{n-2}(\mathbf{n}))^2h^4 - 2f_{n-2}(\mathbf{n})\left(f_n(\mathbf{n}) + D_\Delta(\mathbf{n}) \cdot f_{n-2}(\mathbf{n})\right)h^2 + \\ + \left(f_n(\mathbf{n}) - D_\Delta(\mathbf{n}) \cdot f_{n-2}(\mathbf{n})\right)^2 = 0.\end{aligned} \tag{4.62}$$

Next, trying to rewrite (4.62) as a product of two polynomials quadratic in h we have to guarantee that the discriminant is a perfect square, which is equivalent to the condition

$$D_\Delta(\mathbf{n}) \cdot f_{n-2}(\mathbf{n}) \cdot f_n(\mathbf{n}) = \sigma^2(\mathbf{n}). \tag{4.63}$$

Finally, under the condition on $F^\vee(\mathbf{n},h)$ to be irreducible we arrive at (4.61). □

In addition, similarly to the case of LN curves we may use the partial fraction decomposition to obtain

$$h = \sqrt{-\frac{d_2(\mathbf{n})}{d_1(\mathbf{n})}} \cdot \frac{g(\mathbf{n})}{f(\mathbf{n})} = h_0 + \cdots + h_\ell, \tag{4.64}$$

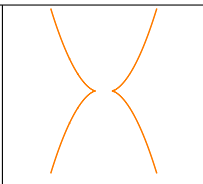
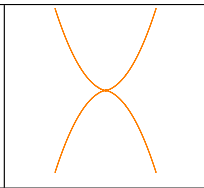
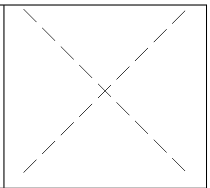
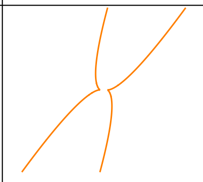
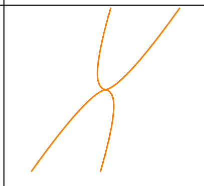
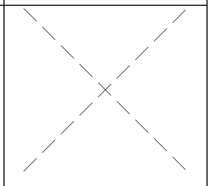
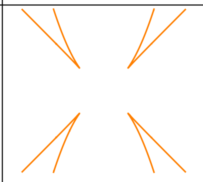
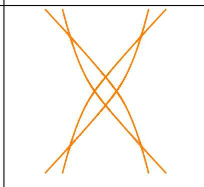
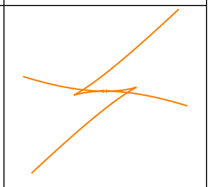
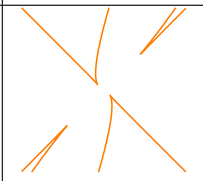
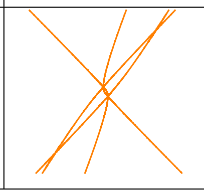
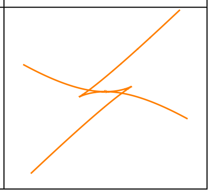
where

$$h_i = \sqrt{-\frac{d_2(\mathbf{n})}{d_1(\mathbf{n})}} \cdot \frac{\alpha_i n_1^{k+d}}{(\beta_i n_1 + \gamma_i n_2)^k} \tag{4.65}$$

for some $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $k \in \mathbb{N}$, and $d = 0$ if $d_1 = D_\Delta$, $d = 2$ if $d_2 = D_\Delta$ and $d = 1$ otherwise. Hence each $\mathcal{V} \in \mathfrak{Q}_\Delta$ is a component of the convolution $\mathcal{Q}_1 \star \cdots \star \mathcal{Q}_\ell$, where

$$\mathcal{Q}_i^\vee : d_1(\mathbf{n})(\beta_i n_1 + \gamma_i n_2)^{2k}h^2 + d_2(\mathbf{n})\alpha_i^2 n_1^{2(k+d)} = 0. \tag{4.66}$$

Table 4.1: Several examples of fundamental QN-curves. Three columns correspond to the dual equations $D_\Delta f^2h^2 + g^4 = 0$, $f^2h^2 + D_\Delta g^2 = 0$ and $\delta_1 f^2h^2 + \delta_2 g^3 = 0$, respectively, where $g = n_1$ and δ_i are nonconstant factors of D_Δ.

D_Δ	f	$(D_\Delta, 1)$	$(1, D_\Delta)$	(δ_1, δ_2)
$n_1^2 + n_2^2$	n_2			
	$n_1 + 2n_2$			
$n_1^2 - n_2^2$	n_2			
	$n_1 + 2n_2$			

Remark 4.42. Surprisingly, we do not need all four types of the dual equations determined by the decomposition $D_\Delta = d_1 \cdot d_2$. It may be shown that any curve in $\mathfrak{Q}^0_\Delta$ can be either written dually in the form $D_\Delta(\mathbf{n})f^2(\mathbf{n})h^2 + g^2(\mathbf{n}) = 0$, or it is a component of the convolution of two such curves – the decomposition can be consequently applied only to this class.

Chapter 5

Summary

In this chapter we will summarize the contribution of the thesis. It can be divided into two main topics.

The first one included in Chapter 3 consists in a general algebraic analysis of convolutions. We presented the summary of methods used for studying the convolutions and afterwards we used these methods to solve the problems, which appear throughout dealing with this operation. The different types of convolution components were introduced – namely simple, special and degenerated. It was shown that the most suitable components are the simple ones, e.g. if they are rational then each their parameterization can be obtained from the parameterizations of input hypersurfaces. Therefore, in order to avoid special and degenerated components in practise, it is convenient to understand their properties, too.

A lot of effort was devoted to an affine invariant of a given algebraic hypersurface, called the convolution degree. In the curve case the formula in closed form enabling the computation of the convolution degree of almost arbitrary curve was presented. The closed relation between the convolution degrees of input hypersurfaces and the upper bound on the number of irreducible components was given. One of another important results from this chapter is the application of Bertini's theorem to prove that the convolution is generically irreducible.

The second part of the thesis was devoted to the hypersurfaces with a low convolution degree. In particular, Chapter 4 is divided into two parts – first dealing with hypersurfaces with convolution degree one, whereas in the second part we study hypersurfaces with convolution degree two. It was shown that any hypersurface with convolution degree one is rational and moreover it admits a special parameterization such that its associated normal vector field is expressed by linear functions. This related our work to the older results on the class of LN hypersurfaces. Nevertheless the detailed algebraic analysis of convolutions of these hypersurface has not been at disposal up to the present day. Moreover the chosen approach enabled us to give more simple proofs of known facts – e.g., the proof of the statement that every hypersurface admitting quadratic polynomial parameterization admits an LN parameterization, too. In the curve case, we found a decomposition of an arbitrary LN curve into the convolution of finite number of simple fundamental LN curves. We concluded by a brief discussion on the approximation techniques of convolutions.

In Section 4.2 we showed that, although hypersurfaces with convolution degree two need not to be rational, they possess the so-called square root parameterization and hence we concluded that every curve with convolution degree two has to be rational, elliptic or hyper-elliptic. Afterwards the properties of these hypersurfaces with respect to convolution were discussed and the criterion on reducibility of convolution was given. In order to show application potential of hypersurfaces of convolution degree two, we identified the so-called QN condition, which naturally generalizes the known LN condition. To each QN hypersurface was assigned the so-called coherent form, which provided an effective criterion on the reducibility and/or rationality of convolution with a given QN hypersurfaces. Moreover their connection with the generalized Blaschke cylinder enables us to write down all the parameterizations coherent to this hypersurface. We concluded this section with two results on QN curves. First using the coherent mappings and the Arrondo-Sendra-Sendra genus formula we gave a genus formula for the irreducible convolution with QN curve. The second result – similarly to the LN curves case, showed how to decompose QN curve into the convolution of LN curve and a finite number of simple fundamental QN curves.

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