Jumping Unbounded Nonlinearities and ALP Condition

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Abstract: We investigate the existence of solutions to the nonlinear problem

$$u''(x) + \lambda_+ u^+(x) - \lambda_- u^-(x) + g(x, u(x)) = f(x)\,, \quad x \in (0, 2\pi)\,,$$
$$u(0) = u(2\pi)\,, \quad u'(0) = u'(2\pi)\,,$$

where the point $[\lambda_+, \lambda_-]$ is a point of the Fučík spectrum $\Sigma = \bigcup_{m=0}^{\infty} \Sigma_m$. We denote φ_m any nontrivial solution to our problem with $g = f = 0$ corresponding to $\lambda_+, \lambda_- \in \Sigma_m$. We assume that $g(x, s) = \gamma(x, s)s + h(x, s)$ and the nonlinearity g satisfies ALP type condition

Key-Words: Second order ODE, periodic, resonance, jumping nonlinearities, Dancer-Fucik spectrum, ALP condition, saddle point theorem.

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1 Introduction

The aim of this article is to provide new existence result for the periodic problem with unbounded jumping nonlinearities

$$u''(x) + \lambda_+ u^+(x) - \lambda_- u^-(x) + g(x, u(x)) = f(x),$$
$$x \in (0, 2\pi)\,, \quad u(0) = u(2\pi)\,, \quad u'(0) = u'(2\pi)\,, \tag{1}$$

where nonlinearity $g\colon [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory's function, $f \in L^1(0, 2\pi)$, $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$.

Studies of the behavior of suspension bridges lead to nonlinear differential equations with jumping nonlinearity. Also, the description of the behavior of electrical circuits leads to nonlinear differential equations. Piezoceramic materials exhibit different types of nonlinearities under different combinations of electric and mechanical fields. When excited near resonance in the presence of weak electric fields, they exhibit typical nonlinearities similar to a Duffing oscillator such as jump phenomena and presence of superharmonics in the response spectra. Transistors have both very non-linear region and a fairly linear one. Semiconductor-based thermistors are quite non-linear and so are light bulbs using tungsten filament. Another common source of non-linearity is ionization as maybe evidenced by operating a neon filled bulb.

To prove the existence results for nonjumping problems ($\lambda_+ = \lambda_-$) authors formulated several conditions. In 1969, a paper by Landesman and Leach [1] for a periodic problem opened the way towards what today is usually called the Landesman-Lazer condition, introduced one year later in [2] for a semilinear problem.

We can also study the periodic problems with friction $u''(x) + r(x)u'(x) + g(x, u(x)) = f(x)$ in [3] or for positive solutions see [4]. One of the latest results in this regard is [5]. The singular periodic problem is investigate in [6] by lower and upper solution. The authors of [7] use phase-plane analysis to prove the existence of a periodic solution to a nonlinear impact oscillator. The reader is referred to [8], [9] for the problem with impulsive differential equations.

A significant alternative to the Landesman-Lazer condition was proposed by Ahmad, Lazer and Paul [10] (ALP condition) in 1976, but for the bounded nonlinearity g. The ALP condition generalizes (see [11]) the classical Landesman-Lazer condition and also the potential Landesman-Lazer condition (see [12]). Therefore to relax the boundedness of g is a problem which attracted several authors' attention (see [13]). In [14] with $f \equiv 0$, the nonlinearity g is allowed to be unbounded and satisfies $|g(x, s)| \leq q(x)|s|^\alpha + h(x)$, where $0 \leq \alpha < 1$, $q, h \in L^2(0, 2\pi)$ with assumption $\lim_{|s|\to\infty} \int_0^{2\pi} G(x, s)\, dx/|s|^{2\alpha} = \infty$, where $G(x, s) = \int_0^s g(x, t)\, dt$.

The existence results for jumping problems ($\lambda_+ \neq \lambda_-$) with bounded nonlinearities g are investigated in [15], [16], with sublinear nonlinearities in [17]. In this article we obtain a solution to (1) for g with linear growth.
For $g \equiv 0$ and $f \equiv 0$ problem (1) becomes

$$\begin{gathered} u''(x) + \lambda_+ u^+(x) - \lambda_- u^-(x) = 0,\ x \in (0, 2\pi), \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi)\,. \end{gathered} \tag{2}$$

It is well known (see [18]) that problem (2) has nontrivial solutions only when the pairs (λ_+, λ_-) lies in the set of points made up of the curves

$$\begin{aligned} \Sigma_0 &= \{\, [\lambda_+, \lambda_-] \in \mathbb{R}^2 \mid \lambda_+ \lambda_- = 0 \,\}\,, \\ \Sigma_m &= \{\, [\lambda_+, \lambda_-] \in \mathbb{R}^2 \mid m \left(\tfrac{1}{\sqrt{\lambda_+}} + \tfrac{1}{\sqrt{\lambda_-}} \right) = 2 \,\}\,, \end{aligned}$$

where $m \in \mathbb{N}$. The set $\Sigma = \bigcup\limits_{m=0}^{\infty} \Sigma_m$ is called the Fučík spectrum.

Using the Landesman-Lazer type conditions authors usually suppose that g satisfies the linear growth restriction $|g(x,s)| \le q(x)|s| + h(x)$ and there are functions $a, A \in L^1(0, 2\pi)$, constants $r, R \in \mathbb{R}$ such that $g(x,s) \ge A(x)$ for a.e. $x \in [0, 2\pi]$ and all $s \ge R$ and $g(x,s) \le a(x)$ for a.e. $x \in [0, 2\pi]$ and all $s \le r$ (see [19]). These conditions imply our assumptions (see also [20]), that is the function g can be decomposed as

$$g(x,s) = \gamma(x,s)s + h(x,s)\,, \tag{3}$$

where

$$0 \le \gamma(x,s) \le q_1(x)\,, \quad |h(x,s)| \le q_2(x) \tag{4}$$

for a.e. $x \in (0, 2\pi)$, for all $s \in \mathbb{R}$, with some $q_1, q_2 \in L^1(0, 2\pi)$. Moreover $\lambda_+ \ge \lambda_-$, $[\lambda_+, \lambda_-] \in \Sigma_m$, $m \in \mathbb{N}$ and there exists $\varepsilon > 0$ such that

$$\begin{aligned} \limsup_{s \to +\infty} \tfrac{g(x,s)}{s} &\le (m+1)^2 - \lambda_+ - \varepsilon\,, \\ \limsup_{s \to -\infty} \tfrac{g(x,s)}{s} &\le (m+1)^2 - \lambda_- - \varepsilon\,. \end{aligned} \tag{5}$$

We denote φ_m any nontrivial solution to (2) corresponding to $[\lambda_+, \lambda_-] \in \Sigma_m$. We shall suppose the following ALP type conditions

$$\lim_{|s| \to \infty} \int_0^{2\pi} [G(x, s\,\varphi_m(x)) - f(x)\, s\,\varphi_m(x)]\, dx = +\infty \tag{6}$$

and

$$\liminf_{|s| \to \infty} \int_0^{2\pi} [H(x, s\,\varphi_m(x)) - f(x)\, s\,\varphi_m(x)]\, dx \ge c_1 \tag{7}$$

with some constant c_1, where $H(x,s) = \int\limits_0^s h(x,t)\, dt$.

If the nonlinearity g is L^1-bounded (as in [10]) then clearly (6) implies (7). We obtain for example the existence result to the equation (1) with the nonlinearity $g(x,s) = s/(1+s^2) + f(x)$ or $g(x,s) = [(m+1)^2 - \lambda_+ - \varepsilon]\,|\sin s|\, s + f(x)$ if $\lambda_+ \ge \lambda_-$.

2 Preliminaries

We shall use the Lebesgue space $L^p(0, 2\pi)$ with the norm $\|u\|_p$. We denote by H the Sobolev space 2π-periodic absolutely continuous functions $u : \mathbb{R} \to \mathbb{R}$ such that $u' \in L^2(0, 2\pi)$ endowed with the norm $\|u\| = \left(\int_0^{2\pi} u^2\, dx + \int_0^{2\pi} (u')^2\, dx \right)^{1/2}$.

By a solution to (1) we mean a function u in $W^{2,1}(0, 2\pi)$ such that the equation (1) is satisfied a.e. on $(0, 2\pi)$ and $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$.

We study (1) by using of variational method. More precisely, we look for critical points of the functional $I : H \to \mathbb{R}$, which is defined by

$$\begin{aligned} I(u) = \frac{1}{2} \int_0^{2\pi} & [(u')^2 - \lambda_+ (u^+)^2 - \lambda_- (u^-)^2]\, dx \\ & - \int_0^{2\pi} [G(x,u) - fu]\, dx\,. \end{aligned} \tag{8}$$

Every critical point $u \in H$ of the functional I satisfies

$$\begin{aligned} &\int_0^{2\pi} [u'v' - (\lambda_+ u^+ - \lambda_- u^-) v]\, dx \\ &- \int_0^{2\pi} [g(x,u)v - fv]\, dx = 0 \quad \text{for all } v \in H\,. \end{aligned}$$

Then u is also a weak solution to (1) and vice versa.

The usual regularity argument for ODE yields immediately (see Fučík [18]) that any weak solution to (1) is also the solution in the sense mentioned above.

We say that I satisfies Palais-Smale condition (PS) if every sequence (u_n) for which I is bounded in H and $I'(u_n) \to 0$ (as $n \to \infty$) contains a convergent subsequence.

To obtain a critical point of the functional I we will use the following variant of Saddle Point Theorem (see [21]), which is proved in Struwe [21, Theorem 8.4].

Theorem 1 *Let V, H^+ be closed subsets in H, $H = V \oplus H^+$ and Q a bounded subset in V with boundary ∂Q. Set $\Gamma = \{h :\ h \in \mathbf{C}(H,H),\ h(u) = u \ \text{ on } \partial Q\,\}$. Suppose $I \in C^1(H, \mathbb{R})$ and*

(i) $H^+ \cap \partial Q = \emptyset$,

(*ii*) $H^+ \cap h(Q) \neq \emptyset$, *for every* $h \in \Gamma$,

(*iii*) *there are constants* μ, ν *such that* $\mu = \inf_{u \in H^+} I(u) > \sup_{u \in \partial Q} I(u) = \nu$,

(*iv*) *I satisfies Palais-Smale condition.*

Then the number

$$\gamma = \inf_{h\in\Gamma} \sup_{u\in Q} I(h(u))$$

defines a critical value $\gamma > \nu$ *of* I.

We say that H^+ and ∂Q *link* if they satisfy conditions i), ii) of the theorem above.

A simple example of a function that has saddle point geometry is the function $f(x,y) = x^2 - y^2$. In finite-dimensional spaces, the Palais–Smale condition for a continuously differentiable real-valued function is satisfied automatically for proper maps: functions which do not take unbounded sets into bounded sets. For nonproper maps and in infinite-dimensional function spaces, however, we need the PS condition because some other notion of compactness is needed in addition to simple boundedness. For example, the function $f(x,y) = (x^2y - x - 1)^2 + (x^2-1)^2$ does not satisfy the PS condition, see the sequence $(\frac{1}{2n} + \frac{1}{2n^2}, n + \frac{1}{n})$.

We use result from [16, section 2] to assert that any nontrivial solution to the boundary-value problem (2) corresponding to $[\lambda_+, \lambda_-] \in \Sigma_m$, $m \in \mathbb{N}$ must be a translate, or phase shift, of a positive multiple of the function $\varphi_m : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi_m(x) = \begin{cases} \sqrt{\lambda_-}\sin(\sqrt{\lambda_+}x), \\ \quad x \in [0, \frac{\pi}{\sqrt{\lambda_+}}), \\ -\sqrt{\lambda_+}\sin(\sqrt{\lambda_-}(x - \frac{\pi}{\sqrt{\lambda_+}})), \\ \quad x \in [\frac{\pi}{\sqrt{\lambda_+}}, \frac{\pi}{\sqrt{\lambda_+}} + \frac{\pi}{\sqrt{\lambda_-}}), \\ \sqrt{\lambda_-}\sin(\sqrt{\lambda_+}(x - \frac{\pi}{\sqrt{\lambda_+}} - \frac{\pi}{\sqrt{\lambda_-}})), \\ \quad x \in [\frac{\pi}{\sqrt{\lambda_+}} + \frac{\pi}{\sqrt{\lambda_-}}, \frac{2\pi}{\sqrt{\lambda_+}} + \frac{\pi}{\sqrt{\lambda_-}}), \\ \quad \vdots \\ -\sqrt{\lambda_+}\sin(\sqrt{\lambda_-}(x - (2\pi - \frac{\pi}{\sqrt{\lambda_-}})), \\ \quad x \in [2\pi - \frac{\pi}{\sqrt{\lambda_-}}, 2\pi] \end{cases}$$

after it has been extended to be 2π-periodic over all of $\mathbb{R}$.

We denote $\theta_1 = \pi/(2\sqrt{\lambda_+})$ and

$$\varphi_\theta(x) = \varphi_m(x + \theta_1 - \theta)\,, \quad x \in [0, 2\pi]\,, \tag{9}$$

where $\theta \in [0, 2\pi]$, then $\varphi_\theta(x)$ is a nontrivial solution to the boundary-value problem (2) corresponding to $[\lambda_+, \lambda_-] \in \Sigma_m$, $m \in \mathbb{N}$.

Let H^- be the subspace of H spanned by $1, \sin x, \cos x, \sin 2x, \ldots, \sin(m-1)x, \cos(m-1)x$. For $K > 0$, $L > 0$, we define sets

$$\begin{aligned} V &= \{u \in H : u = a\varphi_\theta + w,\ \theta \in [0, 2\pi],\ a \in \mathbb{R}_0^+, \\ &\qquad w \in H^-\}\,, \\ Q &= \{u \in V : 0 \le a \le K,\ \|w\| \le L\}\,. \end{aligned} \tag{10}$$

Let H^+ be the subspace of H spanned by $\sin(m+1)x, \cos(m+1)x, \sin(m+2)x, \cos(m+2)x, \ldots$.

Next, we verify the assumptions (i) of Theorem 1 and assumption $H = V \oplus H^+$.

Lemma 1 *It holds*

$$H^+ \cap \partial Q = \emptyset\,. \tag{11}$$

Proof *We suppose for contradiction that there is* $u \in \partial Q \cap H^+$. *We denote* $\langle\cdot,\cdot\rangle$ *the inner product in* $L^2(0, 2\pi)$. *Then*
$0 \overset{u\in H^+}{=} \langle u, \sin mx\rangle \overset{u\in\partial Q}{=} \langle K\varphi_\theta + w, \sin mx\rangle \overset{w\in H^-}{=} K\langle\varphi_\theta, \sin mx\rangle \overset{K>0}{=} \langle\varphi_\theta, \sin mx\rangle\,.$

Similarly $\langle\varphi_\theta, \cos mx\rangle = 0$. *It is easy to see that* $\langle\varphi_\theta, \sin mx\rangle = 0$ *(see figure 1) only for* $\theta = k\pi/m$, $k \in \mathbb{Z}$. *But* $\langle\varphi_{k\pi/m}, \cos mx\rangle \neq 0$ *a contradiction.*

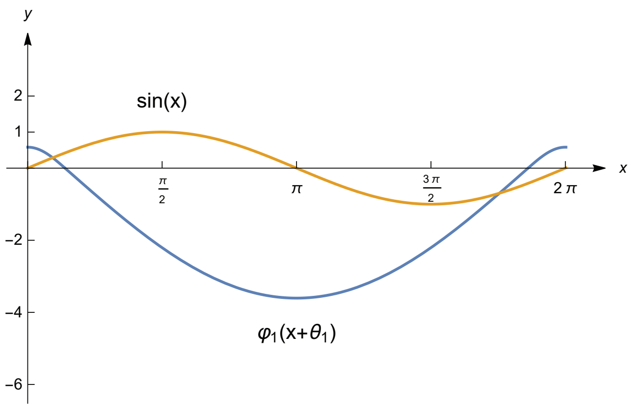

Figure 1: Solution $\varphi_\theta(x) = \varphi_1(x + \theta_1 - \theta)$ to (2) for $\theta = 0$

Lemma 2 *It holds*

$$H = V \oplus H^+. \tag{12}$$

Proof *To prove this lemma, we first need to show that an arbitrary element* u *of* H *can be expressed in the form*

$$u = v + h\,, \tag{13}$$

where $v \in V$ *and* $h \in \mathbb{H}^+$. *To establish (13), we observe that every* $u \in H$ *can be written in the form*

$$u(x) = \overline{u}(x) + a_m \cos mx + b_m \sin mx + \widetilde{u}(x)\,, \quad (14)$$

for all $x \in [0, 2\pi]$, *and some constants* a_m, b_m, *where* $\overline{u} \in H^-$ *and* $\widetilde{u} \in H^+$. *We want to show that we can also write* u *in the form*

$$u(x) = \overline{u}_1(x) + \varrho\varphi_\theta(x) + \widetilde{u}_1(x)\,, \quad (15)$$

for some constants $\varrho > 0$ *and* $\theta \in [0, 2\pi]$, *where* $\overline{u}_1 \in H^-$ *and* $\widetilde{u}_1 \in H^+$. *Taking inner products with* $\cos mx$ *and* $\sin mx$ *in (14) and (15) gives rise to the system*

$$\begin{aligned} \varrho\langle\varphi_\theta, \cos mx\rangle &= \pi a_m \\ \varrho\langle\varphi_\theta, \sin mx\rangle &= \pi b_m\,. \end{aligned} \quad (16)$$

We denote $p(\theta) = \langle\varphi_\theta, \sin mx\rangle$ *then* $p(0) = 0$ *(see figure 1) and*

$$\begin{aligned} p(\theta) &= \int_0^{2\pi} \varphi_m(x + \theta_1 - \theta) \sin mx \, dx \\ &= \left\{ y = x + \theta_1 - \theta \right\} \\ &= \int_{\theta_1-\theta}^{2\pi+\theta_1-\theta} \varphi_m(y) \sin(m(y - \theta_1 + \theta)) \, dy \\ &= \int_0^{2\pi} \varphi_m(y) \sin(m(y - \theta_1 + \theta)) \, dy\,, \end{aligned} \quad (17)$$

since the integrated functions are 2π*-periodic. Hence function* p *satisfies* $p''(\theta) = -m^2 p(\theta)$, *thus* $p(\theta) = c \sin m\theta\,,\ c > 0\,.$

Therefore we can rewrite (16) to the system

$$\begin{aligned} \varrho\, c \cos m\theta &= \pi a_m \\ \varrho\, c \sin m\theta &= \pi b_m\,. \end{aligned} \quad (18)$$

Hence, the system in (16) is solvable for any a_m *and* b_m *in* $\mathbb{R}$ *and there exist* $\varrho_m \geq 0$ *and* $\theta_m \in [0, (2\pi)/m]$ *such that*

$$\varrho_m \varphi_{\theta_m}(x) = h_1(x) + a_m \cos mx + b_m \sin mx + h_2(x), \quad \text{for all } x \in [0, 2\pi]\,, \quad (19)$$

where $h_1 \in H^-$ *and* $h_2 \in H^+$.

Next, solve for $a_m \cos mx + b_m \sin mx$ *in (19) and substitute into the expansion for* u *in (14) to obtain the representation in (15), where* $\overline{u}_1 = \overline{u} - \overline{h}$ *and* $\widetilde{u}_1 = \widetilde{u} - \widetilde{h}$. *We have therefore proved that* $H = V + H^+$. *To complete the proof of (12), we need to show that* $V \cap H^+ = \{0\}$. *We can repeat the steps from the proof of lemma 1. For* $u \in V \cap H^+$ *we obtain:*

$$0 \overset{u \in H^+}{=} \langle u, \sin mx\rangle \overset{u \in V}{=} \langle a\varphi_\theta + w, \sin mx\rangle \overset{w \in H^-}{=} a\langle\varphi_\theta, \sin mx\rangle$$

and similarly $a\langle\varphi_\theta, \cos mx\rangle = 0$. *Hence* $a = 0, u = 0$ *and* $V \cap H^+ = \{0\}$, *the proof is complete. We have proved that* H *is spanned by* V *and* H^+.

We denote the first integral in the functional I by $J(u) = \int_0^{2\pi} [(u')^2 - \lambda_+(u^+)^2 - \lambda_-(u^-)^2] \, dx$. and formulate the following lemma, which is proved in [12, Lemma 2.2].

Lemma 3 *Let* φ *be a solution to (2) with* $[\lambda_+, \lambda_-] \in \Sigma_m, m \in \mathbb{N},\ \lambda_+ \geq \lambda_-$. *We put* $u = a\varphi + w,\ a \geq 0$, $w \in H$. *Then it holds*

$$\int_0^{2\pi} [(w')^2 - \lambda_+ w^2] \, dx \leq J(u) \leq \int_0^{2\pi} [(w')^2 - \lambda_- w^2] \, dx. \quad (20)$$

We will also use the following nonexistence of particular nontrivial solution to a BVP like (1) (see [22, Theorem 8, remarks 2]).

Lemma 4 *Let* $\gamma_\pm$ *be two maps in* $L^\infty(0, 2\pi)$. *There exists* $m \in \mathbb{N}$, *two points* $[\lambda_{+,m}, \lambda_{-,m}] \in \Sigma_m$, $[\lambda_{+,m+1}, \lambda_{-,m+1}] \in \Sigma_{m+1}$ *such that on* $[0, 2\pi]$

$$\lambda_{\pm,m} \lneqq \gamma_\pm(x) \lneqq \lambda_{\pm,m+1} \quad (21)$$

($\lambda_{\pm,m} \neq \gamma_\pm(x)$ *and also* $\gamma_\pm(x) \neq \lambda_{\pm,m+1}$ *on a set of positive measure), then the problem*

$$\begin{aligned} u''(x) + \gamma_+(x)u^+(x) - \gamma_-(x)u_-(x) = 0\,, \\ u(0) = u(2\pi)\,, \quad u'(0) = u'(2\pi) \end{aligned} \quad (22)$$

has only the trivial solution $u(x) \equiv 0$.

3 Main result

Theorem 2 *Let* $[\lambda_+, \lambda_-] \in \Sigma_m$, $m \in \mathbb{N}$, $\lambda_+ \geq \lambda_-$. *Under the assumptions (3), (4),(5), (6) and (7) Problem (1) has at least one solution in* H.

We shall prove that the functional I defined by (8) satisfies the assumptions in Theorem 1 (Saddle Point Theorem).

i) We infer from Lemmas 1, 2 that $H = V \oplus H^+$ and $\partial Q \cap H^+ = \emptyset$.

ii) The proof of the assumption $H^+ \cap h(Q) \neq \emptyset$ $\forall h \in \Gamma$ is similar to the proof in [13, example 8.2].

Let $\pi\colon H \to V$ be the continuous projection of H onto V. We have to show that $0 \in \pi(h(Q))$. For $t \in [0, 1]$, $u \in Q$ we define $h_t(u) = t\pi(h(u)) + (1-t)u$. Function h_t defines a homotopy of $h_0 = id$ with $h_1 = \pi \circ h$. Moreover, $h_t|\partial Q = \mathrm{id}$ for all $t \in [0, 1]$. Hence the topological degree $\deg(h_t, Q, 0)$ is well-defined and by homotopy invariance we have $\deg(\pi \circ$

$h, Q, 0) = \deg(id, Q, 0) = 1$. Hence $0 \in \pi(h(Q))$, as was to be shown.

iii) Firstly, we note that by (4), (5), we get

$$\begin{aligned}
0 &\le \liminf_{|s|\to\infty} \frac{g(x,s)}{s}\,, \\
0 &\le \liminf_{|s|\to\infty} \frac{G(x,s)}{s^2} \\
&\le \limsup_{s\to\pm\infty} \frac{G(x,s)}{s^2} \le \frac{(m+1)^2 - \lambda_\pm - \varepsilon}{2}
\end{aligned} \tag{23}$$

for a.e. $x \in [0, 2\pi]$. Now we estimate the functional I on the space H^+, we prove that

$$\lim_{\|u\|\to\infty} I(u) = \infty \quad \text{for all } u \in H^+ . \tag{24}$$

Since $u \in H^+$, we have

$$\int_0^{2\pi} (u')^2\,dx \ge (m+1)^2 \int_0^{2\pi} u^2\,dx . \tag{25}$$

The definition of I, (23), and (25) yield

$$\begin{aligned}
\liminf_{\|u\|\to\infty} \frac{I(u)}{\|u\|^2} &= \liminf_{\|u\|\to\infty} \frac{1}{\|u\|^2}\Big[\frac{1}{2}\int_0^{2\pi}[(u')^2 - \lambda_+(u^+)^2 \\
&-\lambda_-(u^-)^2]\,dx - \int_0^{2\pi}[G(x,u) - fu]\,dx\Big] \\
&\ge \liminf_{\|u\|\to\infty} \frac{1}{\|u\|^2}\Big[\frac{1}{2}\int_0^{2\pi}[(m+1)^2 u^2 - \lambda_+(u^+)^2 \\
&-\lambda_-(u^-)^2]\,dx - \int_0^{2\pi}\frac{G(x,u)}{u^2}u^2\,dx\Big] \\
&\ge \liminf_{\|u\|\to\infty} \frac{\varepsilon}{2}\frac{\|u\|_2^2}{\|u\|^2}.
\end{aligned} \tag{26}$$

If $\liminf_{\|u\|\to\infty} \|u\|_2^2/\|u\|^2 = 0$ then it follows from the definition of I and (23) that

$$\liminf_{\|u\|\to\infty} \frac{I(u)}{\|u\|^2} = \frac{1}{2}\,. \tag{27}$$

Then (26) and (27) imply $\liminf_{\|u\|\to\infty} I(u) = \infty$. It follows from (24) and the fact that H^+ is compactly embedded in $C[0, 2\pi]$ that there exists a real number, μ, such that $I(u) \ge \mu$ for all $u \in H^+$; in fact, we may take μ to be defined by

$$\mu = \inf_{u\in H^+} I(u)\,. \tag{28}$$

We will next show that we can pick $K > 0$ and $L > 0$ such that $\sup_{u\in\partial Q} I(u) < \mu$, where $Q = \{u \in H : u = a\varphi_\theta + w,\ 0 \le a \le K, w \in H^-, \|w\| \le L, \theta \in [0, 2\pi]\}$, where φ_θ is given in (9). We argue by contradiction. Suppose that $\sup_{\|u\|\to\infty} I(u) = -\infty$ for $u \in \partial Q$ is not true. Then there is a sequence $(u_n) \subset \partial Q$ such that $\|u_n\| \to \infty$ and a constant c_- satisfying

$$\liminf_{n\to\infty} I(u_n) \ge c_-\,. \tag{29}$$

Due to (23) $\liminf_{n\to\infty} \int_0^{2\pi}(G(x,u_n) - fu_n)/\|u_n\|^2\,dx \ge 0$. Hence from the definition of I and (29) we have

$$\liminf_{n\to\infty} \frac{1}{2}\int_0^{2\pi} \frac{(u_n')^2 - \lambda_+(u_n^+)^2 - \lambda_-(u_n^-)^2}{\|u_n\|^2}\,dx \ge 0\,. \tag{30}$$

We denote $v_n = u_n/\|u_n\|$ and we proceed as in [16, pg.24]. Then,

$$v_n \in \partial B \cap V, \qquad \text{for all } n \in \mathbb{N}\,, \tag{31}$$

where B denotes the closed unit ball in H, and V is as defined in (10) ($V = \{u \in H : u = a\varphi_\theta + w,\ 0 \le a, w \in H^-\}$); so that $\partial B \cap V$ lives in a finite dimensional subspace of H (see [16, Remark 3.4]). We also have, that

$$v_n = a_n\varphi_{\theta_n} + z_n\,, \tag{32}$$

where

$$z_n \in B \cap H^-\,, \quad a_n \in [0, 1/r]\,, \tag{33}$$

where $r = \|\varphi_\theta\|$. Using the compactness of $B \cap H^-$ and the closed intervals $[0, 1/r]$ and $[0, 2\pi]$, we may assume, as a consequence of (32), (33), that

$$v_n \to v_0 \quad \text{in } H, \tag{34}$$

where

$v_0 = a_0\varphi_{\theta_0} + z_0,\ a_0 \in [0, 1/r],\ \theta_0 \in [0, 2\pi],\ z_0 \in B\cap H^-$.

Therefore, letting $n \to \infty$, using (30) and (34) we obtain

$$\int_0^{2\pi} [(v_0')^2 - \lambda_+(v_0^+)^2 - \lambda_-(v_0^-)^2]\,dx \ge 0\,. \tag{35}$$

By lemma 3 we have for $v_0 \in V,\ v_0 = a_0\varphi_{\theta_0} + z_0$

$$\begin{aligned}
&\int_0^{2\pi} [(v_0')^2 - \lambda_+(v_0^+)^2 - \lambda_-(v_0^-)^2]\,dx \\
&\le \int_0^{2\pi} [(z_0')^2 - \lambda_- z_0^2]\,dx\,, \quad z_0 \in H^-.
\end{aligned} \tag{36}$$

By (35), (36) we get

$$0 \le \int_0^{2\pi} [(z_0')^2 - \lambda_- z_0^2]\,dx\,. \tag{37}$$

We note that $0 \le \liminf_{|s|\to\infty} g(x,s)/s \le \limsup_{|s|\to\infty} g(x,s)/s$, thus (5) implies $\lambda_+ \le (m +$

$1)^2-\varepsilon$ with some $\varepsilon>0$. Since $1/\sqrt{\lambda_+}+1/\sqrt{\lambda_-}=2/m$ we obtain

$$\begin{aligned}&\frac{1}{\sqrt{\lambda_-}}<\frac{2}{m}-\frac{1}{m+1}=\frac{m+2}{m(m+1)}\\&\Rightarrow \sqrt{\lambda_-}>\frac{m(m+1)}{m+2}>m-1\,.\end{aligned} \tag{38}$$

We denote $\delta=\lambda_- -(m-1)^2>0$. Therefore by (37) we get

$$0\le\int_0^{2\pi}[(z_0')^2-((m-1)^2+\delta)\,z_0^2]\,dx\,. \tag{39}$$

We note that for $z_0\in H^-$ it holds

$$\int_0^{2\pi}[(z_0')^2-(m-1)^2z_0^2]\,dx\le 0\,. \tag{40}$$

Combining (39) with (40) we deduce that $z_0\equiv 0$ and $v_0=a_0\varphi_{\theta_0}$, where $a_0=1/\|\varphi_{\theta_0}\|$ and φ_{θ_0} is a nontrivial solution to the homogeneous boundary-value problem (2) corresponding to $[\lambda_+,\lambda_-]\in\Sigma_m$, we denote $\varphi_{m_0}=a_0\varphi_{\theta_0}$.
Because of the compact imbedding $H\subset C(0,2\pi)$ and (34), we have $v_n\to\varphi_{m_0}(x)$ in $C(0,2\pi)$ and

$$\lim_{n\to\infty}u_n(x)=\begin{cases}+\infty & \text{where } \varphi_{m_0}(x)>0\,,\\ -\infty & \text{where } \varphi_{m_0}(x)<0\,.\end{cases} \tag{41}$$

We return to (29) and firstly estimate by lemma 3 using (40) (with $z_0=w_n\in H^-$) the first integral in $I(u_n)$

$$\begin{aligned}&\int_0^{2\pi}(u_n')^2-\lambda_+(u_n^+)^2-\lambda_-(u_n^-)^2\,dx\\&\le\int_0^{2\pi}[(w_n')^2-\lambda_-w_n^2]\,dx\\&=\int_0^{2\pi}[(w_n')^2+w_n^2-(\lambda_-+1)w_n^2]\,dx\\&=\|w_n\|^2-((m-1)^2+\delta+1)\|w_n\|_2^2\\&\le\|w_n\|^2-\frac{(m-1)^2+\delta+1}{(m-1)^2+1}\|w_n\|^2\\&=-\frac{\delta}{(m-1)^2+1}\|w_n\|^2\end{aligned} \tag{42}$$

since $\|w_n\|^2\le((m-1)^2+1)\|w_n\|_2^2$. By (29) and (42) we obtain

$$\begin{aligned}\liminf_{n\to\infty}\Big(&-\frac{\delta}{2((m-1)^2+1)}\,\|\,w_n\|^2\\&-\int_0^{2\pi}[G(x,u_n)-fu_n]\,dx\Big)\ge c_-\,.\end{aligned}$$

We denote $c_m=\frac{\delta}{2((m-1)^2+1)}>0$, then equivalently

$$\limsup_{n\to\infty}\Big(c_m\|\,w_n\|^2+\int_0^{2\pi}[G(x,u_n)-fu_n]\,dx\Big)\le -c_-. \tag{43}$$

We use the decomposition (3) of $g(x,s)=\gamma(x,s)s+h(x,s)$ and denote $\Gamma(x,s)=\int_0^s\gamma(x,t)\,t\,dt$, we rewrite (43) into

$$\begin{aligned}\limsup_{n\to\infty}\Big(c_m\,\|\,w_n\|^2+\int_0^{2\pi}[\Gamma(x,u_n)\\+H(x,u_n)-fu_n]\,dx\Big)\le -c_-.\end{aligned} \tag{44}$$

By the mean value theorem, (3),(4) and the compact embedding H into $C([0,2\pi])$ ($\|\cdot\|_{C([0,2\pi])}\le c_2\|\cdot\|$) we obtain

$$\begin{aligned}&\int_0^{2\pi}[H(x,u_n)-H(x,a_n\varphi_{m_0})]\,dx\\&=\int_0^{2\pi}[h(x,\xi_n(x))\,w_n)]\,dx\le\|q_2\|_1c_2\|w_n\|\,,\end{aligned} \tag{45}$$

where $\xi_n(x)\in(a_n\varphi_{m_0}(x),u_n(x))$.
Similarly $\int_0^{2\pi}fw_n\le\|f\|_1c_2\|w_n\|$. Therefore by (44), (45) we get $\limsup_{n\to\infty}\Big(c_m\|w_n\|^2-(\|f\|_1+\|q_2\|_1)c_2\|w_n\|+\int_0^{2\pi}[\Gamma(x,u_n)+H(x,a_n\varphi_{m_0})-fa_n\varphi_{m_0}]\,dx\Big)\le -c_-$ and consequently there exists a constant c_3 such that

$$\begin{aligned}&\limsup_{n\to\infty}\int_0^{2\pi}[\Gamma(x,u_n)\\&+H(x,a_n\varphi_{m_0})-fa_n\varphi_{m_0}]\,dx\le c_3\,.\end{aligned} \tag{46}$$

For a.e. $x\in(0,2\pi)$ function $\Gamma(x,s)$ is nonincreasing for $s<0$; $\Gamma(x,0)=0$ and $\Gamma(x,s)$ is nondecreasing for $s>0$. Hence we get

$$\lim_{n\to\infty}\int_0^{2\pi}\Gamma(x,u_n)\,dx=\lim_{n\to\infty}\int_0^{2\pi}\Gamma(x,a_n\varphi_{m_0})\,dx \tag{47}$$

since $\lim_{n\to\infty}u_n(x)=\lim_{n\to\infty}a_n\varphi_{m_0}(x)=+\infty$ for $x\in(0,2\pi)$ such that $\varphi_{m_0}(x)>0$, and $\lim_{n\to\infty}u_n(x)=$

$\lim_{n\to\infty} a_n\varphi_{m_0} = -\infty$ for $x \in (0,2\pi)$ such that $\varphi_{m_0}(x) < 0$. We rewrite condition (6) in the following form

$$\lim_{n\to\infty}\int_0^{2\pi} [\Gamma(x, a_n\varphi_{m_0}(x)) + H(x, a_n\varphi_{m_0}(x)) - f a_n\varphi_{m_0}(x)]\,dx = \infty\,. \tag{48}$$

If the limit in (47) is finite we obtain a contradiction to (46), (48). If the limit in (47) is infinite we obtain a contradiction to (46) and assumption (7). Hence $\sup_{\|u\|\to\infty} I(u) = -\infty$ for $u \in \partial Q$ and we have showed that we can pick $K > 0$ and $L > 0$ such that

$$\mu = \inf_{u\in H^+} I(u) > \sup_{u\in\partial Q} I(u) = \nu\,.$$

iv) For Assumption (iv) of theorem 1, we show that functional I satisfies the Palais-Smale condition.

For contradiction we suppose that the sequence (u_n) is unbounded and there exists a constant c_4 such that

$$\Big|\frac{1}{2}\int_0^{2\pi} (u_n')^2 - \lambda_+(u_n^+)^2 - \lambda_-(u_n^-)^2\,dx - \int_0^{2\pi}[G(x,u_n) - fu_n]\,dx\Big| \le c_4 \tag{49}$$

and

$$\lim_{n\to\infty}\|I'(u_n)\| = 0\,. \tag{50}$$

Let (w_k) be an arbitrary sequence bounded in H. It follows from (50) and the Schwarz inequality

$$\Big|\lim_{\substack{n\to\infty\\k\to\infty}}\int_0^{2\pi}[u_n'w_k' - (\lambda_+u_n^+ - \lambda_-u_n^-)w_k]\,dx - \int_0^{2\pi}[g(x,u_n)w_k - fw_k]\,dx\Big| = |\lim_{\substack{n\to\infty\\k\to\infty}}\langle I'(u_n), w_k\rangle| \le \lim_{\substack{n\to\infty\\k\to\infty}}\|I'(u_n)\|\cdot\|w_k\| = 0\,. \tag{51}$$

Since $\int_0^{2\pi}[(f/\|u_n\|)w_k]\,dx \to 0$ we obtain by (51)

$$\lim_{\substack{n\to\infty\\m\to\infty\\k\to\infty}}\left(\int_0^{2\pi}\Big[\Big(\frac{u_n'}{\|u_n\|}-\frac{u_m'}{\|u_m\|}\Big)w_k' - \Big(\lambda_+\Big(\frac{u_n^+}{\|u_n\|}-\frac{u_m^+}{\|u_m\|}\Big)-\lambda_-\Big(\frac{u_n^-}{\|u_n\|}-\frac{u_m^-}{\|u_m\|}\Big)\Big)w_k\Big]dx - \int_0^{2\pi}\Big[\Big(\frac{g(x,u_n)}{\|u_n\|} - \frac{g(x,u_m)}{\|u_m\|}\Big)w_k\Big]\,dx\right) = 0\,. \tag{52}$$

We put $v_n = u_n/\|u_n\|$ and $w_k = v_n - v_m$ in (52), we conclude

$$\lim_{\substack{n\to\infty\\m\to\infty}}\left(\int_0^{2\pi}(v_n'-v_m')^2\,dx - \int_0^{2\pi}\Big[(\lambda_+(v_n^+-v_m^+)-\lambda_-(v_n^- -v_m^-))(v_n-v_m)\Big]dx - \int_0^{2\pi}\Big[\Big(\frac{g(x,u_n)}{\|u_n\|}-\frac{g(x,u_m)}{\|u_m\|}\Big)(v_n-v_m)\Big]dx\right)=0. \tag{53}$$

Due to compact imbedding $H\subset L^2(0,2\pi), C([0,2\pi])$ there is $v_0 \in H$ such that (up to subsequence) $v_n \rightharpoonup v_0$ weakly in H, $v_n \to v_0$ strongly in $L^2(0,2\pi)$, $C([0,2\pi])$. Due to assumption (3), (4) the sequence $(g(x,u_n)/\|u_n\|)$ is L^1-bounded, thus (53) implies $v_n \to v_0$ strongly in H.

It follows from assumptions (3), (4), (5) (up to subsequence) that

$$\frac{g(x,u_n)}{\|u_n\|} = \frac{\gamma(x,u_n)\,u_n}{\|u_n\|} + \frac{h(x,u_n)}{\|u_n\|} \rightharpoonup \gamma_0^+(x)v_0^+ - \gamma_0^-(x)v_0^- \quad \text{in } L^1(0,2\pi), \tag{54}$$

where $0 \le \gamma_0^+(x) \le (m+1)^2 - \lambda_+ - \varepsilon$, $0 \le \gamma_0^-(x) \le (m+1)^2 - \lambda_- - \varepsilon$ for a.e. $x \in (0,2\pi)$, since the sequence $\gamma_n(x) := \gamma(x,u_n(x))$ is both bounded and equi-integrable in $L^1(0,2\pi)$ (see Dunford, Schwarz [24]). We get from (51) and (54)

$$\int_0^{2\pi}[v_0'w' - ((\lambda_+ + \gamma_0^+)v_0^+ - (\lambda_- + \gamma_0^-)v_0^-)w]\,dx = 0 \ \text{ for all } w \in H. \tag{55}$$

It follows from (54), (55) and from the usual regularity argument for ordinary differential equations (see Fučík [18]) that v_0 is a solution with norm $\|v_0\| = 1$ to the periodic BVP

$$v_0'' - (\lambda_+ + \gamma_0^+)v_0^+ + (\lambda_- + \gamma_0^-)v_0^- = 0$$
$$x \in (0,2\pi),\ \ v_0(0) = v_0(2\pi),\ \ v_0'(0) = v_0'(2\pi), \tag{56}$$

where by (38)

$$m^2 \le \lambda_+ \le \lambda_+ + \gamma_0^+(x) \le (m+1)^2 - \varepsilon,$$
$$(m-1)^2 < (m-1)^2 + \delta = \lambda_- \le \lambda_- + \gamma_0^-(x) \le (m+1)^2 - \varepsilon \tag{57}$$

for a.e. $x \in (0, 2\pi)$. Therefore using lemma 4 with $[\lambda_+, \lambda_-] \in \Sigma_m$, $[(m+1)^2, (m+1)^2] \in \Sigma_{m+1}$ equation (56) and inequalities (57) we obtain

$$\gamma(x, u_n(x)) \to \gamma_0(x) = 0 \ \text{ for a.e } x \in (0, 2\pi)$$
$$\text{and} \quad v_n(x) \to v_0(x) = \frac{\varphi_m(x)}{\|\varphi_m\|}\,, \tag{58}$$

where φ_m is a solution to (2) with $[\lambda_+, \lambda_-] \in \Sigma_m$.

Now we estimate the first integral in (51). We set $u_n = a_n\varphi_m + u_n^\perp$, where $a_n \geq 0$ and $u_n^\perp \in H^- \oplus H^+$. We remark that $u = u^+ - u^-$ and using (21) in the first integral in (51) we denote

$$I_w \equiv \int_0^{2\pi} [(a_n\varphi_m + u_n^\perp)' w_k' - (\lambda_+ u_n^+ - \lambda_- u_n^-) w_k]\, dx$$

and we obtain

$$\begin{aligned} I_w =& \int_0^{2\pi} [(a_n\varphi_m + u_n^\perp)' w_k' \\ & -(\lambda_+ u_n^+ - \lambda_- u_n^-) w_k]\, dx \\ =& \int_0^{2\pi} [a_n\varphi_m' w_k' + (u_n^\perp)' w_k' \\ & -((\lambda_+ - \lambda_-) u_n^+ + \lambda_- u_n)\, w_k]\, dx \\ =& \int_0^{2\pi} [a_n(\lambda_+\varphi_m^+ - \lambda_-\varphi_m^-) w_k + (u_n^\perp)' w_k' \\ & -((\lambda_+ - \lambda_-) u_n^+ + \lambda_- u_n)\, w_k]\, dx \\ =& \int_0^{2\pi} \{a_n[(\lambda_+ - \lambda_-)\varphi_m^+ + \lambda_-\varphi_m]\, w_k \\ & +(u_n^\perp)' w_k' - [(\lambda_+ - \lambda_-)(a_n\varphi_m + u_n^\perp)^+ \\ & +\lambda_-(a_n\varphi_m + u_n^\perp)]\, w_k\}\, dx \\ =& \int_0^{2\pi} [(\lambda_+ - \lambda_-)(a_n\varphi_m^+ - (a_n\varphi_m + u_n^\perp)^+) w_k \\ & +(u_n^\perp)' w_k' - \lambda_- u_n^\perp w_k]\, dx\,. \end{aligned} \tag{59}$$

Similarly

$$\begin{aligned} I_w =& \int_0^{2\pi} [(\lambda_+ - \lambda_-)(a_n\varphi_m^- - (a_n\varphi_m + u_n^\perp)^-) w_k \\ & +(u_n^\perp)' w_k' - \lambda_+ u_n^\perp w_k]\, dx\,. \end{aligned} \tag{60}$$

We add (59) and (60), thus

$$\begin{aligned} 2I_w =& \int_0^{2\pi} [(\lambda_+ - \lambda_-)(|a_n\varphi_m| - |a_n\varphi_m + u_n^\perp|) w_k \\ & + 2(u_n^\perp)' w_k' - (\lambda_+ + \lambda_-) u_n^\perp w_k]\, dx\,. \end{aligned} \tag{61}$$

We set $u_n^\perp = \overline{u}_n + \widetilde{u}_n$ where $\overline{u}_n \in H^-$, $\widetilde{u}_n \in H^+$ and we put $w_k = \overline{u}_n - \widetilde{u}_n + a_n\varphi_m$, $a_n \geq 0$, $(k = n)$ in (61), we get

$$\begin{aligned} 2I_n \equiv& \int_0^{2\pi} ([\lambda_+ - \lambda_-)(|a_n\varphi_m| - |a_n\varphi_m + \overline{u}_n + \widetilde{u}_n|) \\ & \cdot(\overline{u}_n - \widetilde{u}_n) + 2(\overline{u}_n')^2 - 2(\widetilde{u}_n')^2 \\ & -(\lambda_+ + \lambda_-)(\overline{u}_n^2 - \widetilde{u}_n^2)]\, dx \\ & + \int_0^{2\pi} [(\lambda_+ - \lambda_-)(|a_n\varphi_m| - |a_n\varphi_m + u_n^\perp|)\, a_n\varphi_m \\ & + 2(u_n^\perp)' a_n\varphi_m' - (\lambda_+ + \lambda_-) u_n^\perp a_n\varphi_m]\, dx \end{aligned} \tag{62}$$

Hence using $|x| - |y| \leq |x - y|$ and (21) we obtain

$$\begin{aligned} 2I_n \leq& \int_0^{2\pi} [\,(\lambda_+ - \lambda_-)\, |\overline{u}_n + \widetilde{u}_n|\, |\overline{u}_n - \widetilde{u}_n| \\ & + 2(\overline{u}_n')^2 - 2(\widetilde{u}_n')^2 \\ & -(\lambda_+ + \lambda_-)((\overline{u}_n)^2 - (\widetilde{u}_n)^2)\,]\, dx \\ & + \int_0^{2\pi} [(\lambda_+ - \lambda_-)(|a_n\varphi_m| - |a_n\varphi_m + u_n^\perp|) a_n\varphi_m \\ & + 2a_n(\lambda_+\varphi_m^+ u_n^\perp - \lambda_-\varphi_m^- u_n^\perp) \\ & -(\lambda_+ + \lambda_-) u_n^\perp a_n\varphi_m\,]\, dx \\ =& \int_0^{2\pi} [\,(\lambda_+ - \lambda_-)\, |\overline{u}_n^2 - \widetilde{u}_n^2| + 2(\overline{u}_n')^2 \\ & -(\lambda_+ + \lambda_-)(\overline{u}_n)^2 - 2(\widetilde{u}_n')^2 \\ & +(\lambda_+ + \lambda_-)(\widetilde{u}_n)^2\,]\, dx \\ & + \int_0^{2\pi} [(\lambda_+ - \lambda_-)(|a_n\varphi_m| - |a_n\varphi_m + u_n^\perp|) a_n\varphi_m \\ & + u_n^\perp(\lambda_+ - \lambda_-)\, |a_n\varphi_m|\,]\, dx\,. \end{aligned} \tag{63}$$

Inequality $|a^2-b^2| \le a^2+b^2$ and (63) yield

$$\begin{aligned}
2I_n \le\, & 2\Big(\int_0^{2\pi} [\,(\overline{u}_n')^2 - \lambda_-(\overline{u}_n)^2\,]\,dx \\
&+ \int_0^{2\pi} [\,-(\widetilde{u}_n')^2 + \lambda_+(\widetilde{u}_n)^2\,]\,dx\Big) \\
&+(\lambda_+-\lambda_-)\int_0^{2\pi} [(|a_n\varphi_m| - |a_n\varphi_m + u_n^\perp|)a_n\varphi_m \\
&+u_n^\perp\,|a_n\varphi_m|]\,dx \\
\le\, & 2\Big(\int_0^{2\pi} [\,(\overline{u}_n')^2 - \lambda_-(\overline{u}_n)^2\,]\,dx \\
&+ \int_0^{2\pi} [\,-(\widetilde{u}_n')^2 + \lambda_+(\widetilde{u}_n)^2\,]\,dx\Big) \\
&+2\,(\lambda_+-\lambda_-)\int_{M_n} (u_n^\perp)^2\,dx\,,
\end{aligned} \tag{64}$$

where $M_n = \{x \in [0,2\pi] : \varphi_m(\varphi_m + u_n^\perp/a_n) < 0\}$. The last inequality in (64) follows from the following estimates

$$\begin{aligned}
&(|a_n\varphi_m| - |a_n\varphi_m + u_n^\perp|)a_n\varphi_m + u_n^\perp|a_n\varphi_m| \\
&= \begin{cases} 0 \quad (\text{if } a_n\varphi_m(a_n\varphi_m + u_n^\perp) > 0) & x \notin M_n \\ \text{sign}\,(\varphi_m)\,2\,(a_n\varphi_m + u_n^\perp)a_n\varphi_m & x \in M_n \end{cases} \\
&\le\, 2(u_n^\perp)^2
\end{aligned}$$

since $a_n\varphi_m < 0$ and $a_n\varphi_m + u_n^\perp > 0$ imply $u_n^\perp > a_n\varphi_m + u_n^\perp$, $u_n^\perp > -a_n\varphi_m > 0$ and therefore $-(a_n\varphi_m + u_n^\perp)\,a_n\varphi_m \le (u_n^\perp)^2$.
We use $|x|-|y| \ge -|x-y|$ in (62) obtain similarly

$$\begin{aligned}
I_n \ge & \int_0^{2\pi} [\,(\overline{u}_n')^2 - \lambda_+(\overline{u}_n)^2 - (\widetilde{u}_n')^2 + \lambda_-(\widetilde{u}_n)^2\,]\,dx \\
& -(\lambda_+-\lambda_-)\int_{M_n} (u_n^\perp)^2\,dx\,.
\end{aligned} \tag{65}$$

Using $\|\cdot\|_{C([0,2\pi])} \le c_2\|\cdot\|$ we get

$$\int_{M_n} (u_n^\perp)^2\,dx \le \mu(M_n)\,c_2\|u_n^\perp\|^2 \quad \text{and} \quad \mu(M_n) \to 0\,. \tag{66}$$

Since by (58) we have

$$\frac{u_n}{\|u_n\|} = \frac{(\varphi_m + u_n^\perp/a_n)}{\|\varphi_m + u_n^\perp/a_n\|} \to \frac{\varphi_m}{\|\varphi_m\|} \quad \text{and} \quad \frac{u_n^\perp}{a_n} \rightrightarrows 0\,.$$

We write $u_n = \overline{u}_n + a_n\varphi_m + \widetilde{u}_n$, $\overline{u}_n \in H^-$, $\widetilde{u}_n \in H^+$. We put $w_k = (\overline{u}_n + a_n\varphi_m - \widetilde{u}_n)/(a_n\|u_n^\perp\|^{\frac12})$ in (51) then using (64) we obtain

$$\begin{aligned}
\liminf_{n\to\infty} & \frac{1}{a_n\|u_n^\perp\|^{\frac12}}\Big\{\int_0^{2\pi} [\,(\overline{u}_n')^2 - \lambda_-(\overline{u}_n)^2\,]\,dx \\
&+ \int_0^{2\pi} [\,-(\widetilde{u}_n')^2 + \lambda_+(\widetilde{u}_n)^2\,]\,dx \\
&+ (\lambda_+-\lambda_-)\int_{M_n} (u_n^\perp)^2\,dx + \int_0^{2\pi} [\gamma(x,u_n)(\widetilde{u}_n)^2]\,dx \\
&- \int_0^{2\pi} [\gamma(x,u_n)(\,\overline{u}_n + a_n\varphi_m)^2 \\
&+(h(x,u_n)-f)\,(\,\overline{u}_n + a_n\varphi_m - \widetilde{u}_n)]\,dx\Big\} \ge 0\,.
\end{aligned} \tag{67}$$

We note that it holds $\|\overline{u}_n\|^2 \le ((m-1)^2+1)\|\overline{u}_n\|_2^2$, $\|\widetilde{u}_n\|^2 \ge ((m+1)^2+1)\|\widetilde{u}_n\|_2^2$ and using (66) we get

$$\begin{aligned}
&\int_0^{2\pi} [(\overline{u}_n')^2 - \lambda_-(\overline{u}_n)^2]\,dx + \int_0^{2\pi} [-(\widetilde{u}_n')^2 + \lambda_+(\widetilde{u}_n)^2]\,dx \\
&+ (\lambda_+-\lambda_-)\int_{M_n} (u_n^\perp)^2\,dx + \int_0^{2\pi} [\gamma(x,u_n)(\widetilde{u}_n)^2]\,dx \\
&= \|\overline{u}_n\|^2 - (\lambda_- + 1)\|\overline{u}_n\|_2^2 - \|\widetilde{u}_n\|^2 + (\lambda_+ + 1)\|\widetilde{u}_n\|_2^2 \\
&+ (\lambda_+-\lambda_-)\int_{M_n} (u_n^\perp)^2\,dx + \int_0^{2\pi} [\gamma(x,u_n)(\widetilde{u}_n)^2]\,dx \\
&\le \frac{(m-1)^2 - \lambda_-}{(m-1)^2+1}\|\overline{u}_n\|^2 + \frac{\lambda_+ - (m+1)^2}{(m+1)^2+1}\|\widetilde{u}_n\|^2 \\
&+ (\lambda_+-\lambda_-)\mu(M_n)c_2\|u_n^\perp\|^2 \\
&+ \int_0^{2\pi} \gamma(x,u_n)\,dx\,c_2\|\widetilde{u}_n\|^2.
\end{aligned}$$

Hence and from (57), (58) and (66) it follows

$$\begin{aligned}
&\int_0^{2\pi} [\,(\overline{u}_n')^2 - \lambda_-(\overline{u}_n)^2\,]\,dx + \int_0^{2\pi} [\,-(\widetilde{u}_n')^2 \\
&+ \lambda_+(\widetilde{u}_n)^2\,]\,dx + (\lambda_+-\lambda_-)\int_{M_n} (u_n^\perp)^2\,dx \\
&+ \int_0^{2\pi} [\gamma(x,u_n)(\widetilde{u}_n)^2]\,dx \\
&\le \frac{-\delta/2}{(m-1)^2+1}\|\overline{u}_n\|^2 + \frac{-\varepsilon/2}{(m+1)^2+1}\|\widetilde{u}_n\|^2 \\
&\le -\varrho\|u_n^\perp\|^2
\end{aligned} \tag{68}$$

with some $\varrho > 0$. Therefore (67) and (68) imply

$$\begin{aligned}
\liminf_{n\to\infty} & \frac{1}{a_n\|u_n^\perp\|^{\frac12}}\Big\{-\int_0^{2\pi} [\gamma(x,u_n)(\,\overline{u}_n + a_n\varphi_m)^2 \\
&+ (\,h(x,u_n) - f\,)\,(\,\overline{u}_n + a_n\varphi_m - \widetilde{u}_n\,)]\,dx\Big\} \ge 0\,.
\end{aligned} \tag{69}$$

Consequently

$$\liminf_{n\to\infty}\int_0^{2\pi}\Big[\frac{h(x,u_n)-f)}{\|u_n^\perp\|^{\frac12}}((\overline{u}_n-\widetilde{u}_n)/a_n+\varphi_m)\Big]\,dx$$
$$\geq \liminf_{n\to\infty}\int_0^{2\pi}\Big[\frac{\gamma(x,u_n)}{\|u_n^\perp\|^{\frac12}}a_n(\overline{u}_n/a_n+\varphi_m)^2\,dx\Big]\geq 0. \tag{70}$$

Now we put $w_k = (\overline{u}_n-\widetilde{u}_n)/(\|u_n^\perp\|^2)$ in (51) to obtain

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{\|u_n^\perp\|^2}\Big\{\int_0^{2\pi}[\,(\overline{u}'_n)^2-\lambda_-(\overline{u}_n)^2\,]\,dx\\&+\int_0^{2\pi}[-(\widetilde{u}'_n)^2+\lambda_+(\widetilde{u}_n)^2\,]\,dx\\&+(\lambda_+-\lambda_-)\int_{M_n}(u_n^\perp)^2\,dx\\&+\int_0^{2\pi}[\gamma(x,u_n)((\widetilde{u}_n)^2-(\overline{u}_n)^2)]\,dx\\&-\int_0^{2\pi}[(\gamma(x,u_n)a_n\varphi_m+h(x,u_n)-f)\\&\cdot(\overline{u}_n-\widetilde{u}_n)]\,dx\Big\}\geq 0.\end{aligned}\tag{71}$$

We suppose for contradiction that the sequence $(u_n^\perp)$ is unbounded then due to (68) and (71) there exists $\varrho>0$ such that

$$-\varrho+\liminf_{n\to\infty}\Big\{-\int_0^{2\pi}\Big[\frac{\gamma(x,u_n)}{\|u_n^\perp\|^{\frac12}}a_n\varphi_m\frac{\overline{u}_n-\widetilde{u}_n}{\|u_n^\perp\|^{\frac32}}\Big]\,dx\Big\}\geq 0 \tag{72}$$

or equivalently

$$-\varrho\geq\limsup_{n\to\infty}\int_0^{2\pi}\Big[\frac{\gamma(x,u_n)}{\|u_n^\perp\|^{\frac12}}\,a_n\varphi_m\,\frac{\overline{u}_n-\widetilde{u}_n}{\|u_n^\perp\|^{\frac32}}\Big]\,dx\,. \tag{73}$$

We note that $u_n^\perp/a_n \rightrightarrows 0$ and we get by (70) (for $\|u_n^\perp\|\to\infty$)

$$\lim_{n\to\infty}\int_0^{2\pi}\frac{\gamma(x,u_n)}{\|u_n^\perp\|^{\frac12}}a_n\varphi_m^2\,dx=0\,. \tag{74}$$

We denote $S_n=\Big\{x\in[0,2\pi]\ \mid\ |\varphi_m(x)|\leq(\overline{u}_n(x)-\widetilde{u}_n(x))/(\|u_n^\perp\|^{3/2})\Big\}$ then $\lim\limits_{n\to\infty}\mu(S_n)=0$ and

$$\begin{aligned}&\int_{[0,2\pi]\setminus S_n}\Big[\frac{\gamma(x,u_n)}{\|u_n^\perp\|^{\frac12}}\,a_n\,\Big|\varphi_m\,\frac{\overline{u}_n-\widetilde{u}_n}{\|u_n^\perp\|^{\frac32}}\Big|\Big]\,dx\\&\leq\int_{[0,2\pi]\setminus S_n}\frac{\gamma(x,u_n)}{\|u_n^\perp\|^{\frac12}}a_n\varphi_m^2\,.\end{aligned}\tag{75}$$

By (51) (with $w_k=(\overline{u}_n-\widetilde{u}_n)/\|u_n^\perp\|^2$), (65) we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{\|u_n^\perp\|^2}\Big\{\int_0^{2\pi}[\,(\overline{u}'_n)^2-\lambda_+(\overline{u}_n)^2\,]\,dx\\&+\int_0^{2\pi}[\,-(\widetilde{u}'_n)^2+\lambda_-(\widetilde{u}_n)^2\,]\,dx\\&-(\lambda_+-\lambda_-)\int_{M_n}(u_n^\perp)^2\,dx\\&+\int_0^{2\pi}[\gamma(x,u_n)((\widetilde{u}_n)^2-(\overline{u}_n)^2)]\,dx\\&-\int_0^{2\pi}[\gamma(x,u_n)a_n\varphi_m+(h(x,u_n)-f)\\&\cdot(\overline{u}_n-\widetilde{u}_n)]\,dx\Big\}\leq 0\,.\end{aligned}$$

Hence there exists a constant c_5 such that $\liminf\limits_{n\to\infty}\frac{1}{\|u_n^\perp\|^2}\int_0^{2\pi}[\gamma(x,u_n)a_n\varphi_m(\overline{u}_n-\widetilde{u}_n)]\,dx\geq c_5$. Thus $\limsup_{n\to\infty}\int_{S_n}\Big[(\gamma(x,u_n)/\|u_n^\perp\|^{1/2})\,a_n\varphi_m\,(\overline{u}_n-\widetilde{u}_n)/(\|u_n^\perp\|^{3/2})\Big]\,dx\geq 0$ since $\mu(S_n)\to 0$. Hence and by (74), (75) we get

$$\limsup_{n\to\infty}\int_0^{2\pi}\Big[\frac{\gamma(x,u_n)}{\|u_n^\perp\|^{\frac12}}\,a_n\varphi_m\,\frac{\overline{u}_n-\widetilde{u}_n}{\|u_n^\perp\|^{\frac32}}\Big]\,dx\geq 0 \tag{76}$$

a contradiction to (73). This implies that the sequence $(u_n^\perp)$ is bounded. We use (20) from Lemma 3 with $w=u_n^\perp$ and we obtain

$$\begin{aligned}&\int_0^{2\pi}[((u_n^\perp)')^2-\lambda_+(u_n^\perp)^2]\,dx\\&\leq J(u_n)\leq\int_0^{2\pi}[((u_n^\perp)')^2-\lambda_-(u_n^\perp)^2]\,dx\end{aligned}\tag{77}$$

where $J(u_n)=\int_0^{2\pi}\big[(u'_n)^2-\lambda_+u_n^2-\lambda_-u_n^2\big]\,dx$. Hence boundedness of $(u_n^\perp)$ implies with (49) that there exists a constant c_6 such that

$$\Big|\int_0^{2\pi}[G(x,u_n)-fu_n]\,dx\Big|\leq c_6\quad\text{for all } n\in\mathbb{N}\,. \tag{78}$$

We again use the decomposition $G(x,s)=\Gamma(x,s)+H(x,s)$ to rewrite (78) into

$$\Big|\int_0^{2\pi}[\Gamma(x,u_n)+H(x,u_n)-f(u_n^\perp+a_n\varphi_m)\,dx\Big|\leq c_6$$
$$\text{for all } n\in\mathbb{N}\,. \tag{79}$$

We use (45) boundedness of $(u_n^\perp)$ and (79) to obtain a constant c_7 such that

$$\Big|\int_0^{2\pi}[\Gamma(x,u_n)+H(x,a_n\varphi_m)-fa_n\varphi_m\,dx\Big|\le c_7$$

for all $n\in\mathbb{N}$.

(80)

Using (47) and (80) we obtain a contradiction to assumptions (6) (see (48)), (7), hence sequence (u_n) is bounded. Then there exists $u_0\in H$ such that $u_n\rightharpoonup u_0$ in H, $u_n\to u_0$ in $L^2(0,2\pi)$, $C(0,2\pi)$ (taking a subsequence if it is necessary). It follows from equality (39) that

$$\begin{aligned}&\lim_{\substack{n\to\infty\\ m\to\infty\\ k\to\infty}}\Big\{\int_0^{2\pi}[(u_n-u_m)'w_k'\\ &-(\lambda_+(u_n^+-u_m^+)-\lambda_-(u_n^--u_m^-))w_k]\,dx\\ &-\int_0^{2\pi}[g(x,u_n)-g(x,u_m)]w_k\,dx\Big\}=0\,.\end{aligned}\tag{81}$$

The nonlinearity g is the Carathéodory's function, thus strong convergence $u_n\to u_0$ in $C(0,2\pi)$ imply

$$\lim_{\substack{n\to\infty\\ m\to\infty}}\int_0^{2\pi}[g(x,u_n)-g(x,u_m)](u_n-u_m)\,dx=0\,.\tag{82}$$

If we set $w_k=u_n$, $w_k=u_m$ in (81) and subtract these equalities, then by (82) we obtain

$$\begin{aligned}&\lim_{\substack{n\to\infty\\ m\to\infty}}\int_0^{2\pi}[(u_n'-u_m')^2-(\lambda_+(u_n^+-u_m^+)\\ &-\lambda_-(u_n^--u_m^-))(u_n-u_m)]\,dx=0\,.\end{aligned}\tag{83}$$

Hence the strong convergence $u_n\to u_0$ in $L^2(0,2\pi)$ implies the strong convergence $u_n\to u_0$ in H. This shows that J satisfies Palais-Smale condition and the proof of Theorem 2 is complete.

Conclusion

For simplicity, we will now assume that $f(x,s)=0$. As we mentioned in the introduction, ALP-condition

$$\lim_{|s|\to\infty}\int_0^{2\pi}[G(x,s\,\varphi_m(x))]\,dx=+\infty\tag{6}$$

generalizes the classical Landesman-Lazer condition (see [1]) and the potential Landesman-Lazer condition ([12]).

However, condition (6) cannot be used in the case where it sets the so-called strong resonance, i.e., the nonlinearity g satisfies $\lim_{|s|\to\infty}g(x,s)=0$ and $G(x,s)=\int_0^s g(x,t)\,dt$ bounded as $s\to+\infty$.

Another open problem is the restriction given by assumption (5), which implies that with point $[\lambda_+,\lambda_-]$ we can only move along a limited part of the Σ_m curve of the Fučík spectrum.

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