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Postupné vlny pro bistabilní a monostabilní Fisher–Kolmogorovu rovnici s nespojitou difuzí závislou na hustotě

Mgr. Michaela Zahradníková

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Travelling waves for bistable and monostable Fisher–Kolmogorov equation with discontinuous density dependent diffusion

Mgr. Michaela Zahradníková

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Declaration

I hereby declare that this thesis is my own work, unless clearly stated otherwise.

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Michaela Zahradníková

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Abstract

This thesis is devoted to the travelling wave solutions of the generalized Fisher–Kolmogorov equation with discontinuous density dependent diffusion that can degenerate or have singularities at equilibrium points. The reaction term is either bistable or monostable and it is a continuous, possibly non-Lipschitz function. We present our recent results concerning the existence, uniqueness and asymptotic behaviour of travelling and standing waves in the bistable case. Expected results for the monostable case are included as open problems. The work on these problems is the main subject of our current research.

The novelty of the results in this thesis consists in the fact that the diffusion term allows for discontinuities of the first kind in finite number of points as well as for degenerations and/or singularities at equilibrium points. It is shown how the latter generalization of density dependent diffusion is compensated by the speed of vanishing of the reaction term in these equilibria. Another added value of this thesis is the fact that bistable balanced and unbalanced as well as the monostable cases are treated in a unified and rather general way based on the Carathéodory's theory of the first order ODEs.

Keywords:

travelling wave, generalized Fisher–Kolmogorov equation, density dependent diffusion, discontinuous diffusion, degenerate and singular diffusion, bistable and monostable reaction, non-Lipschitz reaction, asymptotic behaviour

Contents

Chapter 1

Introduction

Let us consider the semilinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(u). \tag{1.1}$$

An important question related to this equation is the existence of the so called *travelling wave* solutions. Customarily a travelling wave is a wave which travels without the change of shape and the speed of propagation of this shape is constant, usually denoted by c. In other words, if we observed this wave from a travelling frame moving at speed c it would appear stationary. Mathematically this means that a travelling wave solution $u(x,t)$ is of the form

$$u(x,t) = U(x-ct) = U(z), \quad z = x - ct, \tag{1.2}$$

where c is the propagation speed that generally has to be determined, U is the profile of the travelling wave and z is the moving coordinate, sometimes also called the *wave variable*. The wave (1.2) travels in positive x-direction while a wave of the form $u(x,t) = U(x+ct)$ moves in the negative x-direction. Equivalently, this distinction could be done based on the sign of c in (1.2) by altering the usual notion of speed used in physics to include negative values as well. In addition, travelling waves are assumed to be bounded for all z with the limits $U(-\infty) := \lim_{z\to-\infty} U(z)$, $U(+\infty) := \lim_{z\to+\infty} U(z)$ finite and unequal, see Figure 1.1.

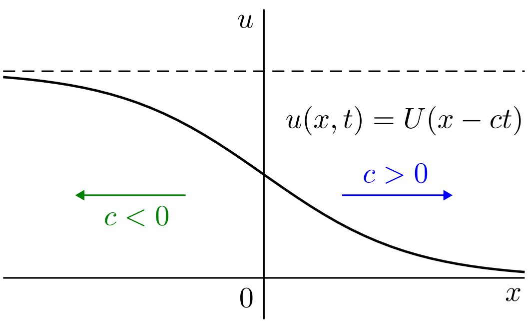

Figure 1.1: Travelling wave with a speed of propagation c

When we look for travelling wave solutions of the form (1.2), we have

$$\frac{\partial u}{\partial t} = -c\frac{\mathrm{d}U}{\mathrm{d}z}, \quad \frac{\partial u}{\partial x} = \frac{\mathrm{d}U}{\mathrm{d}z},$$

so the original partial differential equation (1.1) becomes an ordinary one

$$U''(z) + cU'(z) + g(U(z)) = 0. \tag{1.3}$$

The usual approach is to then study solutions of (1.3) in the phase plane (U, V) where

$$U' = V, \quad V' = -cV - g(U).$$

For a rather special $g(U) = U(1-U)$ the linear stability analysis and the phase plane trajectories of (1.3) can be found in [17].

It is a classical result from the theory of linear parabolic equations that if $g \equiv 0$ there are no physically realistic travelling wave solutions. More specifically, the requirement on boundedness of U would yield $U(z) = k$, $k \in \mathbb{R}$, which is not a wave solution. The fact that travelling wave solutions might appear is therefore a consequence of the particular form of g.

For special forms of the *reaction term* $g = g(s)$ the equation (1.1) models a variety of biological and chemical phenomena. In what follows we restrict our attention to reaction terms suggested by some of these applications.

1.1 Motivation

The classical application of (1.1) is a problem from population genetics modelling the propagation of advantageous genes formulated in 1937 by R. A. Fisher, [12]. The same genetical context was considered in [1, 2] where the authors introduce a classification based on relevant features of the function $g = g(s)$ including the one proposed by Fisher.

Consider a population of diploid individuals distributed in a one-dimensional habitat and suppose that the gene at a specific locus in a specific chromosome pair occurs in two forms, called alleles. Then the population is divided into three classes or genotypes depending on the alleles they carry. Individuals that carry only one kind of allele are called *homozygotes* and they constitute two of these classes. The remaining class consists of individuals that carry one of each allele and they are called *heterozygotes.* The linear densities as well as the viabilities of these three genotypes generally differ. In [1] it is explained how the modelling of such population leads to the equation (1.1) and the general assumptions on g are derived, namely that $g \in C^1[0, 1]$, $g(0) = g(1) = 0$. The authors then proceed by distinguishing the following three cases in which the properties of g are specified. The *heterozygote intermediate case*, in which the viability of the heterozygotes is between the viabilities of the homozygotes, corresponds to

$$g'(0) > 0, \quad g(s) > 0 \text{ in } (0, 1).$$

This is the case that appeared in the classical studies of Fisher. Mathematical treatment was provided in the same year by Kolmogorov, Petrovsky and Piskunov, [15]. Hence, (1.1) with this kind of reaction term is often regarded as Fisher–Kolmogorov, Fisher–KPP or simply FK equation. The *heterozygote superiority* occurs when the viability of the heterozygotes is larger than the viabilities of the homozygotes. In this case the relevant properties of g are

$$g'(0) > 0, \ g'(1) > 0$$
$$g(s) > 0 \text{ in } (0, s_*), \ g(s) < 0 \text{ in } (s_*, 1)$$

for some $s_* \in (0, 1)$. Finally, if the viabilities of the homozygotes exceed the viability of the heterozygotes we have *heterozygote inferiority.* Appropriate characteristic of g is then

$$g'(0) > 0, \quad g(s) < 0 \text{ in } (0, s_*), \ g(s) > 0 \text{ in } (s_*, 1) \text{ for some } s_* \in (0, 1),$$
$$\int_0^1 g(s) \, \mathrm{d}s > 0.$$

The equation (1.1) is relevant in other contexts as well. Therefore, it would be inconvenient to only use the terminology used for different forms of g according to this model. We will use the following notion instead. We say that the reaction term g is *monostable* if

$$g(0) = g(1) = 0, \quad g > 0 \text{ in } (0,1), \tag{1.4}$$

i.e., the only points at which g vanishes are 0 and 1. Note that this property of g yields that $U \equiv 0$ and $U \equiv 1$ are the only stationary solutions of (1.3) and they are called equilibria. If g has another intermediate zero at some point $s_* \in (0,1)$ and

$$g(0) = g(s_*) = g(1), \quad g < 0 \text{ in } (0, s_*), \quad g > 0 \text{ in } (s_*, 1), \tag{1.5}$$

we call it *bistable* reaction term. Another aspect related to bistable g is the sign of $\int_0^1 g(s)\,\mathrm{d}s$. If this integral is equal to zero we say that g is a *bistable balanced nonlinearity.* Otherwise, the bistable nonlinearity is called *unbalanced.* The notion of "monostable" and "bistable" reaction term originates from the stability of the stationary points 0 and 1, cf. [21], suggesting that either one or both of them are stable, respectively. Since the stability analysis is beyond the scope of this work and it can be quite involved for more general functions g, we will use this terminology strictly in the sense of the shape of g given by (1.4) or (1.5). These are the usual properties of functions that satisfy the stability criteria, see e.g. [17, 19].

In a variety of biological phenomena, the appearance of travelling waves is not unusual, especially when it comes to developmental processes in which simple diffusion would be too slow for transmitting information over significant distances. Other examples, mentioned in [17], include modelling of insect dispersal, the progressive wave of the rabies epizootic epidemic, movement of microorganisms to a food source etc.

If $g(s) \le 0$ instead of $g(s) < 0$ for $s \in (0, s_*)$ in (1.5), the equation (1.1) models flame propagation in chemical reactor theory. In contrast with the population dynamics model, where u denotes the relative density of the population of one allele, in the combustion model u represents a normalized temperature and s_* represents a critical temperature at which an exothermic reaction starts (see, e.g., [13]). The bistable equation with reaction term (1.5) was also suggested in [4] as a model for a nerve which has been treated with certain toxins. In [18] this equation serves as a model for a bistable active transmission line. Other possible interpretations may be found in [21].

1.2 Basic results and generalizations

Let us assume that $g \in C^1[0,1]$, $g(0) = g(1) = 0$. Any solution $u(x,t) = U(x - ct)$ of (1.1) with range $(0,1)$ that satisfies boundary conditions

$$U(-\infty) = 1 \text{ and } U(+\infty) = 0 \tag{1.6}$$

or

$$U(-\infty) = 0 \text{ and } U(+\infty) = 1 \tag{1.7}$$

is necessarily monotone, cf. [11, Lemma 2.1]. More precisely, for all $z \in \mathbb{R}$ we have $U'(z) < 0$ or $U'(z) > 0$, respectively.

In [15] it was shown that if g is a monostable reaction term, there exists a number $c^* > 0$ such that (1.1) possesses travelling wave solutions for any wave speed c with $|c| \ge c^*$. The sign of c is determined by the boundary conditions above. Namely, it is positive for decreasing solutions satisfying (1.6) and negative for increasing ones with (1.7). This means that for each value of $c \geq c^* > 0$ there exists a pair of travelling wave solutions $u_1(x,t) = U_1(x-ct)$ and $u_2(x,t) = U_2(x+ct)$ where u_1 is decreasing in x and u_2 is increasing, see Figure 1.2.

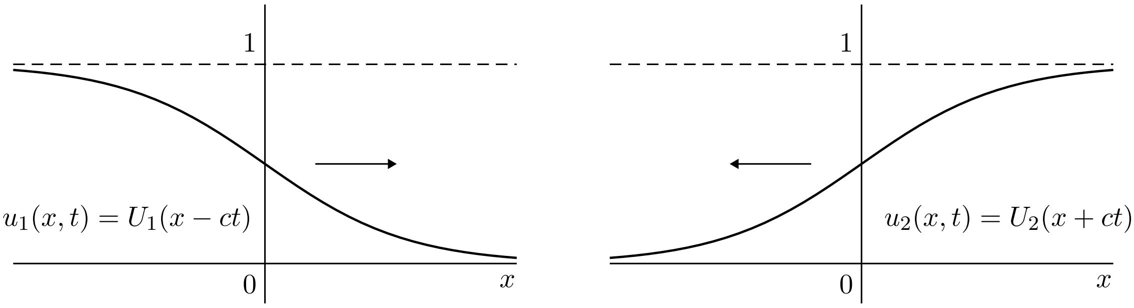

Figure 1.2: Decreasing and increasing travelling wave solution of (1.1) with wave speed $c \geq c^* > 0$

If g is a bistable reaction term of the form (1.5) then by [11, 14] equation (1.1) possesses only two travelling wave solutions "connecting" constant equilibria 0 and 1, one decreasing and one increasing. These solutions are unique except for translation and they travel in opposite directions. As in the previous case, we denote the decreasing solution by $u_1(x,t) = U_1(x-ct)$ and the increasing one by $u_2(x,t) = U_2(x+ct)$. If $\int_0^1 g(s)\,\mathrm{d}s > 0$, then $c > 0$, i.e., the decreasing wave moves to the right and the increasing wave moves to the left. On the other hand, $\int_0^1 g(s)\,\mathrm{d}s < 0$ yields $c < 0$ and the direction in which the waves propagate is now interchanged.

Finally, the bistable balanced condition $\int_0^1 g(s)\,\mathrm{d}s = 0$ corresponds to the case when $c = 0$, i.e., equation (1.1) possesses nonconstant stationary solutions $u(x,t) = u(x)$, also called the *standing waves*. Both decreasing and increasing solutions exist, again unique up to translation. For special reaction terms they can be found in a closed form. For example, for

$$g(s) = s(1-s)\left(s - \frac{1}{2}\right)$$

we obtain

$$u(x) = \pm\frac{1}{2}\tanh\left(\frac{x}{x\sqrt{2}}\right) + \frac{1}{2},$$

cf. [5, 19].

A natural extension of (1.1) is to consider equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[d(u)\frac{\partial u}{\partial x}\right] + g(u), \tag{1.8}$$

where $d = d(s)$ is the density dependent diffusion coefficient. Note that if d is constant, the above equation reduces to (1.1) by suitable rescaling. In many population models, density dependent dispersal has been observed such as migration to regions of lower density as the population gets more crowded. Usual properties of $d = d(s)$ are $d \in C^1[0,1]$, $d > 0$ in $[0,1]$. Certain models suggest that it is also reasonable to assume that d vanishes at least at one point, typically $d(0) = 0$, see e.g. [17]. This case was studied in [16] for (1.8) with a monostable reaction term $g \in C[0,1]$. The authors assume (1.6) and show that if $d'(0) > 0$ there exists a continuum of travelling waves with wave speeds exceeding a threshold value $c^* > 0$ and discuss the appearance of *sharp-type profiles* when $c = c^*$. This notion reflects the fact that the leading edge of the nonincreasing wave $U(x-c^*t)$ reaches 0 at a finite z^* with negative slope $U'(z^*)$ and the authors also provide an estimate for c^*. Analogous results for particular form of (1.8) with $d(s)=s$ and $g(s)=s(1-s)$, $s\in[0,1]$, were derived in [17] where it was shown that $c^*=1/\sqrt{2}$ in this case.

Further generalizations usually deal with the problem of deriving existence results for (1.1) or (1.8) with more general functions $d=d(s)$ and $g=g(s)$. There are numerous articles covering this topic and we mention only a few that are most relevant to the subject of this thesis. In [7] equation (1.8) is studied in the monostable case assuming only $d\in C[0,1]$, $g\in C[0,1]$. Moreover, the diffusion coefficient d might degenerate or have singularities near the endpoints 0 and 1. Density dependent diffusion coefficient which is discontinuous appeared in [20]. Recently, equations of the form

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(d(u)\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)+g(u),\quad p>1, \tag{1.9}$$

also appeared in literature, see, e.g., [10] and the references therein. For $d\equiv 1$ and a continuous bistable reaction term g, travelling waves of (1.9) were studied in [6]. Analysis of standing wave solutions can be found in [5] for special forms of bistable g. The author discusses the existence of new-type smooth solutions that reach the equilibria on a finite interval.

In this thesis we study travelling and standing waves of (1.9) with rather general reaction and diffusion terms. We present results obtained for the bistable case in our published papers [8, 9], included in Appendices A and B. In [8] we focused on the standing wave solutions while in [9] travelling waves were investigated. We introduce a unified approach to the proof of the existence of such solutions that is applicable in the monostable case as well. It is a part of our current research to derive similar results as in [9] for monostable reaction term g.

The thesis is organized as follows. In Chapter 2 we specify the properties of functions $d=d(s)$ and $g=g(s)$ and establish a common ground for investigation of solutions to (1.9) regardless of the type of reaction term. This includes the definition of solution and reduction of (1.9) to a first order problem. Chapter 3 is then devoted to the main results concerning the existence, monotonicity and asymptotic behaviour of solutions. Expected results for the monostable case are also presented.

Chapter 2

Quasilinear diffusion equation

We are concerned with the travelling wave solutions of the quasilinear reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(d(u)\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + g(u), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+. \tag{2.1}$$

Here $\mathbb{R}^+ := [0, +\infty)$, $1 < p < +\infty$ and the properties of the diffusion coefficient $d = d(s)$ as well as the reaction term $g = g(s)$ will be specified below.

For $p = 2$, $d \equiv 1$ and appropriate $g \in C^1[0,1]$ this problem reduces to the semilinear Fischer-Kolmogorov equation (1.1) discussed in the previous chapter. Therefore, (2.1) can be regarded as generalized FK equation. We will consider diffusion and reaction terms similar to (but more general than) those that appear in classical applications.

The diffusion coefficient $d : [0,1] \to \mathbb{R}$ is a nonnegative lower semicontinuous function with $d > 0$ in $(0,1)$. There exist $0 = s_0 < s_1 < s_2 < \cdots < s_n < s_{n+1} = 1$ such that $d\big|_{(s_i,s_{i+1})} \in C(s_i, s_{i+1})$, $i = 0, \ldots, n$, and d has discontinuity of the first kind (finite jump) at s_i, $i = 1, \ldots, n$. Examples are sketched in Figure 2.1.

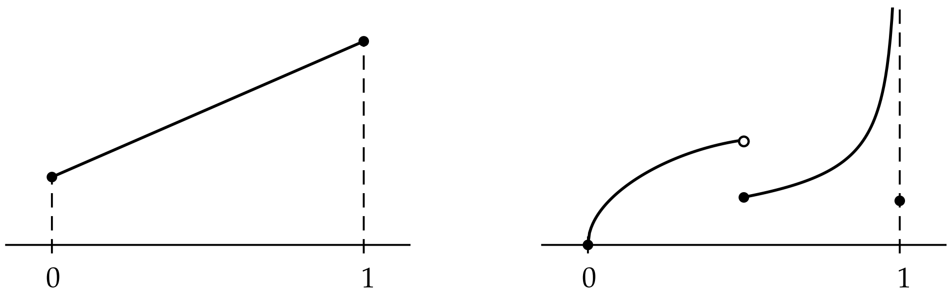

Figure 2.1: Illustration of admissible diffusion coefficients d with qualitatively different properties

The reaction term $g : [0,1] \to \mathbb{R}$, $g \in C[0,1]$, is either bistable or monostable, i.e., either of the form

$$g(0) = g(s_*) = g(1) = 0 \text{ for } s_* \in (0,1), \quad g < 0 \text{ on } (0, s_*), \quad g > 0 \text{ on } (s_*, 1)$$

or

$$g(0) = g(1) = 0, \quad g > 0 \text{ on } (0,1),$$

respectively. In the bistable case we replace the classical balanced (unbalanced) condition, namely

$$\int_0^1 g(s)\,\mathrm{d}s = 0 \quad (\neq 0),$$

by a more general one

$$\int_0^1 (d(s))^{\frac{1}{p-1}}\, g(s)\,\mathrm{d}s = 0 \quad (\neq 0)$$

and we show that it determines the sign of the speed of propagation of the travelling wave. Note that we only require g to be a continuous function on $[0,1]$. In particular, $g'(0+)$, $g'(1-)$ can be zero or infinite (with proper sign depending on the type of reaction), see Figures 2.2, 2.3 below.

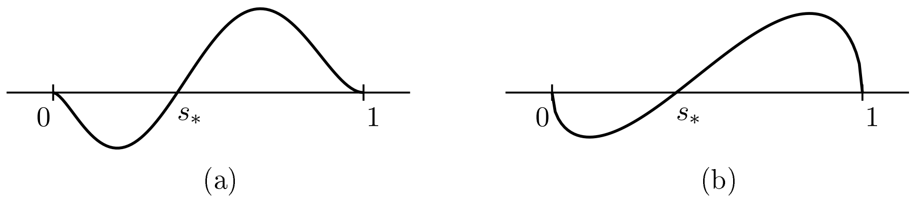

Figure 2.2: (a) Smooth bistable reaction term with $g'(0+) = g'(1-) = 0$; (b) non-Lipschitz bistable reaction term with $g'(0+) = g'(1-) = -\infty$

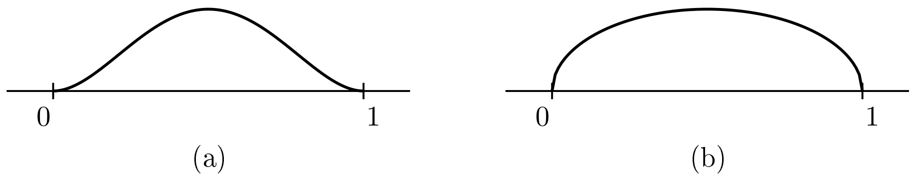

Figure 2.3: (a) Smooth monostable reaction term with $g'(0+) = g'(1-) = 0$; (b) non-Lipschitz monostable reaction term with $g'(0+) = +\infty$, $g'(1-) = -\infty$

We study the existence of travelling wave solutions, i.e., solutions of the form

$$u(x,t) = U(x - ct)$$

where $c \in \mathbb{R}$ is the speed of propagation and U is the travelling wave profile. Using the moving coordinate $z = x - ct$ we can rewrite (2.1) as an ordinary differential equation

$$\left(d(U(z))\,\left|U'(z)\right|^{p-2}\, U'(z)\right)' + cU'(z) + g(U(z)) = 0 \tag{2.2}$$

where for the sake of simplicity we write $(\,\cdot\,)'$ instead of $\frac{\mathrm{d}}{\mathrm{d}z}(\,\cdot\,)$.

In our general setup, especially since d need not be continuous, we first have to clarify what we mean by *solution* of (2.2) and in what sense does it satisfy this equation.

2.1 Definition of solution

Let $U : \mathbb{R} \to [0,1]$ be a monotone continuous function. We denote

$$M_U := \{z \in \mathbb{R} : U(z) = s_i,\ i = 1, 2, \ldots, n\}, \quad N_U := \{z \in \mathbb{R} : U(z) = 0 \text{ or } U(z) = 1\}.$$

Then M_U and N_U are closed sets, M_U is a union of a finite number of points or intervals, $N_U = (-\infty, z_0] \cup [z_1, +\infty)$, where $-\infty \leq z_0 < z_1 \leq +\infty$ and we use the convention $(-\infty, z_0] = \emptyset$ if $z_0 = -\infty$ and $[z_1, +\infty) = \emptyset$ if $z_1 = +\infty$.

Definition 2.1. A monotone continuous function $U : \mathbb{R} \to [0,1]$ is called a *solution* of (2.2) if

(a) For any $z \notin M_U \cup N_U$ the derivative $U'(z)$ exists and it is finite.

(b) For any $z \in \partial M_U$ there exist finite one sided derivatives $U'(z-)$, $U'(z+)$ and

$$L(z) := \left|U'(z-)\right|^{p-2} U'(z-) \lim_{\xi \to z-} d(U(\xi)) = \left|U'(z+)\right|^{p-2} U'(z+) \lim_{\xi \to z+} d(U(\xi)).$$

(c) Function $v : \mathbb{R} \to \mathbb{R}$ defined by

$$v(z) := \begin{cases} d(U(z))\, |U'(z)|^{p-2}\, U'(z), & z \notin M_U \cup N_U, \\ 0, & z \in N_U \cup \operatorname{int} M_U, \\ L(z), & z \in \partial M_U \end{cases}$$

is continuous and for any $z, \hat{z} \in \mathbb{R}$

$$v(\hat{z}) - v(z) + c\,(U(\hat{z}) - U(z)) + \int_z^{\hat{z}} g(U(\xi))\,\mathrm{d}\xi = 0. \tag{2.3}$$

Moreover, $\lim_{z\to\pm\infty} v(z) = 0$ if either $\lim_{z\to-\infty} U(z) = 1$ and $\lim_{z\to+\infty} U(z) = 0$ or $\lim_{z\to-\infty} U(z) = 0$ and $\lim_{z\to+\infty} U(z) = 1$.

Since the definition is quite complex, let us explain the main idea behind it in a less precise but more intuitive way. For this purpose, suppose for now that $c = 0$, i.e., let us look for nonconstant *stationary* solutions of (2.1), the so called standing waves, which satisfy the equation

$$\left(d(u(x))\left|u'(x)\right|^{p-2} u'(x)\right)' = -g(u(x)). \tag{2.4}$$

On the right-hand side of (2.4) there is a continuous function, indicating that it is reasonable to require for the *whole* term in brackets on the left-hand side (which we denote $v = v(x)$) to be differentiable. Because of this, it is sufficient to assume that only one-sided derivatives of u exist for $x \in M_u$ as long as they are properly "compensated" by the discontinuous diffusion coefficient d. The resulting product v attains one value, but the individual terms taken as one-sided limits can be unequal. More precisely, a *transition condition*

$$\left|u'(x-)\right|^{p-2} u'(x-) \lim_{\xi\to x-} d(u(\xi)) = \left|u'(x+)\right|^{p-2} u'(x+) \lim_{\xi\to x+} d(u(\xi))$$

must hold, as it is required in part (b) of Definition 2.1. Consequently, the solution $u = u(x)$ is not a smooth function in general. This fact, however, is not purely associated with d having discontinuities in $(0,1)$. Degenerations of the diffusion coefficient at one or both endpoints of the interval $[0,1]$ might also result in u being only continuous. We will not go into more detail regarding possible degenerations or singularities of d for now but we invite the reader to employ similar reasoning as above to develop a better understanding for part (c) of Definition 2.1. To visualize the transition condition, let us assume for simplicity that d has only one point of discontinuity $s_1 \in (0,1)$, it is smooth and bounded on $(0, s_1)$, $(s_1, 1)$ and $d(s_1-) < d(s_1+)$. Then $M_u = \{\xi_1\}$ and the profile of a corresponding nondecreasing solution $u = u(x)$ is for illustrative purposes sketched in Figure 2.4.

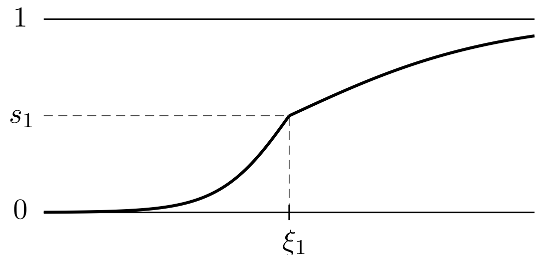

Figure 2.4: Profile of a nondecreasing solution $u = u(x)$ for d discontinuous at s_1

The case $c = 0$ is singular in the sense that the function v from Definition 2.1 is not only continuous but also differentiable on $\mathbb{R}$, cf. Remark 2.4. It also follows from (2.4) and the motivation described above. For $c \neq 0$ we get from (2.2)

$$\left(d(U(z))\left|U'(z)\right|^{p-2}U'(z)\right)' + cU'(z) = -g(U(z)).$$

Similarly to the stationary case, we can see that the left-hand side as a whole is equal to a continuous function. This time, it is a sum of two terms, suggesting that $v'(z-)$, $v'(z+)$ exist but do not have to be equal.

We now proceed with remarks regarding some observations that follow from the definition of solution.

Remark 2.2. Constant functions $U_0 \equiv 0$, $U_1 \equiv 1$ together with $U_* \equiv s_*$ in the case of bistable reaction are solutions of (2.2). It follows from the properties of d and g that those are the only constant solutions and they are called equilibria.

Remark 2.3. Let $p = 2$, $d \equiv 1$ and $g \in C^1[0,1]$. Let $U = U(z)$ be a solution in the sense of Definition 2.1. Then $M_U = \emptyset$, $N_U = \emptyset$, and (2.2) holds pointwise, i.e., $U \in C^2(\mathbb{R})$ and it is a classical solution. For more general d we have to employ the first integral (2.3) because of the lack of differentiability of a solution U.

Remark 2.4. Let $z \notin M_U \cup N_U, \hat{z} = z + h, h \neq 0$. Divide (2.3) by h and let $h \to 0$. Then, by Definition 2.1 (a), the derivative $U'(z)$ exists and

$$v'(z) + cU'(z) + g(U(z)) = 0. \tag{2.5}$$

In particular, v is differentiable in z.

In the case of standing wave solutions $u(x,t) \equiv u(x)$ we obtain that v is continuously differentiable and the equation

$$v'(x) + g(u(x)) = 0 \tag{2.6}$$

holds for all $x \in \mathbb{R}$.

Remark 2.5. Let $z \in M_U$, $\hat{z} = z + h$, $h < 0$. Divide (2.3) by h and let $h \to 0$. Then by Definition 2.1 we get

$$v'(z-) + cU'(z-) + g(U(z)) = 0.$$

In particular, $v'(z-)$ exists and it is finite. Similarly, we derive

$$v'(z+) + cU'(z+) + g(U(z)) = 0.$$

2.2 Equivalent first order ODE

The main tool for proving the existence of solutions of (2.2) is to show that this second order problem can be transformed into a first order one, as suggested in [10]. We then derive existence results by investigating the equivalent first order ODE which has the same form for both bistable and monostable g. For the purposes of this chapter we will mostly not distinguish between the two but we would like to point out in advance that the type of reaction term g will play an important role when we formulate the main results.

The idea behind what follows is to restrict our attention to solutions $U = U(z)$ that are strictly monotone whenever $U(z) \in (0,1)$. We then use the fact that an inverse of the solution exists and we transform (2.2) to an equation with the independent variable $U \in (0,1)$.

Let $U : \mathbb{R} \to [0,1]$ be a nonincreasing solution of (2.2) satisfying boundary conditions

$$\lim_{z\to-\infty} U(z) = 1 \qquad \text{and} \qquad \lim_{z\to+\infty} U(z) = 0 \tag{2.7}$$

such that U is strictly decreasing at any point $z \in \mathbb{R}$ where $U(z) \in (0,1)$. Then there exist $-\infty \leq z_0 < z_1 \leq +\infty$ such that $U(z) = 1, z \in (-\infty, z_0]$ and $U(z) = 0, z \in [z_1, +\infty)$. In order to highlight the basic structure of what follows we will assume for now that d is continuous on the entire interval $(0,1)$. Then $M_U = \emptyset$ and according to Definition 2.1 (a) we have $U \in C^1(z_0, z_1)$, $U'(z) < 0$ for all $z \in (z_0, z_1)$. Therefore, there exists strictly decreasing inverse function U^{-1} : $(0,1) \to (z_0, z_1)$, $z = U^{-1}(U)$, such that $U^{-1} \in C^1(0,1)$. Hence we make the change of variables (cf. [10])

$$w(U) = v(U^{-1}(U)), \quad U \in (0,1). \tag{2.8}$$

It follows from Remark 2.4 that $w \in C^1(0,1)$ and we have

$$\frac{\mathrm{d}}{\mathrm{d}z} v(z) = \frac{\mathrm{d}}{\mathrm{d}z} w(U(z)) = \frac{\mathrm{d}w}{\mathrm{d}U}\,(U(z))\, U'(z). \tag{2.9}$$

From $v(z) = -d(U(z))\,|U'(z)|^{p-1}$ we deduce that

$$U'(z) = -\left|\frac{v(z)}{d(U(z))}\right|^{p'-1}, \qquad p' = \frac{p}{p-1}\,. \tag{2.10}$$

From (2.8), (2.9) and (2.10),

$$\frac{\mathrm{d}v}{\mathrm{d}z} = -\frac{\mathrm{d}w}{\mathrm{d}U}\,(U(z)) \left|\frac{v(z)}{d(U(z))}\right|^{p'-1} = -\frac{\mathrm{d}w}{\mathrm{d}U}\left|\frac{w(U)}{d(U)}\right|^{p'-1}.$$

Therefore, the equation (2.5), namely

$$v'(z) + cU'(z) + g(U(z)) = 0, \quad z \in (z_0, z_1),$$

becomes

$$-\frac{\mathrm{d}w}{\mathrm{d}U}\left|\frac{w(U)}{d(U)}\right|^{p'-1} - c\left|\frac{w(U)}{d(U)}\right|^{p'-1} + g(U) = 0, \quad U \in (0,1).$$

This is equivalent to

$$|w|^{p'-1}\,\frac{\mathrm{d}w}{\mathrm{d}U} = -c\,|w|^{p'-1} + (d(U))^{p'-1}\,g(U), \tag{2.11}$$

or, since $w < 0$ in $(0,1)$,

$$\frac{1}{p'}\frac{\mathrm{d}}{\mathrm{d}U}\,|w|^{p'} = c\,|w|^{p'-1} - (d(U))^{p'-1}\,g(U). \tag{2.12}$$

Set

$$f(U) = (d(U))^{\frac{1}{p-1}}\, g(U) \tag{2.13}$$

and write t instead of U and $y(t) = |w(t)|^{p'}$. Then (2.12) transforms to

$$y'(t) = p' \left[c\,(y(t))^{\frac{1}{p}} - f(t) \right], \quad t \in (0,1). \tag{2.14}$$

From (2.7) and Definition 2.1 (c) we deduce that $v(z) \to 0$ as $z \to z_0+$ or $z \to z_1-$ which is equivalent to $\lim\limits_{t\to 0+} w(t) = \lim\limits_{t\to 1-} w(t) = 0$. Therefore, $y = y(t)$ satisfies the boundary conditions

$$y(0) = y(1) = 0. \tag{2.15}$$

On the other hand, let us suppose that $y = y(t)$ is a positive solution of (2.14), (2.15). Set $w(s) := -(y(s))^{\frac{1}{p'}}$. Then w satisfies (2.11) and (2.12). For $U, s_* \in (0,1)$ set

$$z(U) = z(s_*) - \int\limits_{s_*}^{U} \left| \frac{d(s)}{w(s)} \right|^{\frac{1}{p-1}} \mathrm{d}s. \tag{2.16}$$

Then the function $z = z(U)$ is continuous strictly decreasing in $(0,1)$ and maps the interval $(0,1)$ onto (z_0, z_1), where $-\infty \le z_0 < z_1 \le +\infty$. Let us denote by $U : (z_0, z_1) \to (0,1)$ the inverse function to $z = z(U)$. Then $U(z(s_*)) = s_*$, U is continuous strictly decreasing,

$$\lim_{z\to z_0+} U(z) = 1 \quad \text{and} \quad \lim_{z\to z_1-} U(z) = 0.$$

From (2.16) we deduce

$$\frac{\mathrm{d}U}{\mathrm{d}z} = \frac{1}{\frac{\mathrm{d}z(U)}{\mathrm{d}U}} = -\left| \frac{w(U)}{d(U)} \right|^{\frac{1}{p-1}}, \quad U \in (0,1),$$

i.e., $U \in C^1(z_0, z_1)$, $U'(z) < 0$ and

$$-d(U(z)) \left| \frac{\mathrm{d}U(z)}{\mathrm{d}z} \right|^{p-1} = w(U(z)) =: v(z), \tag{2.17}$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}z} \left[d(U(z)) \left| \frac{\mathrm{d}U}{\mathrm{d}z} \right|^{p-2} \frac{\mathrm{d}U}{\mathrm{d}z} \right] = \frac{\mathrm{d}}{\mathrm{d}z} w(U(z)) = \frac{\mathrm{d}w}{\mathrm{d}U} \frac{\mathrm{d}U(z)}{\mathrm{d}z}. \tag{2.18}$$

From (2.11), (2.17) we derive that

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}U} &= -\left|w(U)\right|^{-(p'-1)} \left(-c\left|w(U)\right|^{p'-1} + (d(U))^{p'-1} g(U) \right) \\
&= -c + \left|w(U)\right|^{-(p'-1)} (d(U))^{p'-1} g(U) \\
&= -c + (d(U(z)))^{-(p'-1)} \left| \frac{\mathrm{d}U(z)}{\mathrm{d}z} \right|^{-(p-1)(p'-1)} (d(U(z)))^{p'-1} g(U(z)) \\
&= -c + \left| \frac{\mathrm{d}U(z)}{\mathrm{d}z} \right|^{-1} g(U(z)).
\end{aligned}$$

Let us substitute this into (2.18):

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[d(U(z))\left|\frac{\mathrm{d}U}{\mathrm{d}z}\right|^{p-2}\frac{\mathrm{d}U}{\mathrm{d}z}\right]=\left[-c+\left|\frac{\mathrm{d}U(z)}{\mathrm{d}z}\right|^{-1}g(U(z))\right]\frac{\mathrm{d}U(z)}{\mathrm{d}z}$$
$$=-c\frac{\mathrm{d}U(z)}{\mathrm{d}z}-g(U(z)),$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[d(U(z))\left|\frac{\mathrm{d}U}{\mathrm{d}z}\right|^{p-2}\frac{\mathrm{d}U}{\mathrm{d}z}\right]+c\frac{\mathrm{d}U(z)}{\mathrm{d}z}+g(U(z))=0,\quad z\in(z_0,z_1).$$

It follows from (2.17) and continuity of U that

$$\lim_{z\to z_0+}d(U(z))\left|U'(z)\right|^{p-2}U'(z)=\lim_{z\to z_1-}d(U(z))\left|U'(z)\right|^{p-2}U'(z)=0.$$

We may summarize the above reasoning in the following equivalence.

Proposition 2.6. *A function $U:\mathbb{R}\to[0,1]$ is a monotone nonincreasing solution of* (2.2), (2.7) *which is strictly decreasing at any point $z\in\mathbb{R}$ where $U(z)\in(0,1)$ if and only if $y:[0,1]\to\mathbb{R}$ is a positive solution of* (2.14), (2.15).

The statement of this proposition remains true even for more general d with properties listed in the beginning of this chapter. Indeed, the main idea behind the transformation suggests that the solution U does not have to be differentiable on the interval (z_0,z_1) as long as it is invertible. If d has jumps in a finite number points $s_i\in(0,1)$, $i=1,2,\dots,n$, then $M_U=\{\xi_1,\xi_2,\dots,\xi_n\}$ where $U(\xi_i)=s_i$. In particular, $\operatorname{int}M_U=\emptyset$, $M_U=\partial M_U$ and for all $z\in M_U$ we have $U'(z-)<0$, $U'(z+)<0$. The function U is continuous and piecewise C^1 in the sense that $U\big|_{(\xi_i,\xi_{i+1})}\in C^1(\xi_i,\xi_{i+1})$. The proof is then conducted similarly as above. We derive the same set of equations on intervals where U and U^{-1} are differentiable and we deal with one-sided limits at ξ_i and s_i separately to establish the equivalence. Consequently, the first order equation (2.14) holds pointwise only for $t\in(0,1)\setminus\bigcup_{i=1}^{n}\{s_i\}$ and using the continuity of solution we conclude that U satisfies the transition condition

$$\left|U'(\xi_i-)\right|^{p-2}U'(\xi_i-)\lim_{s\to s_i+}d(s)=\left|U'(\xi_i+)\right|^{p-2}U'(\xi_i+)\lim_{s\to s_i-}d(s)$$

as a final part of the proof. We refer the reader to our paper [9, p. 4–6] in Appendix B for the detailed execution of these steps.

Next, we discuss the sign of the speed of propagation c. Let $y(t)>0$, $t\in(0,1)$ be a positive solution of (2.14), (2.15). Integrating (2.14) and using (2.15) we obtain

$$0=y(1)-y(0)=\int_0^1 y'(t)\,\mathrm{d}t=p'\left[c\int_0^1(y(t))^{\frac{1}{p}}\,\mathrm{d}t-\int_0^1 f(t)\,\mathrm{d}t\right]$$

and hence

$$c=\frac{\int_0^1 f(t)\,\mathrm{d}t}{\int_0^1(y(t))^{\frac{1}{p}}\,\mathrm{d}t},\tag{2.19}$$

where f is given by (2.13). It follows immediately that the sign of c is ultimately determined by the sign of

$$\int_0^1 f(t)\,\mathrm{d}t=\int_0^1(d(t))^{\frac{1}{p-1}}g(t)\,\mathrm{d}t,$$

justifying the following lemma.

Lemma 2.7. *Let us assume that*

$$\int_0^1 (d(t))^{\frac{1}{p-1}}\, g(t)\,\mathrm{d}t > 0 \quad (< 0)$$

and BVP (2.14), (2.15) *has a positive solution. Then* $c > 0$ (< 0).

Remark 2.8. Suppose that the following balanced condition holds

$$\int_0^1 (d(s))^{\frac{1}{p-1}}\, g(s)\,\mathrm{d}s = 0.$$

Then $c = 0$ and

$$y(t) = -p' \int_0^t (d(s))^{\frac{1}{p-1}}\, g(s)\,\mathrm{d}s, \quad t \in (0,1) \tag{2.20}$$

is a unique positive solution of (2.14), (2.15) with $c = 0$ (cf. Theorem 3.12). The solution given by (2.20) leads to the standing wave. Its profile $u = u(x)$ satisfies the equation

$$\big(d(u(x))|u'(x)|^{p-2}u'(x)\big)' + g(u(x)) = 0, \quad x \in \mathbb{R}.$$

Remark 2.9. If we were to look for nondecreasing solutions instead of nonincreasing ones, the procedure leading up to the first order problem would be the same. Let us denote the speed of propagation of a nondecreasing travelling wave by C. Since $U'(z) > 0$ whenever U' exists and $U'(z-) > 0$, $U'(z+) > 0$ if $z = \xi_i$, $i = 1, 2, \ldots, n$, instead of equation (2.12) we would arrive at

$$\frac{1}{p'}\frac{\mathrm{d}}{\mathrm{d}U}\,|w|^{p'} = -C\,|w|^{p'-1} - (d(U))^{p'-1}\, g(U),$$

where $|w| = w$. The corresponding first order equation can be written in the form (2.14) if we set $c = -C$. Therefore, the existence results regarding nonincreasing solutions also hold for nondecreasing travelling waves which travel in the opposite direction.

Chapter 3

Main results

In this chapter we present results concerning the existence and uniqueness of travelling wave solutions to the generalized Fischer-Kolmogorov equation (2.1). We also discuss the monotonicity of solutions since in some cases less assumptions are needed to conclude the properties of solutions listed in Theorems 3.9, 3.13 below. Finally, we investigate the asymptotic profile of solutions and provide classification based on qualitatively different types of behaviour.

We assume the diffusion coefficient $d = d(s)$ as well as the reaction term $g = g(s)$ to be as in Chapter 2 and we distinguish among three cases described therein: bistable unbalanced, bistable balanced and monostable case. Our primary focus is the bistable case which we treated in [8], [9]. The monostable case is a topic of our current research and the main results remain yet to be proven in full detail. Therefore, we now present just conjectures formulated as open problems and include complete proofs only where they are available.

3.1 Existence results

In Section 2.2 we have shown that the second order boundary value problem

$$\begin{cases} \left(d(U(z))\,|U'(z)|^{p-2}\,U'(z)\right)' + cU'(z) + g(U(z)) = 0, \quad z \in \mathbb{R}, \\ \lim\limits_{z\to-\infty} U(z) = 1, \quad \lim\limits_{z\to+\infty} U(z) = 0 \end{cases} \tag{3.1}$$

can be reduced to a first order problem (2.14), (2.15), see Proposition 2.6. More precisely, these problems are equivalent in the sense that the solution of one of them determines uniquely the solution to the other. This allows us to derive existence results for (3.1) by studying the first order problem

$$\begin{cases} y'(t) = p'\left[c\left(y^+(t)\right)^{\frac{1}{p}} - f(t)\right], \quad t \in (0,1), \\ y(0) = y(1) = 0. \end{cases} \tag{3.2}$$

Here $y^+(t) = \max\{y(t), 0\}$ denotes the positive part of y, $p > 1$ and $p' > 1$ are conjugate numbers and recall that

$$f(t) = (d(t))^{\frac{1}{p-1}}\,g(t). \tag{3.3}$$

Note that (3.2) is not an overdetermined problem since besides the positive solution $y = y(t)$ we also look for unknown speed of propagation $c \in \mathbb{R}$.

Let us also recall that the diffusion coefficient $d = d(t)$ is generally only piecewise continuous with jumps at a finite number of points. Therefore, we need to employ the concept of solution of the first order ODE in the sense of Carathéodory, see [3, Chapter 2]. Since the particular form of $f = f(t)$ will be important later, we only assume for now that $f \in L^1(0,1)$. For $(t,y,c) \in [0,1] \times \mathbb{R}^2$ and $f = f(t)$ we set

$$h(t,y,c) := p' \left[c \left(y^+\right)^{\frac{1}{p}} - f(t) \right]$$

and consider the following two initial value problems which depend on a parameter $c \in \mathbb{R}$:

$$y'(t) = h(t, y(t), c), \quad y(0) = 0 \tag{3.4}$$

and

$$y'(t) = h(t, y(t), c), \quad y(1) = 0. \tag{3.5}$$

In both cases we look for a solution $y = y(t)$, $t \in [0,1]$. Therefore, (3.4) is referred to as a *forward initial value problem*, while (3.5) is referred to as a *backward initial value problem.* Note that $f \in L^1(0,1)$ implies that $h = h(t,y,c)$ satisfies Carathéodory's conditions, i.e., for a.e. $t \in [0,1]$ fixed, $h(t,\cdot,\cdot)$ is continuous with respect to y and c and for every $y \in \mathbb{R}$ and $c \in \mathbb{R}$ fixed, $h(\cdot, y, c)$ is measurable with respect to t. In what follows, for any fixed $c \in \mathbb{R}$, $y_c = y_c(t)$ denotes the solution in the sense of Carathéodory of the forward and backward initial value problem (3.4) and (3.5), respectively. In particular, y_c is absolutely continuous in $[0,1]$ and the equation holds a.e. in $[0,1]$. We first mention the following global existence result.

Lemma 3.1. [9, Lemma 4.2] *Let $f \in L^1(0,1)$, $c \in \mathbb{R}$. Then there exists at least one global solution $y_c = y_c(t)$ of both* (3.4) *and* (3.5) *defined on the entire interval* $[0,1]$.

The uniqueness of the solution in the above lemma does not hold in general due to the fact that the function $y \mapsto c\left(y^+\right)^{\frac{1}{p}}$, $y \in \mathbb{R}$, does not satisfy the Lipschitz condition at 0. However, it is nondecreasing for $c \geq 0$ and nonincreasing for $c \leq 0$. Therefore, it satisfies one-sided Lipschitz condition in either case and we have the following uniqueness results separately for the forward and backward initial value problems, depending on the sign of c.

Lemma 3.2. [9, Lemma 4.4] *Let $f \in L^1(0,1)$. If $c \leq 0$ then* (3.4) *has exactly one solution $y_c = y_c(t)$, $t \in [0,1]$. If $c \geq 0$ then* (3.5) *has exactly one solution $y_c = y_c(t)$, $t \in [0,1]$.*

Thanks to the uniqueness result we have also continuous dependence of solutions on the parameter c.

Lemma 3.3. [9, Lemma 4.5] *Let $f \in L^1(0,1)$, $c_0 \geq 0$. Then $c \to c_0 > 0$ or $c \to 0+$ if $c_0 = 0$ implies that solutions $y_c = y_c(t)$ of the backward initial value problem* (3.5) *converge uniformly in* $[0,1]$ *(i.e., in the topology of* $C[0,1]$*) to y_{c_0}. Similar result holds for $c_0 \leq 0$ and the forward initial value problem* (3.4)*.*

The basic idea now is to use the shooting method with c as a parameter. To this end, we further focus on parameters $c \in [0, +\infty)$ and the backward initial value problem (3.5). It follows from (2.19) that this restriction is justified under the assumption

$$\int_0^1 f(t) \, \mathrm{d}t \geq 0. \tag{3.6}$$

We know that for any $c \in [0, +\infty)$ there is a unique solution of (3.5), $y_c = y_c(t)$, $t \in [0,1]$. Our goal is to show that there exists $c \geq 0$ such that $y_c > 0$ in $(0,1)$ and $y_c(0) = 0$. In order to do that we have to investigate in more detail the dependence of solution $y_c = y_c(t)$ of the backward initial value problem (3.5) on the parameter c.

Let us introduce the notion of the *defect* $P_c\varphi$ of a function $\varphi = \varphi(t)$ with respect to the differential equation $y' = h(t, y, c)$, see e.g. [22, §9.II]:

$$(P_c\varphi)(t) := \varphi'(t) - h(t, \varphi(t), c).$$

The following comparison argument is one of our basic tools.

Lemma 3.4. [9, Lemma 4.6] *Let $f \in L^1(\varrho, 1)$, $0 \le \varrho < 1$, $c \ge 0$, $\varphi(1) \le \psi(1)$, $P_c\varphi \ge P_c\psi$ a.e. in $[\varrho, 1]$. Then either $\varphi < \psi$ in $(\varrho, 1]$ or there exists $\xi \in (\varrho, 1]$ such that $\varphi = \psi$ in $[\xi, 1]$ and $\varphi < \psi$ in (ϱ, ξ). In particular, $\varphi \le \psi$ in $[\varrho, 1]$.*

Let us now assume that $f \in L^1(0, 1)$ has the following property

$$\int_t^1 f(s)\,\mathrm{d}s > 0 \quad \text{for all } 0 < t < 1.$$

This condition is clearly satisfied if

$$f(t) > 0, \quad t \in (0, 1) \tag{3.7}$$

or if (3.6) holds and

$$f(t) \le 0 \quad \text{for } t \in (0, s_*), \quad f(t) > 0 \quad \text{for } t \in (s_*, 1). \tag{3.8}$$

Note that for $f = f(t)$ given by (3.3), monostable reaction term g yields (3.7) while for g bistable (3.8) holds and f has exactly one intermediate zero. We allow the first inequality in (3.8) to be non-strict since it does not affect obtained results. Moreover, it corresponds to the case $g(t) \le 0$ for $t \in (0, s_*)$, $g(t) > 0$ for $t \in (s_*, 1)$ used in the modelling of flame propagation in chemical reactor theory, cf. Chapter 1. Using Lemma 3.4 we can then prove the following result, see [9, proof of Corollary 4.7].

Corollary 3.5. *Let $f \in L^1(0, 1)$ be such that (3.7) or (3.6), (3.8) hold and $0 \le c_1 < c_2$. Then*

$$y_{c_1}(t) > y_{c_2}(t), \quad t \in (0, 1).$$

In particular, we have the following weak comparison at the terminal value 0: $y_{c_1}(0) \ge y_{c_2}(0)$.

The next step is to investigate the function

$$\mathcal{S} : c \mapsto y_c(0).$$

For this task we now need to specify the assumptions on $f = f(t)$. The distinction among the following three cases is based on the notion from Chapter 2, suggesting which type of reaction term g leads to the desired properties of f.

3.1.1 Bistable unbalanced case

Let $f \in L^1(0, 1)$ be such that (3.8) holds and

$$\int_0^1 f(t)\,\mathrm{d}t > 0.$$

Under these assumptions on $f = f(t)$ we prove that there exists a unique real number $c_* > 0$ and an absolutely continuous function $y_{c_*} = y_{c_*}(t)$ such that $y_{c_*} > 0$ on $(0, 1)$, $y_{c_*}(0) = y_{c_*}(1) = 0$ and the equation in (3.2) is satisfied in the sense of Carathéodory.

First, we mention the following two corollaries of Lemma 3.4.

Corollary 3.6. [9, Corollary 4.10] *Let* $f \in L^1(0,1)$, $f(t) \le 0$, $t \in (0, s_*)$, $f(t) > 0$, $t \in (s_*, 1)$. *Then there exists* $c_\# > 0$ *such that* $y_{c_\#}(0) < 0$.

Corollary 3.7. [9, Corollary 4.11] *Let* $f \in L^1(0,1)$, $f(t) \le 0$, $t \in (0, s_*)$, $f(t) > 0$, $t \in (s_*, 1)$ *and* $y_c = y_c(t)$ *be a solution of* (3.5) *with* $c > 0$. *If* $y_c(0) \ge 0$ *then* $y_c(t) > 0$ *for* $t \in (0,1)$.

It follows from the properties of f that

$$y_0(t) = p' \int_t^1 f(s) \, \mathrm{d}s > 0$$

for all $t \in [0,1)$. In particular, $y_0(0) > 0$. On the other hand, from Corollary 3.6 there exists $c_\# > 0$ such that $y_{c_\#}(0) < 0$. The continuous dependence on parameter c in Lemma 3.3, intermediate value theorem and the monotonicity of function $\mathcal{S} : c \mapsto y_c(0)$ in Corollary 3.5 imply that there exist $0 < c_1 \le c_2 < c_\#$ such that $\mathcal{S}(c) = 0$ for all $c \in [c_1, c_2]$, $\mathcal{S}(c) > 0$, $0 \le c < c_1$ and $\mathcal{S}(c) < 0$, $c > c_2$.

Further analysis of the solution of the backward initial value problem (3.5) at the terminal value zero shows that $c_1 = c_2$. Note that this result cannot be concluded from the reasoning above since according to Corollary 3.5 we only know that the function $\mathcal{S}$ is nonincreasing. Therefore, a strong comparison argument at 0 needs to be derived separately. We refer the reader to [9, Proof of Theorem 4.1] for more details and summarize the results for the first order problem (3.2) below.

Theorem 3.8. [9, Theorem 4.1] *Let* $f \in L^1(0,1)$, $f(t) \le 0$, $t \in (0, s_*)$, $f(t) > 0$, $t \in (s_*, 1)$ *for some* $s_* \in (0,1)$ *and*

$$\int_0^1 f(t) \, \mathrm{d}t > 0. \tag{3.9}$$

Then there is a unique number $c > 0$ *and an absolutely continuous function* $y = y(t)$, $t \in [0,1]$, *such that* $y(0) = y(1) = 0$, $y(t) > 0$, $t \in (0,1)$, *and*

$$y'(t) = p' \left[c \left(y^+(t) \right)^{\frac{1}{p}} - f(t) \right]$$

for a.a. $t \in (0,1)$.

In combination with Proposition 2.6 this result is a tool to prove the existence and uniqueness of the travelling speed c and monotone nonincreasing travelling wave profile U satisfying (3.1).

Theorem 3.9. [9, Theorem 4.12] *Let* d *be as in Chapter 2 and* $g \in C[0,1]$ *such that* $g(0) = g(s_*) = g(1) = 0$ *for* $s_* \in (0,1)$,

$$g(s) \le 0, \ s \in (0, s_*), \quad g(s) > 0, \ s \in (s_*, 0)$$

and

$$\int_0^1 (d(s))^{\frac{1}{p-1}} g(s) \, \mathrm{d}s > 0. \tag{3.10}$$

Then there is a unique value of $c = c_*$ *and a unique nonincreasing travelling wave profile* $U = U(z)$, $z \in \mathbb{R}$, *such that* U *solves the BVP* (3.1). *Furthermore,* $c_* > 0$ *and*

(i) *there exist* $-\infty \le z_0 < 0 < z_1 \le +\infty$ *such that* $U(z) = 1$ *for* $z \in (-\infty, z_0]$, $U(z) = 0$ *for* $z \in [z_1, +\infty)$;

(ii) U *is strictly decreasing in* (z_0, z_1), $U(0) = s_*$ *(see Figure 3.1);*

(iii) *for* $i = 0, 1, 2, \ldots, n, n+1$ *let* $\xi_i \in [z_0, z_1]$ *be such that* $U(\xi_i) = s_i$*, then* U *is a piecewise* C^1*- function in the sense that* U *is continuous,*

$$U\big|_{(\xi_i,\xi_{i+1})} \in C^1(\xi_i, \xi_{i+1}), \quad i = 0, 1, \ldots, n,$$

and the limits $U'(\xi_i-) := \lim_{z\to\xi_i-} U'(z)$*,* $U'(\xi_i+) := \lim_{z\to\xi_i+} U'(z)$ *exist finite for all* $i = 1, 2, \ldots, n$*;*

(iv) *for any* $i = 1, 2, \ldots, n$ *the following transition condition holds:*

$$\left|U'(\xi_i-)\right|^{p-2} U'(\xi_i-) \lim_{s\to s_i+} d(s) = \left|U'(\xi_i+)\right|^{p-2} U'(\xi_i+) \lim_{s\to s_i-} d(s).$$

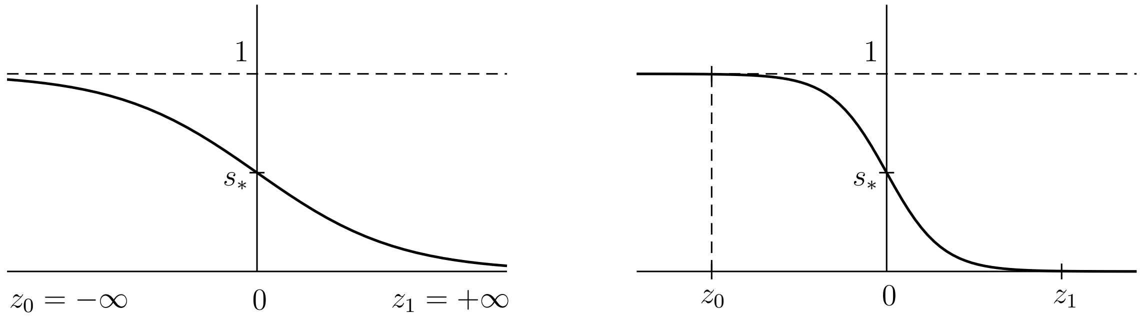

Figure 3.1: Nonincreasing solutions $U = U(z)$ of the BVP (3.1)

Remark 3.10. The inequality (3.10) was motivated by modelling heterozygote inferior case. On the other hand, inequality

$$\int_0^1 (d(s))^{\frac{1}{p-1}} g(s)\,\mathrm{d}s < 0 \tag{3.11}$$

leads to negative travelling speed of propagation $c_* < 0$ and it can be treated in a similar way. However, in this case the main tool is a shooting argument applied to the forward initial value problem and the strong comparison argument must be derived at the terminal value 1.

We can also prove similar results for nondecreasing travelling wave profile U satisfying

$$\lim_{z\to-\infty} U(z) = 0 \quad \text{and} \quad \lim_{z\to+\infty} U(z) = 1.$$

In this case the assumption (3.10) leads to $c_* < 0$ while (3.11) leads to $c_* > 0$, respectively, cf. Remark 2.9.

Remark 3.11. Notice that condition $U(0) = s_*$ has just a normalizing character. Indeed, since the equation (2.2) is autonomous then given any $\xi \in \mathbb{R}$ the translation $V(z) = U(z - \xi)$, $z \in \mathbb{R}$, is also a solution of (2.2) which satisfies $V(\xi) = s_*$. We will use this fact in the remaining cases as well.

3.1.2 Bistable balanced case

Let, now, $f \in L^1(0, 1)$ be such that

$$f(t) < 0 \quad \text{for } t \in (0, s_*), \quad f(t) > 0 \quad \text{for } t \in (s_*, 1)$$

for some $s_* \in (0,1)$ and assume that

$$\int_0^1 f(t)\,\mathrm{d}t = 0. \tag{3.12}$$

This is a special form of f which satisfies (3.6), (3.8). Therefore, the general results regarding the backward initial value problem (3.5) up to Corollary 3.5 remain valid. In contrast with the previous case, the situation is now much simpler.

Indeed, from (2.19) we see that (3.12) forces $c = 0$. The solution of

$$\begin{cases} y'(t) = -p' f(t), \quad t \in (0,1), \quad t \in (0,1) \\ y(0) = 0 \end{cases} \tag{3.13}$$

is then

$$y_0(t) = -p' \int_0^t f(s)\,\mathrm{d}s > 0, \quad t \in (0,1).$$

and due to (3.12) it satisfies $y_0(1) = 0$.

Let us summarize the above reasoning in the following analogue of Theorem 3.8.

Theorem 3.12. *Let $f \in L^1(0,1)$, $f(t) < 0$, $t \in (0,s_*)$, $f(t) > 0$, $t \in (s_*,1)$ for some $s_* \in (0,1)$ and*

$$\int_0^1 f(t)\,\mathrm{d}t = 0.$$

Then $c = 0$ and $y_0(t) = -p' \int_0^t f(s)\,\mathrm{d}s$ is a unique solution of the BVP (3.2).

In the context of the second order BVP (3.1), this result yields that the bistable balanced condition, namely

$$\int_0^1 (d(s))^{\frac{1}{p-1}}\, g(s)\,\mathrm{d}s = 0, \tag{3.14}$$

is sufficient for the existence of a standing wave solution which is unique up to translation. On the other hand, if we have a solution $u = u(x)$ to the second order BVP (3.1) with $c = 0$ then by Proposition 2.6 and Lemma 2.7 we obtain that (3.14) must hold, i.e., this condition is also necessary.

In [8] we proved this result without direct reference to the first order problem (3.2) since for $c = 0$ the reduction described in Section 2.2 leads to the equation in (3.13). It is apparent from above that similar approach as for $c \neq 0$ would be unnecessarily complicated.

The bistable balanced condition yields both types of monotone standing waves – nonincreasing and nondecreasing. From Remark 2.9 it can be easily seen that in both cases we arrive at the same equivalent equation since $c = C = 0$. In [8] we focused primarily on nondecreasing standing waves. For reasons of consistency with the previous case we modify the statement of [8, Theorem 3.2] for nonincreasing solutions.

Theorem 3.13. [8, Theorem 3.2 and Remark 3.4] *Let d be as in Chapter 2 and $g \in C[0,1]$ such that $g(0) = g(s_*) = g(1) = 0$ for $s_* \in (0,1)$,*

$$g(s) < 0,\ s \in (0,s_*), \quad g(s) > 0,\ s \in (s_*,0)$$

Then the BVP (3.1) with $c = 0$ has a nonincreasing solution if and only if

$$\int_0^1 (d(s))^{\frac{1}{p-1}}\, g(s)\,\mathrm{d}s = 0. \tag{3.15}$$

If (3.15) *holds then there is a unique solution* $u = u(x)$ *of*

$$\begin{cases} \left(d(u(x))\,|u'(x)|^{p-2}\,u'(x)\right)' + g(u(x)) = 0 \\ \lim\limits_{x\to-\infty} u(x) = 1, \quad \lim\limits_{x\to+\infty} u(x) = 0 \end{cases}$$

such that the following conditions hold:

(i) *there exist* $-\infty \le x_0 < 0 < x_1 \le +\infty$ *such that* $u(x) = 1$ *for* $x \le x_0$, $u(x) = 0$ *for* $x \ge x_1$ *and* $0 < u(x) < 1$ *for* $x \in (x_0, x_1)$*;*

(ii) u *is strictly decreasing in* (x_0, x_1), $u(0) = s_*$*;*

(iii) *for* $i = 1, 2, \ldots, n$ *let* $\xi_i \in \mathbb{R}$ *be such that* $u(\xi_i) = s_i$, $\xi_0 = x_0$ *and* $\xi_{n+1} = x_1$. *Then* u *is a piecewise* C^1*-function in the sense that* u *is continuous,*

$$u\big|_{(\xi_i,\xi_{i+1})} \in C^1(\xi_i, \xi_{i+1}), \quad i = 0, 1, \ldots, n,$$

and the limits $u'(\xi_i-) := \lim\limits_{x\to\xi_i-} u'(x)$, $u'(\xi_i+) := \lim\limits_{x\to\xi_i+} u'(x)$ *exist finite for all* $i = 1, 2, \ldots, n$*;*

(iv) *for any* $i = 1, 2, \ldots, n$, *the following transition condition holds:*

$$\left|u'(\xi_i-)\right|^{p-2} u'(\xi_i-) \lim_{s\to s_i+} d(s) = \left|u'(\xi_i+)\right|^{p-2} u'(\xi_i+) \lim_{s\to s_i-} d(s).$$

3.1.3 Monostable case

Finally, we include the expected results regarding the existence of solution for the first order problem (3.2) in the case of $f \in L^1(0,1)$, $f(t) > 0$ for $t \in (0,1)$. They extend those from [10] where the same equation was studied assuming $f \in C[0,1]$, $f(0) = f(1) = 0$ and $f > 0$ in $(0,1)$.

Open problem 1. *Let* $f(t) > 0$, $t \in (0,1)$, *and*

$$0 < \mu := \sup_{t\in(0,1)} \frac{f(t)}{t^{p'-1}} < +\infty. \tag{3.16}$$

Then there exists a number $c^* \in (0, (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}]$ *such that the BVP* (3.2) *has a unique positive solution if and only if* $c \ge c^*$.

Open problem 2. *Let* $f(t) > 0$, $t \in (0,1)$, *and*

$$0 < \nu := \liminf_{t\to0+} \frac{f(t)}{t^{p'-1}}. \tag{3.17}$$

If

$$c < (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \nu^{\frac{1}{p'}} \tag{3.18}$$

then the BVP (3.2) *has no solution. In particular, if*

$$\lim_{t\to0+} \frac{f(t)}{t^{p'-1}} = +\infty,$$

the BVP (3.2) *has no solution for any* $c \in \mathbb{R}$.

Remark 3.14. Let μ and ν be defined as in Open prolems 1 and 2, respectively. Then we conclude that the minimal value of the "critical" speed $c^* > 0$ must satisfy

$$(p')^{\frac{1}{p'}} p^{\frac{1}{p}} \nu^{\frac{1}{p'}} \le c^* \le (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}.$$

The above results would then lead to the following existence theorem for the second order BVP (3.1) with monostable reaction term g.

Theorem 3.15. *Let d be as in Chapter 2, $g \in C[0,1]$, $g(0) = g(1) = 0$, $g > 0$ on $(0,1)$ and assume that the function*

$$f(t) = (d(t))^{\frac{1}{p-1}} g(t)$$

satisfies (3.16), (3.17). *Then there exists a unique value $c^* > 0$ such that the BVP* (3.1) *has*

(i) *a unique nonincreasing solution for every wavespeed $c \ge c^*$ which has the properties* (i)–(iv) *from Theorem 3.9;*

(ii) *no solution for $0 < c < c^*$.*

Moreover,

$$(p')^{\frac{1}{p'}} p^{\frac{1}{p}} \nu^{\frac{1}{p'}} \le c^* \le (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}$$

where μ and ν are defined as in (3.16) *and* (3.17), *respectively.*

3.2 Monotonicity of solutions

In order to derive the equivalence between

$$\begin{cases} \Big(d(U(z))\,|U'(z)|^{p-2}\,U'(z)\Big)' + cU'(z) + g(U(z)) = 0, \\ \lim\limits_{z\to-\infty} U(z) = 1, \qquad \lim\limits_{z\to+\infty} U(z) = 0 \end{cases} \tag{3.19}$$

and the first order boundary value problem (3.2), it was necessary to assume strict monotonicity of solutions at points where they do not attain values 0 and 1. The transformation presented in Section 2.2 relied on the fact that solutions are invertible on (z_0, z_1), where $\lim_{z\to z_0+} U(z) = 1$, $\lim_{z\to z_1-} U(z) = 0$, and constant outside of this interval if either one or both endpoints are finite. More precisely, for nonincreasing solutions we had $U(z) = 1$ for $z \le z_0$, $U(z) \in (0,1)$ for $z \in (z_0, z_1)$ and $U(z) = 0$ for $z \ge z_1$.

The assumption about strict monotonicity is necessary in the bistable unbalanced case, but can be omitted in the other two cases, as we will show in this section. We prove that monotone standing wave solutions are in fact strictly monotone between 0 and 1, i.e., whenever they attain values in $(0,1)$ (see 3.2.1). In the monostable case we obtain an even stronger result, stating that every continuous function monotone or nonmonotone, satisfying (3.19) is actually nonincreasing and strictly decreasing at points where it does not attain values 0 and 1 (see 3.2.2).

3.2.1 Stationary solutions

Throughout this section we will assume that d and g are as in Chapter 2 and g is a bistable reaction term. Let us recall that stationary solutions of (2.1) satisfy the equation

$$\big(d(u(x))|u'(x)|^{p-2}u'(x)\big)' + g(u(x)) = 0, \quad x \in \mathbb{R}, \quad p > 1 \tag{3.20}$$

and they exist if and only if the following bistable balanced condition holds

$$\int_0^1 (d(s))^{\frac{1}{p-1}} g(s)\,\mathrm{d}s = 0.$$

Since the propagation speed of these solutions is zero, the first integral (2.3) in Definition 2.1 can be written in the form

$$v(y) - v(x) + \int_x^y g(u(\sigma))\,\mathrm{d}\sigma = 0 \quad \text{for any } x, y \in \mathbb{R}. \tag{3.21}$$

Remark 3.16. Let u be a solution of (3.20) satisfying boundary conditions

$$\lim_{x\to-\infty} u(x) = 1, \quad \text{and} \quad \lim_{x\to+\infty} u(x) = 0 \tag{3.22}$$

or

$$\lim_{x\to-\infty} u(x) = 0, \quad \text{and} \quad \lim_{x\to+\infty} u(x) = 1. \tag{3.23}$$

By Definition 2.1, in either case, passing to the limit for $x \to -\infty$ and $y \to +\infty$ in (3.21) and writing x in place of y in the first case, we obtain that for every $x \in \mathbb{R}$ the following two equations hold:

$$v(x) + \int_{-\infty}^x g(u(\sigma))\,\mathrm{d}\sigma = 0 \tag{3.24}$$

and

$$v(x) - \int_x^{+\infty} g(u(\sigma))\,\mathrm{d}\sigma = 0. \tag{3.25}$$

Proposition 3.17. *Let u be a nonincreasing solution of* (3.20), (3.22) *with a bistable reaction term g. Then u is strictly decreasing at any point $x \in \mathbb{R}$ such that $u(x) \in (0,1)$.*

Proof. Let $u = u(x)$ be a nonincreasing solution of the BVP (3.20), (3.22) such that $u(0) = s_*$. Since the equation is autonomous this condition is just a normalization of the solution. It follows from (3.22) that

$$-\infty \le x_0 := \inf\{x \in \mathbb{R} : u(x) < 1\} < 0$$

is well defined. By (3.24) and continuity of v we have

$$0 < x_1 := \sup\{x \in \mathbb{R} : v(\xi) < 0 \text{ for all } \xi \in (x_0, x)\} \le +\infty\,.$$

Since $d(s) > 0, s \in (0,1)$, it follows from the definition of $v(x)$ that u is a strictly decreasing function in (x_0, x_1) and therefore the following limit

$$\bar{u}(x_1) := \lim_{x\to x_1-} u(x)$$

is well defined. If $x_1 = +\infty$ then by the second condition in (3.22) it must be $\bar{u}(x_1) = 0$. On the other hand, if $x_1 < +\infty$, we have $\bar{u}(x_1) = u(x_1)$, $v(x_1) = 0$ and $s_* > u(x_1) \ge 0$. We rule out the case $u(x_1) > 0$. Indeed, $v(x_1) = 0$ implies $u'(x_1-) = u'(x_1+) = u'(x_1) = 0$. From $s_* > u(x_1)$ and (2.6) in Remark 2.4 we deduce $v'(x_1) = -g(u(x_1)) > 0$. Therefore, there exists $\delta > 0$ such that for all $x \in (x_1, x_1 + \delta)$ we have $v(x) > 0$ and hence also $u'(x-) > 0$ and $u'(x+) > 0$. This contradicts our assumption that u is nonincreasing. □

Remark 3.18. Modifying the proof of Proposition 3.17 we obtain the same result for nondecreasing solutions, cf. [8] where we considered the BVP (3.20), (3.23). Another possible approach to the proof in both cases is to use (3.25) instead of (3.24) and proceed "from right to left", i.e., starting from x_1, as suggested in [8, Remark 3.4].

3.2.2 Solutions of the monostable equation

Let g be a monostable reaction term. We assume that a continuous function $U = U(z)$ (monotone or nonmonotone) satisfies (3.19) in the sense that (a)–(c) of Definition 2.1 hold. The reader should keep in mind that the closed sets M_U and N_U might be of more complicated structure if U is not a monotone function. Then passing to the limit for $z \to -\infty$ in (2.3) and writing z in place of $\hat{z}$, we obtain that

$$v(z) + c\,(U(z) - 1) + \int_{-\infty}^{z} g(U(\sigma))\,\mathrm{d}\sigma = 0 \tag{3.26}$$

holds for any $z \in \mathbb{R}$. On the other hand, passing to the limit for $\hat{z} \to +\infty$ in (2.3), we obtain that

$$v(z) + cU(z) - \int_{z}^{+\infty} g(U(\sigma))\,\mathrm{d}\sigma = 0 \tag{3.27}$$

holds for any $z \in \mathbb{R}$. Passing to the limit for $\hat{z} \to +\infty$ and $z \to -\infty$ in (2.3) yields

$$-c + \int_{-\infty}^{+\infty} g(U(\sigma))\,\mathrm{d}\sigma = 0.$$

Since $g > 0$ in $(0, 1)$, it follows from here that $c > 0$.

Lemma 3.19. *Let $U = U(z)$, $z \in \mathbb{R}$, be a solution of* (3.19) *and assume $\xi \in N_U$. Then the following two alternatives occur:*

(i) *if $U(\xi) = 0$ then $U(z) = 0$ for every $z \ge \xi$;*

(ii) *if $U(\xi) = 1$ then $U(z) = 1$ for every $z \le \xi$.*

Proof. (i) Let $U(\xi) = 0$ and there exists $\xi_* > \xi$ such that $U(\xi_*) > 0$. Taking ξ_* closer to ξ if necessary, we may assume that also $U(\xi_*) < 1$. Then $g(U(\xi_*)) > 0$ and therefore $\int_{\xi}^{+\infty} g(U(\sigma))\,\mathrm{d}\sigma > 0$. From the definition of v we get $v(\xi) = 0$ and from (3.27) with $z = \xi$ we deduce $\int_{\xi}^{+\infty} g(U(\sigma))\,\mathrm{d}\sigma = 0$, a contradiction.
(ii) Assume $U(\xi) = 1$ and there is some $\xi_* < \xi$ such that $U(\xi_*) < 1$. Taking ξ_* closer to ξ if necessary, we can guarantee also $U(\xi_*) > 0$. Hence $g(U(\xi_*)) > 0$ and so $\int_{-\infty}^{\xi} g(U(\sigma))\,\mathrm{d}\sigma > 0$. From the definition of v we have $v(\xi) = 0$ and from (3.26) with $z = \xi$ we deduce $\int_{-\infty}^{\xi} g(U(\sigma))\,\mathrm{d}\sigma = 0$, a contradiction. □

Lemma 3.20. *Let $U = U(z)$, $z \in \mathbb{R}$, be a solution of* (3.19). *Then U is nonincreasing in $\mathbb{R}$. Moreover, for $z \notin N_U$ we have $U'(z) < 0$ if $z \notin M_U$ and $U'(z-) < 0$, $U'(z+) < 0$ if $z \in M_U$.*

Proof. Let $\xi \notin N_U$ be such that $U'(\xi-) = 0$. Then it follows from Remarks 2.4, 2.5, depending on whether $z \notin M_U \cup N_U$ or $z \in M_U$, respectively, that

$$v'(\xi-) = -g(U(\xi)) < 0.$$

Since $v(\xi) = 0$, there exists a left neighbourhood $\mathcal{U}_-(\xi)$ of the point ξ such that for all $z \in \mathcal{U}_-(\xi)$ we have $v(z) > 0$. Taking $\mathcal{U}_-(\xi)$ smaller if necessary, we may assume that $N_U \cap \mathcal{U}_-(\xi) = \emptyset$. Since $d(U(z)) > 0$, $z \in \mathcal{U}_-(\xi)$, from $v(z) > 0$ we deduce that for any $z \in \mathcal{U}_-(\xi)$ we have also

$U'(z-) > 0$, $U'(z+) > 0$. However, this implies that $U(z) < U(\xi)$, $z \in \mathcal{U}_-(\xi)$. Since, by Definition 2.1, $U'(\xi+) = 0$, we deduce that there is also a right neighbourhood $\mathcal{U}_+(\xi)$ of ξ such that $U(z) < U(\xi)$, $z \in \mathcal{U}_+(\xi)$. Therefore, ξ is the point of strict local maximum for U. Since $U(z) \to 1$ as $z \to -\infty$ and $U(\xi) < 1$, there is $\xi_* \in (-\infty, \xi)$ such that $U(\xi) \leq U(\xi_*) < 1$. Let $\xi^* \in [\xi_*, \xi]$ be a global minimizer for U over the compact interval $[\xi_*, \xi]$. But ξ is a strict local maximizer of U and hence $\xi^* \in (\xi_*, \xi)$, $U(\xi^*) < U(\xi)$ and $U'(\xi^*) = 0$. If also $U(\xi^*) > 0$, we prove as above that ξ^* must be a strict local maximizer for U, a contradiction. The case $U(\xi^*) = 0$ would lead to a contradiction with the previous lemma which would force $U(z) = 0$ for every $z \geq \xi^*$ and, in particular, also $U(\xi) = 0$ contradicting the choice of $\xi \notin N_U$. □

We may summarize the assertion of Lemmas 3.19 and 3.20 in the following proposition.

Proposition 3.21. *Let U be a continuous function satisfying* (3.19) *in the sense of Definition 2.1 with a monostable reaction term g. Then $U = U(z)$ is a nonincreasing function in $\mathbb{R}$ that is strictly decreasing at any point $z \in \mathbb{R}$ such that $U(z) \in (0, 1)$.*

3.3 Asymptotic behaviour of solutions

Apart from the existence and monotonicity results, we are also interested in the asymptotic behaviour of solutions as they approach equilibria 0 and 1. As shown below, it strongly depends on the behaviour of the diffusion term $d = d(s)$ and reaction term $g = g(s)$ as $s \to 0+$ and $s \to 1-$. Our primary concern is to determine whether the solution actually attains one or both of the values 0 and 1. In other words, we investigate the finiteness of endpoints of the interval (z_0, z_1) on which the solution is strictly monotone. To this end, we assume that both d and g are of power-type in the neighbourhood of 0 and 1 and their shape and properties are consistent with the assumptions from Chapter 2. We then provide a classification based on the different configurations between d, g and $p > 1$ that lead to z_0 and/or z_1 finite or infinite.

In this section we only deal with the bistable reaction term. The asymptotic analysis of solutions in the monostable case is currently under investigation. In the bistable balanced case, corresponding to the standing wave solutions, we are able to obtain additional information about the behaviour of solutions at the "points of detachment" from the steady states 0 and 1. More precisely, if these values are attained, we distinguish among different types of solution according to the one-sided derivatives, and if not, the distinction is done based on "how fast" solutions approach the equilibria. As for the travelling waves in the bistable unbalanced case, the analysis is more involved. Consequently, it is less precise and does not lead to a similar distinction among the types of solutions as in the balanced case.

3.3.1 Asymptotic analysis of the standing wave profile

We begin by summarizing the results from [8, Section 4] where the analysis was done for *nondecreasing* standing waves. By a simple modification we then obtain results also for *nonincreasing* solutions as they are the main focus of this work.

Nondecreasing standing waves

The starting point is the asymptotic analysis of

$$x(u)=\int_{s_*}^{u}\left(\frac{d(s)}{w(s)}\right)^{\frac{1}{p-1}}\mathrm{d}s=\left(\frac{1}{p'}\right)^{\frac{1}{p}}\int\limits_{s_*}^{u}\frac{(d(s))^{\frac{1}{p-1}}}{\left(-\int\limits_0^s(d(\sigma))^{\frac{1}{p-1}}\,g(\sigma)\,\mathrm{d}\sigma\right)^{1/p}}\,\mathrm{d}s, \tag{3.28}$$

i.e., the inverse of the solution $u=u(x)$ which is normalized by $u(0)=s_*$. Since $x=x(u)$ is strictly increasing and maps the interval $(0,1)$ onto (x_0,x_1) where

$$x_0=x(0)=\int\limits_{s_*}^{0}\left(\frac{d(s)}{w(s)}\right)^{\frac{1}{p-1}}\mathrm{d}s=\left(\frac{1}{p'}\right)^{\frac{1}{p}}\int\limits_{s_*}^{0}\frac{(d(s))^{\frac{1}{p-1}}}{\left(-\int\limits_0^s(d(\sigma))^{\frac{1}{p-1}}\,g(\sigma)\,\mathrm{d}\sigma\right)^{1/p}}\,\mathrm{d}s, \tag{3.29}$$

$$x_1=x(1)=\int\limits_{s_*}^{1}\left(\frac{d(s)}{w(s)}\right)^{\frac{1}{p-1}}\mathrm{d}s=\left(\frac{1}{p'}\right)^{\frac{1}{p}}\int\limits_{s_*}^{1}\frac{(d(s))^{\frac{1}{p-1}}}{\left(-\int\limits_0^s(d(\sigma))^{\frac{1}{p-1}}\,g(\sigma)\,\mathrm{d}\sigma\right)^{1/p}}\,\mathrm{d}s, \tag{3.30}$$

the fact that x_0 and x_1 are finite or infinite depends on the asymptotic behaviour of the diffusion coefficient $d=d(s)$ and reaction term $g=g(s)$ near the equilibria 0 and 1. Assuming that d and g are of power-type in the neighbourhood of 0 and 1 allows us to carry out some simple estimates on $x=x(u)$. Applying the inverse function then yields the asymptotics for $u=u(x)$ presented below.

For the sake of brevity, for $t_0\in\mathbb{R}$ we write

$$h_1(t)\sim h_2(t)\text{ as }t\to t_0\quad\text{if and only if}\quad\lim_{t\to t_0}\frac{h_1(t)}{h_2(t)}\in(0,+\infty).$$

Theorem 3.22. *Let $g(s)\sim(-s^{\alpha})$, $d(s)\sim s^{\beta}$ as $s\to 0+$ where $\alpha>0$, $\beta\in\mathbb{R}$ satisfy*

$$\alpha+\frac{\beta}{p-1}>-1 \tag{3.31}$$

for given $p>1$.

(I) *If $\alpha-\beta\ge p-1$ then $x_0=-\infty$. Moreover, for $\alpha-\beta=p-1$ we have*

$$u(x)\sim \mathrm{e}^{x}\to 0+\qquad\text{for }x\to-\infty$$

and for $\alpha-\beta>p-1$ we have

$$u(x)\sim|x|^{\frac{p}{p-1-(\alpha-\beta)}}\to 0+\qquad\text{for }x\to-\infty.$$

(II) *If $\alpha-\beta<p-1$ then $x_0>-\infty$ and for $x\to x_0+$ we have*

$$u(x)\sim(x-x_0)^{\frac{p}{p-1-(\alpha-\beta)}}.$$

As for the derivatives, we then have

(a) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+}\sim\ (x-x_0)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}}\to 0\quad\text{for }x\to x_0+\qquad\text{if }\alpha-\beta>-1,$$

(b) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+}\sim\ (x-x_0)^{0}\to k>0\quad\text{for }x\to x_0+\qquad\text{if }\alpha-\beta=-1,$$

(c) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+}\sim\ (x-x_0)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}}\to+\infty\quad\text{for }x\to x_0+\qquad\text{if }\alpha-\beta<-1.$$

The proof of this theorem follows from the reasoning in [8, Section 4] where detailed discussion is provided. We briefly mention that following the assumptions on d and g from Theorem 3.22 we conclude that

$$x(u) \sim \int_{s_*}^{u} s^{\frac{\beta-\alpha-1}{p}} \mathrm{d}s \tag{3.32}$$

for $u \to 0+$. Convergence or divergence of the integral

$$\int_0^{s_*} s^{\frac{\beta-\alpha-1}{p}} \mathrm{d}s \tag{3.33}$$

then leads to the distinction between two qualitatively different cases (I) and (II) in Theorem 3.22. Condition (3.31) ensures the integrability of $s \mapsto (d(s))^{\frac{1}{p-1}} g(s)$ on $(0,1)$.

Remark 3.23. To visualize conditions from Theorem 3.22, we introduce the sets

$$\begin{aligned}
\mathcal{A}_1 &:= \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0,\ \alpha - \beta = p-1\},\\
\mathcal{A}_2 &:= \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0,\ \alpha + \beta/(p-1) > -1,\ \alpha - \beta > p-1\}
\end{aligned}$$

corresponding to case (I) and

$$\begin{aligned}
\mathcal{B}_1 &:= \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0,\ -1 < \alpha - \beta < p-1\},\\
\mathcal{B}_2 &:= \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0,\ \alpha - \beta = -1\},\\
\mathcal{B}_3 &:= \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0,\ \alpha - \beta < -1\}
\end{aligned}$$

corresponding to case (II). For $p = 2$ these sets are depicted in Figure 3.2. For different values of p these sets are separately sketched in [8].

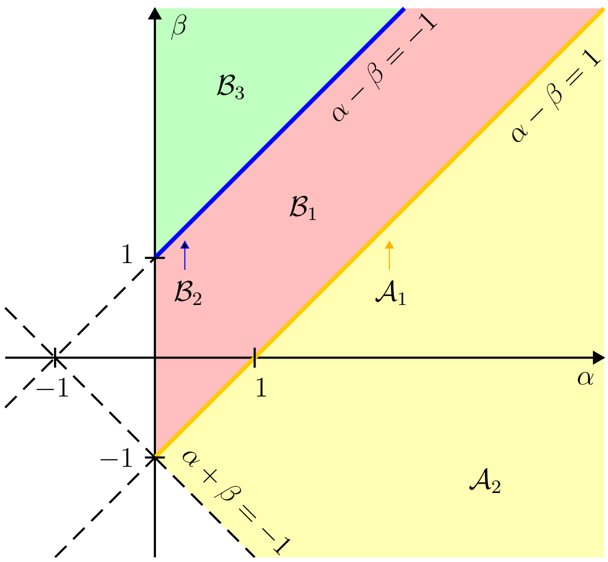

Figure 3.2: Visualization of the sets $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ for $p = 2$

Remark 3.24. From part (II) in Theorem 3.22 we observe that if $x_0 > -\infty$ the solution $u = u(x)$ is smooth in the neighbourhood of x_0 only in case (a), i.e., if $(\alpha,\beta) \in \mathcal{B}_1$, since $u(x) = 0$ for $x \in (-\infty, x_0]$. In the other two cases we only get continuous solutions instead of smooth ones as a consequence of allowing the diffusion coefficient $d = d(s)$ to degenerate as

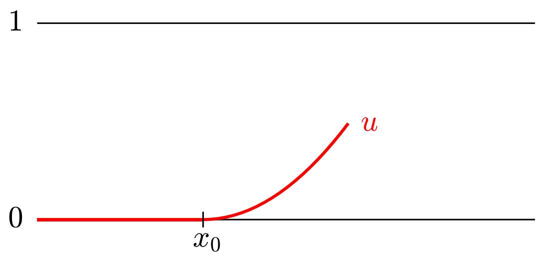

Figure 3.3: $(\alpha, \beta) \in \mathcal{B}_1$

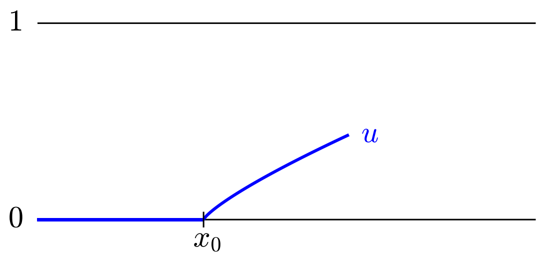

Figure 3.4: $(\alpha, \beta) \in \mathcal{B}_2$

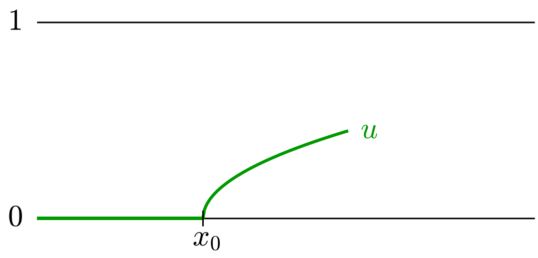

Figure 3.5: $(\alpha, \beta) \in \mathcal{B}_3$

$s \to 0+$. The asymptotic behaviour of these solutions near the point x_0 is illustrated in Figures 3.3–3.5 in colours corresponding to the sets $\mathcal{B}_i$, $i = 1, 2, 3$ from Figure 3.2.

On the other hand, it is interesting to observe that $x_0 = -\infty$ occurs even when the diffusion coefficient degenerates or has a singularity near 0 if this fact is compensated by a proper degeneration of the reaction term g.

We conclude the results concerning nondecreasing standing waves with the asymptotics near 1 which can be derived similarly, cf. [8, Section 4].

Theorem 3.25. *Let $g(s) \sim (1-s)^\gamma$, $d(s) \sim (1-s)^\delta$ as $s \to 1-$ where $\gamma > 0$, $\delta \in \mathbb{R}$ satisfy*

$$\gamma + \frac{\delta}{p-1} > -1$$

for given $p > 1$.

(I) *If $\gamma - \delta \geq p - 1$ then $x_1 = +\infty$. Moreover, for $\gamma - \delta = p - 1$ we have*

$$u(x) \sim 1 - \mathrm{e}^{-x} \to 1- \qquad \text{for } x \to +\infty$$

and for $\gamma - \delta > p - 1$ we have

$$u(x) \sim 1 - |x|^{\frac{p}{p-1-(\gamma-\delta)}} \to 1- \qquad \text{for } x \to +\infty.$$

(II) *If $\gamma - \delta < p - 1$ then $x_1 < +\infty$ and for $x \to x_1-$ we have*

$$u(x) \sim 1 - (x_1 - x)^{\frac{p}{p-1-(\gamma-\delta)}}.$$

As for the derivatives, we then have

(a) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1-} \sim (x_1-x)^{\frac{\gamma-\delta+1}{p-1-(\gamma-\delta)}} \to 0 \quad \textit{for } x \to x_1- \qquad \textit{if } \gamma-\delta > -1,$$

(b) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1-} \sim (x_1-x)^0 \to k > 0 \quad \textit{for } x \to x_1- \qquad \textit{if } \gamma-\delta = -1,$$

(c) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1-} \sim (x_1-x)^{\frac{\gamma-\delta+1}{p-1-(\gamma-\delta)}} \to +\infty \quad \textit{for } x \to x_1- \qquad \textit{if } \gamma-\delta < -1.$$

Remark 3.26. The dependence on parameters γ, δ and p is the same as for asymptotics near 0, i.e., it leads to the same distinction for x_1 finite or infinite and also regarding the derivatives as $x \to x_1-$. Introducing analogous sets as in Remark 3.23 and sketching them in the plane (γ,δ) we would get the same layout as in Figure 3.2. We want to highlight this fact since it will not be the case for the travelling wave solutions, see below.

Nonincreasing standing waves

To derive asymptotic analysis for nonincreasing standing waves, we again investigate the behaviour of the inverse function $x = x(u)$ which now has the form

$$x(u) = -\int_{s_*}^{u} \left|\frac{d(s)}{w(s)}\right|^{\frac{1}{p-1}} \mathrm{d}s = -\left(\frac{1}{p'}\right)^{\frac{1}{p}} \int_{s_*}^{u} \frac{(d(s))^{\frac{1}{p-1}}}{\left(-\int_0^s (d(\sigma))^{\frac{1}{p-1}}\, g(\sigma)\,\mathrm{d}\sigma\right)^{1/p}}\,\mathrm{d}s. \tag{3.34}$$

This follows from (2.16), Remark 2.9 and the fact that $c = 0$ which yields that the right-hand side is the same as in (3.28) but with opposite sign. This suggests that the asymptotic analysis of nonincreasing standing waves should be similar as in the previous case. Since $x = x(u)$ is now strictly decreasing on $(0,1)$, (3.29), (3.30) become

$$x_0 = x(1) = -\int_{s_*}^{1} \left|\frac{d(s)}{w(s)}\right|^{\frac{1}{p-1}} \mathrm{d}s = -\left(\frac{1}{p'}\right)^{\frac{1}{p}} \int_{s_*}^{1} \frac{(d(s))^{\frac{1}{p-1}}}{\left(-\int_0^s (d(\sigma))^{\frac{1}{p-1}}\, g(\sigma)\,\mathrm{d}\sigma\right)^{1/p}}\,\mathrm{d}s,$$

$$x_1 = x(0) = -\int_{s_*}^{0} \left|\frac{d(s)}{w(s)}\right|^{\frac{1}{p-1}} \mathrm{d}s = -\left(\frac{1}{p'}\right)^{\frac{1}{p}} \int_{s_*}^{0} \frac{(d(s))^{\frac{1}{p-1}}}{\left(-\int_0^s (d(\sigma))^{\frac{1}{p-1}}\, g(\sigma)\,\mathrm{d}\sigma\right)^{1/p}}\,\mathrm{d}s.$$

Assuming the same behaviour of $d = d(s)$ and $g = g(s)$ near 0 as above, we obtain

$$x(u) \sim -\int_{s_*}^{u} s^{\frac{\beta-\alpha-1}{p}}\,\mathrm{d}s$$

and proceeding similarly as in [8, Section 4] results in the following asymptotics for nonincreasing standing wave solutions as $x \to x_1$:

Theorem 3.27. *Let $g(s) \sim (-s^\alpha)$, $d(s) \sim s^\beta$ as $s \to 0+$ where $\alpha > 0$, $\beta \in \mathbb{R}$ satisfy*

$$\alpha + \frac{\beta}{p-1} > -1$$

for given $p > 1$.

(I) *If* $\alpha - \beta \geq p - 1$ *then* $x_1 = +\infty$. *Moreover, for* $\alpha - \beta = p - 1$ *we have*

$$u(x) \sim \mathrm{e}^{-x} \to 0+ \qquad \text{for } x \to +\infty$$

and for $\alpha - \beta > p - 1$ *we have*

$$u(x) \sim x^{\frac{p}{p-1-(\alpha-\beta)}} \to 0+ \qquad \text{for } x \to +\infty.$$

(II) *If* $\alpha - \beta < p - 1$ *then* $x_1 < +\infty$ *and for* $x \to x_1-$ *we have*

$$u(x) \sim (x_1 - x)^{\frac{p}{p-1-(\alpha-\beta)}}.$$

As for the derivatives, we then have

(a) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1-} \sim -(x_1 - x)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}} \to 0 \quad \text{for } x \to x_1- \qquad \text{if } \alpha - \beta > -1,$$

(b) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1-} \sim -(x_1 - x)^{0} \to k < 0 \quad \text{for } x \to x_1- \qquad \text{if } \alpha - \beta = -1,$$

(c) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1+} \sim -(x_1 - x)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}} \to -\infty \quad \text{for } x \to x_1- \qquad \text{if } \alpha - \beta < -1.$$

Analogously we can derive the asymptotics near 1 as well, i.e., the asymptotic behaviour of solution $u = u(x)$ as $x \to x_0$.

Theorem 3.28. *Let* $g(s) \sim (1-s)^{\gamma}$, $d(s) \sim (1-s)^{\delta}$ *as* $s \to 1-$ *where* $\gamma > 0$, $\delta \in \mathbb{R}$ *satisfy*

$$\gamma + \frac{\delta}{p-1} > -1$$

for given $p > 1$.

(I) *If* $\gamma - \delta \geq p - 1$ *then* $x_0 = -\infty$. *Moreover, for* $\gamma - \delta = p - 1$ *we have*

$$u(x) \sim 1 - \mathrm{e}^{x} \to 1- \qquad \text{for } x \to -\infty$$

and for $\gamma - \delta > p - 1$ *we have*

$$u(x) \sim 1 - |x|^{\frac{p}{p-1-(\gamma-\delta)}} \to 1- \qquad \text{for } x \to -\infty.$$

(II) *If* $\gamma - \delta < p - 1$ *then* $x_0 > -\infty$ *and for* $x \to x_0+$ *we have*

$$u(x) \sim 1 - (x - x_0)^{\frac{p}{p-1-(\gamma-\delta)}}.$$

As for the derivatives, we then have

(a) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+} \sim -(x - x_0)^{\frac{\gamma-\delta+1}{p-1-(\gamma-\delta)}} \to 0 \quad \text{for } x \to x_0+ \qquad \text{if } \gamma - \delta > -1,$$

(b) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+} \sim -(x - x_0)^{0} \to k < 0 \quad \text{for } x \to x_0+ \qquad \text{if } \gamma - \delta = -1,$$

(c) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+} \sim -(x - x_0)^{\frac{\gamma-\delta+1}{p-1-(\gamma-\delta)}} \to -\infty \quad \text{for } x \to x_0+ \qquad \text{if } \gamma - \delta < -1.$$

3.3.2 Asymptotic analysis of the travelling wave profile

We now focus on the asymptotic analysis of nonincreasing travelling wave solutions in the case of bistable reaction g whose existence was established in Theorem 3.9. The main idea is the same as for standing waves but the analysis yields less precise results, since the solution of the equation (2.14) cannot be obtained in a closed form by simple integration of

$$f(t) = (d(t))^{\frac{1}{p-1}} g(t)$$

as in the stationary case $c = 0$.

It follows from (2.16) that the inverse function to a profile $U = U(z)$ corresponding to the speed $c_* > 0$ and normalized by $U(0) = s_*$ is given by

$$z(U) = -\int_{s_*}^{U} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t, \quad U \in (0,1), \tag{3.35}$$

where $y_{c_*} = y_{c_*}(t)$ is the unique positive solution of (3.2). In order to find the asymptotic behaviour of $z = z(U)$ as $U \to 1-$ and $U \to 0+$ we first need to establish the asymptotics of $y_{c_*} = y_{c_*}(t)$ as $t \to 1-$ and $t \to 0+$, respectively. We refer the reader to [9, Section 5] where this task is handled in detail and we present the final results below.

Let us briefly mention that there is a considerable difference in the asymptotics near 1 and 0. In the first case, the reasoning resulting in Theorems 3.29, 3.30 is based on the use of Lemma 3.4 which does not hold in the latter case due to the lack of uniqueness for the forward initial value problem (3.4). However, analogous result can be derived if we restrict on the set of positive solutions in the neighbourhood of 0 but it has to be proven separately, see [9, Lemmas 5.4, 5.6] and the proof therein.

Asymptotics near 1

Theorem 3.29. [9, Theorem 5.1] *Let $g(t) \sim (1-t)^\gamma$, $d(t) \sim (1-t)^\delta$ as $t \to 1-$ where $\gamma > 0$ and $\delta \in \mathbb{R}$ are such that*

$$-1 < \gamma + \frac{\delta}{p-1} \leq \frac{1}{p-1}$$

for given $p > 1$. If

$$\frac{\gamma - \delta + 1}{p} < 1$$

then $z_0 > -\infty$. If

$$\frac{\gamma - \delta + 1}{p} \geq 1$$

then $z_0 = -\infty$.

Theorem 3.30. [9, Theorem 5.2] *Let $g(t) \sim (1-t)^\gamma$, $d(t) \sim (1-t)^\delta$ as $t \to 1-$ where $\gamma > 0$ and $\delta \in \mathbb{R}$ are such that*

$$\gamma + \frac{\delta}{p-1} > \frac{1}{p-1}$$

for given $p > 1$. If $\gamma < 1$ then $z_0 > -\infty$. If $\gamma \geq 1$ then $z_0 = -\infty$.

Remark 3.31. [9, Remark 5.3] To visualize conditions from Theorems 3.29, 3.30, we introduce the following sets:

$$\mathcal{M}_1^1 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, -1 < \gamma + \frac{\delta}{p-1} \le \frac{1}{p-1}, \gamma - \delta + 1 < p\},$$
$$\mathcal{M}_1^2 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, -1 < \gamma + \frac{\delta}{p-1} \le \frac{1}{p-1}, \gamma - \delta + 1 \ge p\},$$
$$\mathcal{M}_1^3 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, \gamma + \frac{\delta}{p-1} > \frac{1}{p-1}, \gamma < 1\},$$
$$\mathcal{M}_1^4 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, \gamma + \frac{\delta}{p-1} > \frac{1}{p-1}, \gamma \ge 1\}.$$

Then $z_0 > -\infty$ if and only if $(\gamma,\delta) \in \mathcal{M}_1^1 \cup \mathcal{M}_1^3$ and $z_0 = -\infty$ if and only if $(\gamma,\delta) \in \mathcal{M}_1^2 \cup \mathcal{M}_1^4$. See Figure 3.6 for geometric interpretation. Our results generalize those from [6, Section 6].

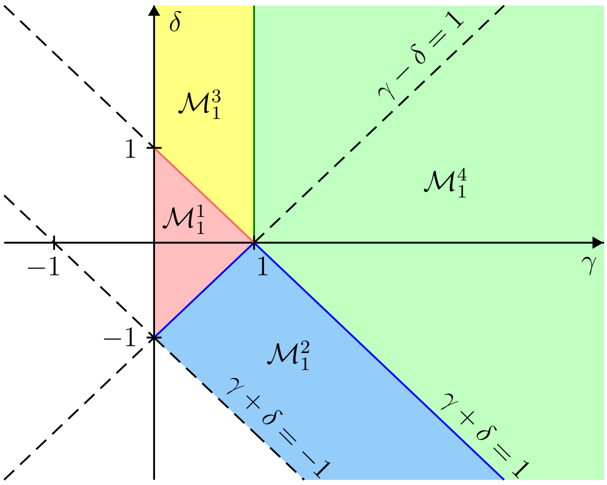

Figure 3.6: Visualization of the sets $\mathcal{M}_1^1$, $\mathcal{M}_1^2$, $\mathcal{M}_1^3$ and $\mathcal{M}_1^4$ for $p = 2$

Asymptotics near 0

Theorem 3.32. [9, Theorem 5.7] *Let $g(t) \sim (-t^\alpha)$, $d(t) \sim t^\beta$ as $t \to 0+$ where $\alpha > 0$ and $\beta \in \mathbb{R}$ are such that*

$$-1 < \alpha + \frac{\beta}{p-1} \le \frac{1}{p-1}$$

for given $p > 1$. If

$$\frac{\alpha - \beta + 1}{p} < 1$$

then $z_1 < +\infty$. If

$$\frac{\alpha - \beta + 1}{p} \ge 1$$

then $z_1 = +\infty$.

Theorem 3.33. [9, Theorem 5.8] *Let $g(t) \sim (-t^\alpha)$, $d(t) \sim t^\beta$ as $t \to 0+$ where $\alpha > 0$ and $\beta \in \mathbb{R}$ are such that*

$$\alpha + \frac{\beta}{p-1} > \frac{1}{p-1}$$

for given $p > 1$. If $\beta > 2 - p$ then $z_1 < +\infty$. If $\beta \le 2 - p$ then $z_1 = +\infty$.

Remark 3.34. [9, Remark 5.9] To visualize conditions from Theorems 3.32, 3.33, we introduce the sets:

$$\mathcal{M}_0^1 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, -1 < \alpha + \frac{\beta}{p-1} \le \frac{1}{p-1}, \alpha - \beta + 1 < p\},$$
$$\mathcal{M}_0^2 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, -1 < \alpha + \frac{\beta}{p-1} \le \frac{1}{p-1}, \alpha - \beta + 1 \ge p\},$$
$$\mathcal{M}_0^3 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, \alpha + \frac{\beta}{p-1} > \frac{1}{p-1}, \beta > 2-p\},$$
$$\mathcal{M}_0^4 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, \alpha + \frac{\beta}{p-1} > \frac{1}{p-1}, \beta \le 2-p\}.$$

Then $z_1 < +\infty$ if and only if $(\alpha,\beta) \in \mathcal{M}_0^1 \cup \mathcal{M}_0^3$ and $z_1 = +\infty$ if and only if $(\alpha,\beta) \in \mathcal{M}_0^2 \cup \mathcal{M}_0^4$. The reader is invited to see Figure 3.7 for geometric interpretation and compare the sets $\mathcal{M}_0^1$, $\mathcal{M}_0^2$, $\mathcal{M}_0^3$, $\mathcal{M}_0^4$ and $\mathcal{M}_1^1$, $\mathcal{M}_1^2$, $\mathcal{M}_1^3$, $\mathcal{M}_1^4$.

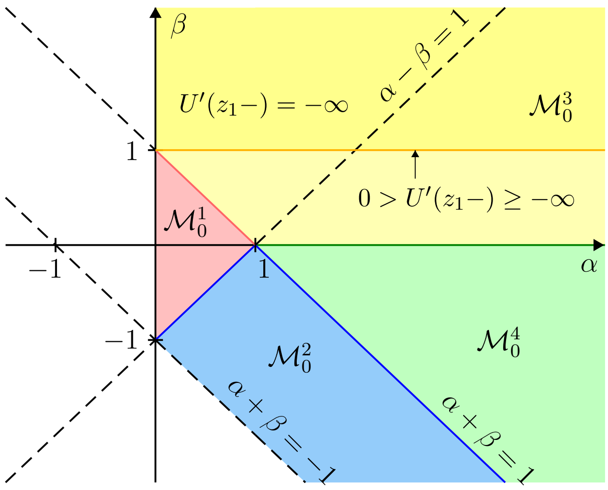

Figure 3.7: Visualization of the sets $\mathcal{M}_0^1$, $\mathcal{M}_0^2$, $\mathcal{M}_0^3$ and $\mathcal{M}_0^4$ for $p = 2$

Remark 3.35. Let us assume $z_0 > -\infty$, i.e., $(\gamma,\delta) \in \mathcal{M}_1^1 \cup \mathcal{M}_1^3$. Then $U'(z_0-) = 0$ and it follows from Definition 2.1 that $U'(z_0+)$ exists finite or infinite. Since U is a nonincreasing function, we have $-\infty \le U'(z_0+) \le 0$. If $z_1 < +\infty$, i.e., $(\alpha,\beta) \in \mathcal{M}_0^1 \cup \mathcal{M}_0^3$ then by similar reasons $U'(z_1+) = 0$ and $-\infty \le U'(z_1-) \le 0$.

In case $(\alpha,\beta) \in \mathcal{M}_0^3$ we can obtain more precise information about the smoothness of U at z_1. In particular,

(a) if $\beta > 1$ then $U'(z_1-) = -\infty$,

(b) if $\beta = 1$ then $0 > U'(z_1-) \ge -\infty$,

see [9, Remark 5.10]. In the remaining cases $(\alpha,\beta) \in \mathcal{M}_0^1$ and $(\gamma,\delta) \in \mathcal{M}_1^1 \cup \mathcal{M}_1^3$ analogous information as above cannot be derived. This is a big difference between the travelling wave and the standing wave.

Resumé

V této práci jsme se zabývali zobecněnou Fisher-Kolmogorovou rovnicí s nespojitou difuzí závislou na hustotě a jednotnou metodou důkazu existence postupných vln pro různé typy reakčního členu. Výsledky pro bistabilní reakční člen byly publikovány ve dvou článcích, které jsou přílohou této práce.

První kapitola je věnována úvodu do problematiky postupných vln pro reakčně-difuzní rovnici a motivaci této úlohy v podobě klasické aplikace v modelování populace diploidních jedinců. Dále uvádíme přehled známých výsledků pro monostabilní a bistabilní reakční člen, a to včetně zobecnění.

Ve druhé kapitole formulujeme studovanou úlohu a definujeme pojem řešení. Následně dokazujeme ekvivalenci s úlohou prvního řádu, jejíž studium je výchozím bodem k získání výsledků pro úlohu původní.

Třetí kapitola obsahuje hlavní výsledky a je rozdělena do tří podkapitol. První podkapitola je věnována větám o existenci a základních vlastnostech řešení. Po uvedení obecných výsledků pro úlohu prvního řádu je text dále členěn podle tvaru uvažovaného reakčního členu. Očekávané výsledky pro monostabilní reakční člen jsou zahrnuty v podobě otevřených problémů, které představují směr dalšího výzkumu. Ve druhé podkapitole diskutujeme možné zeslabení předpokladů na monotonii řešení. Třetí podkapitola je zaměřena na asymptotiku stacionárních a nestacionárních řešení v bistabilním případě.

Resume

In this thesis we were concerned with the generalized Fisher–Kolmogorov equation with discontinuous density dependent diffusion and we presented a unified approach to the proof of existence of travelling wave solutions for different types of the reaction term. Results obtained in the case of bistable reaction term were published in two articles which are included at the end of this thesis.

The first chapter serves as an introduction into the task of finding travelling wave solutions for a reaction–diffusion equation and motivation of such problem arising in the modelling of diploid individuals. We also provide an overview of known results for monostable and bistable reaction term, including generalizations of these results.

In the second chapter we present the problem we consider and define its solution. Next we prove the equivalence with a first order problem which we study in order to derive results for the original one.

The third chapter contains main results and is divided into three sections. The first section concerns the existence results and basic properties of solutions. Following the general results for the first order problem the text is further divided based on the type of the reaction term. Expected results for the monostable case are included as open problems which represent the direction of further research. In the next section we discuss possible relaxation of requirements regarding the monotonicity of solution. The final section is focused on the asymptotics of stationary and time-dependent solutions in the bistable case.

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were obtained during series of joint discussions. I provided basic ideas and sketches of proofs of the main statements which were then revised and executed in detail by M. Zahradníková. She also significantly contributed to the asymptotic analysis of solutions including visualization.

The overall contribution of M. Zahradníková was approximately 50% in both papers.

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Pavel Drábek

Appendix A

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Bistable equation with discontinuous density dependent diffusion with degenerations and singularities

Dedicated to the memory of Professor Josef Daněček, our friend and mentor

Pavel Drábek ✉ and **Michaela Zahradníková**

Department of Mathematics, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 8, 30100 Plzeň, Czech Republic

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Abstract. In this article we introduce rather general notion of the stationary solution of the bistable equation which allows to treat discontinuous density dependent diffusion term with singularities and degenerations, as well as degenerate or non-Lipschitz balanced bistable reaction term. We prove the existence of new-type solutions which do not occur in case of the "classical" setting of the bistable equation. In the case of the power-type behavior of the diffusion and bistable reaction terms near the equilibria we provide detailed asymptotic analysis of the corresponding solutions and illustrate the lack of smoothness due to the discontinuous diffusion.

Keywords: density dependent diffusion, bistable balanced nonlinearity, asymptotic behavior, discontinuous diffusion, degenerate and singular diffusion, degenerate non-Lipschitz reaction.

2020 Mathematics Subject Classification: 35Q92, 35K92, 34C60, 34A12.

1 Introduction

Let us consider the bistable equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(u) \tag{1.1}$$

in $\mathbb{R}$, where the reaction term $g : [0,1] \to \mathbb{R}$ is continuous and there exists $s_* \in (0,1)$ such that

$$g(0) = g(s_*) = g(1) = 0, \quad g(s) < 0 \text{ for } s \in (0, s_*), \quad g(s) > 0 \text{ for } s \in (s_*, 1).$$

Equation (1.1) appears in many mathematical models in population dynamics, genetics, combustion or nerve propagation, see e.g. [1,2] and references therein.

✉ Corresponding author. Email: pdrabek@kma.zcu.cz

This kind of reaction is called *bistable*, cf. [3,7–9]. We distinguish between two different cases of bistable reactions which lead to different type of solutions to (1.1). Namely, when

$$\int_0^1 g(s)\,\mathrm{d}s = 0, \tag{1.2}$$

we say that g is *balanced bistable nonlinearity* while in case

$$\int_0^1 g(s)\,\mathrm{d}s \neq 0$$

the bistable nonlinearity g is called *unbalanced*. In the former case the equation (1.1) possesses (time independent) *stationary solutions* which connect constant equilibria $u_0 \equiv 0$ and $u_1 \equiv 1$, i.e., solutions $u = u(x)$ of (1.1) satisfying

$$\lim_{x\to-\infty} u(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} u(x) = 1 \tag{1.3}$$

or

$$\lim_{x\to-\infty} u(x) = 1 \quad \text{and} \quad \lim_{x\to+\infty} u(x) = 0. \tag{1.4}$$

On the other hand, the latter case leads to the (time dependent) *nonstationary travelling wave solutions* connecting u_0 and u_1, see e.g. [6,10].

The stationary solutions of (1.1) satisfying (1.3) or (1.4) can be found in the closed form for special reaction terms. For example, for

$$g(s) = s(1-s)\left(s - \frac{1}{2}\right)$$

we get stationary solution of (1.1), (1.3) in the following form

$$u(x) = \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right) + \frac{1}{2},$$

cf. [4]. Then solution $u = u(x) \in (0,1)$, $x \in \mathbb{R}$, is a strictly increasing function which approaches equilibria u_0 and u_1 at an exponential rate:

$$u(x) \sim \mathrm{e}^{x} \quad \text{as } x \to -\infty \quad \text{and} \quad 1 - u(x) \sim \mathrm{e}^{-x} \quad \text{as } x \to +\infty. \tag{1.5}$$

If we consider the quasilinear bistable equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + g(u), \tag{1.6}$$

where $p > 1$ and g is balanced bistable nonlinearity then the structure of stationary solutions to (1.6), (1.3) or (1.6), (1.4) may be considerably different as shown in [4]. For example, if

$$g(s) = s^{\alpha}(1-s)^{\alpha}\left(s - \frac{1}{2}\right), \quad s \in (0,1), \quad \alpha > 0,$$

we distinguish between the following two qualitatively different cases:

Case 1: $\alpha + 1 \geq p$,

Case 2: $\alpha + 1 < p$.

In Case 1 solution $u = u(x)$ of (1.6), (1.3) is again a strictly increasing continuously differentiable function which assumes values in $(0,1)$. However, (1.5) holds only in the case $\alpha + 1 = p$. In the case $\alpha + 1 > p$ we have

$$u(x) \sim |x|^{\frac{p}{p-(\alpha+1)}} \quad \text{as } x \to -\infty \quad \text{and} \quad 1 - u(x) \sim |x|^{\frac{p}{p-(\alpha+1)}} \quad \text{as } x \to +\infty .$$

In Case 2 there exist real numbers $x_0 < x_1$ such that for all $x \in (x_0, x_1)$ we have $u(x) \in (0,1)$, u is strictly increasing continuously differentiable, $u(x) = 0$ for all $x \in (-\infty, x_0]$ and $u(x) = 1$ for all $x \in [x_1, +\infty)$. Moreover,

$$u(x) \sim (x - x_0)^{\frac{p}{p-(\alpha+1)}} \quad \text{as } x \to x_0+ \quad \text{and} \quad 1 - u(x) \sim (x_1 - x)^{\frac{p}{p-(\alpha+1)}} \quad \text{as } x \to x_1- \ .$$

Our ambition in this paper is to study similar properties for the quasilinear bistable equation with density dependent diffusion coefficient $d = d(s)$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(d(u)\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + g(u), \tag{1.7}$$

where the properties of $d = d(s)$ are specified in the next section.

2 Preliminaries

Let $p > 1$, $g : [0,1] \to \mathbb{R}$, $g \in C[0,1]$ be such that $g(0) = g(s_*) = g(1) = 0$ for $s_* \in (0,1)$ and

$$g(s) < 0,\ s \in (0, s_*), \quad g(s) > 0,\ s \in (s_*, 1).$$

The diffusion coefficient $d : [0,1] \to \mathbb{R}$ is supposed to be a nonnegative lower semicontinuous function and $d > 0$ in $(0,1)$. There exist $0 = s_0 < s_1 < s_2 < \cdots < s_n < s_{n+1} = 1$ such that $d|_{(s_i,s_{i+1})} \in C(s_i, s_{i+1})$, $i = 0, \ldots, n$, and d has discontinuity of the first kind (finite jump) at s_i, $i = 1, \ldots, n$.

For $p = 2$ and $d(s) \equiv 1$ in $[0,1]$ equation (1.7) reduces to the bistable equation (1.1) with constant diffusion coefficient and *bistable reaction* term g. In this paper we deal with diffusion which allows for *singularities* and for *degenerations* both at 0 and/or 1. We also consider d to be a *discontinuous* function. Last but not least, reaction term g can degenerate in 0 and/or in 1. In particular, we admit $g'(0) = 0$ and/or $g'(1) = 0$, as well as $g'(0) = -\infty$ and/or $g'(1) = -\infty$. This in turn yields that our solution is not a C^1-function in $\mathbb{R}$ and it does not satisfy the equation pointwise in the classical sense. For this purpose we have to employ the first integral of the second order differential equation. Since our primary interest in this paper is the investigation of stationary solutions to (1.7) which are monotone (i.e., nonincreasing or nondecreasing) between the equilibria 0 and 1, we provide rather general definition of monotone solutions to the second order ODE

$$\left(d(u)\,|u'|^{p-2}\,u'\right)' + g(u) = 0, \tag{2.1}$$

where, for the sake of simplicity, we write $(\,\cdot\,)'$ instead of $\frac{\mathrm{d}}{\mathrm{d}x}(\,\cdot\,)$.

Let $u : \mathbb{R} \to [0,1]$ be a monotone continuous function. We denote

$$M_u := \{x \in \mathbb{R} : u(x) = s_i,\ i = 1, 2, \ldots, n\}, \qquad N_u := \{x \in \mathbb{R} : u(x) = 0 \text{ or } u(x) = 1\}.$$

Then M_u and N_u are closed sets, M_u is a union of a finite number of points or intervals,

$$N_u = (-\infty, x_0] \cup [x_1, +\infty),$$

where $-\infty \leq x_0 < x_1 \leq +\infty$ and we use the convention $(-\infty, x_0] = \emptyset$ if $x_0 = -\infty$ and $[x_1, +\infty) = \emptyset$ if $x_1 = +\infty$.

Definition 2.1. A monotone continuous function $u : \mathbb{R} \to [0,1]$ is a *solution* of equation (2.1) if

(a) For any $x \notin M_u \cup N_u$ there exists finite derivative $u'(x)$ and for any $x \in \operatorname{int} M_u \cup \operatorname{int} N_u$ we have $u'(x) = 0$.

(b) For any $x \in \partial M_u$ there exist finite one sided derivatives $u'(x-), u'(x+)$ and

$$L(x) := |u'(x-)|^{p-2} u'(x-) \lim_{y \to x-} d(u(y)) = |u'(x+)|^{p-2} u'(x+) \lim_{y \to x+} d(u(y)).$$

(c) Function $v : \mathbb{R} \to \mathbb{R}$ defined by

$$v(x) := \begin{cases} d(u(x))\, |u'(x)|^{p-2} u'(x), & x \notin M_u \cup N_u\,, \\ 0, & x \in N_u \cup \operatorname{int} M_u\,, \\ L(x), & x \in \partial M_u \end{cases}$$

is continuous and for any $x, y \in \mathbb{R}$

$$v(y) - v(x) + \int_x^y g(u(\xi))\, \mathrm{d}\xi = 0. \tag{2.2}$$

Moreover, $\lim_{x\to\pm\infty} v(x) = 0$ if either $\lim_{x\to-\infty} u(x) = 0$ and $\lim_{x\to+\infty} u(x) = 1$ or $\lim_{x\to-\infty} u(x) = 1$ and $\lim_{x\to+\infty} u(x) = 0$.

Remark 2.2. Constant functions

$$u_0(x) = 0, \quad u_*(x) = s_*, \quad u_1(x) = 1, \quad x \in \mathbb{R},$$

are solutions of (2.1). It follows from the properties of d and g that those are the only constant solutions of (2.1) and they are called equilibria.

Remark 2.3. If we set $y = x + h, h \neq 0$ in (2.2), multiply both sides of (2.2) by $\frac{1}{h}$ and pass to the limit for $h \to 0$, we obtain that v is continuously differentiable and the equation

$$v'(x) + g(u(x)) = 0 \tag{2.3}$$

holds for all $x \in \mathbb{R}$.

Remark 2.4. Let u be a solution of (2.1) in the sense of Definition 2.1. If $M_u \neq \emptyset$, i.e., d is not continuous in $(0,1)$, then $M_u = \partial M_u$, $\operatorname{int} M_u = \emptyset$ unless $s_i = s_*$ for some $i = 1, 2, \ldots, n$. In this case u can be constant on some interval (a, b), $-\infty \leq a < b \leq +\infty$, and equal to s_*. The equation (2.1) would then be satisfied pointwise for all $x \in (a, b)$ and $(a, b) \subset \operatorname{int} M_u$. Furthermore, it follows from the continuity of v that if $a > -\infty$ or $b < +\infty$ we have $u'(a) = u'(b) = 0$ because $d(s_*) > 0$. Also note that for $x \in \partial N_u$ one sided derivatives $u'(x-), u'(x+)$ exist but one of them can be infinite.

If u is strictly monotone between 0 and 1 then $M_u = \{\xi_1, \xi_2, \ldots, \xi_n\}$ where $u(\xi_i) = s_i$, $i = 1, 2, \ldots, n$.

Remark 2.5. Let $p = 2$, $d \equiv 1$ and $g \in C^1[0,1]$. Let $u = u(x)$ be a solution in the sense of Definition 2.1. Then $M_u = \emptyset$ if u is not a constant, $N_u = \emptyset$, and (2.1) holds pointwise, i.e., $u \in C^2(\mathbb{R})$ and it is a classical solution, cf. [1], [2] or [6].

3 Existence results

We are concerned with the existence of solutions of the equation (2.1) which satisfy the "boundary conditions"

$$\lim_{x\to-\infty} u(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} u(x) = 1. \tag{3.1}$$

Remark 3.1. Let u be a solution of the BVP (2.1), (3.1). Passing to the limit for $x \to -\infty$ in (2.2) and writing x in place of y, we derive that for arbitrary $x \in \mathbb{R}$ we have

$$v(x) + \int_{-\infty}^{x} g(u(\xi))\,\mathrm{d}\xi = 0. \tag{3.2}$$

Theorem 3.2. *Let d and g be as in Section 2 and recall that $p > 1$. Then the BVP (2.1), (3.1) has a nondecreasing solution if and only if*

$$\int_0^1 (d(s))^{\frac{1}{p-1}} g(s)\,\mathrm{d}s = 0. \tag{3.3}$$

If (3.3) holds then there is a unique solution $u = u(x)$ of (2.1), (3.1) such that the following conditions hold (see Figure 3.1):

(i) *there exist $-\infty \le x_0 < 0 < x_1 \le +\infty$ such that $u(x) = 0$ for $x \le x_0$, $u(x) = 1$ for $x \ge x_1$ and $0 < u(x) < 1$ for $x \in (x_0, x_1)$;*

(ii) *u is strictly increasing in (x_0, x_1), $u(0) = s_*$;*

(iii) *for $i = 1, 2, \ldots, n$ let $\xi_i \in \mathbb{R}$ be such that $u(\xi_i) = s_i$, $\xi_0 = x_0$ and $\xi_{n+1} = x_1$. Then u is a piecewise C^1-function in the sense that u is continuous,*

$$u\big|_{(\xi_i,\xi_{i+1})} \in C^1(\xi_i, \xi_{i+1}), \quad i = 0, 1, \ldots, n,$$

and the limits $u'(\xi_i-) := \lim_{x\to\xi_i-} u'(x)$, $u'(\xi_i+) := \lim_{x\to\xi_i+} u'(x)$ exist finite for all $i = 1, 2, \ldots, n$;

(iv) *for any $i = 1, 2, \ldots, n$, the following transition condition holds:*

$$\left(u'(\xi_i-)\right)^{p-1} \lim_{s\to s_i-} d(s) = \left(u'(\xi_i+)\right)^{p-1} \lim_{s\to s_i+} d(s).$$

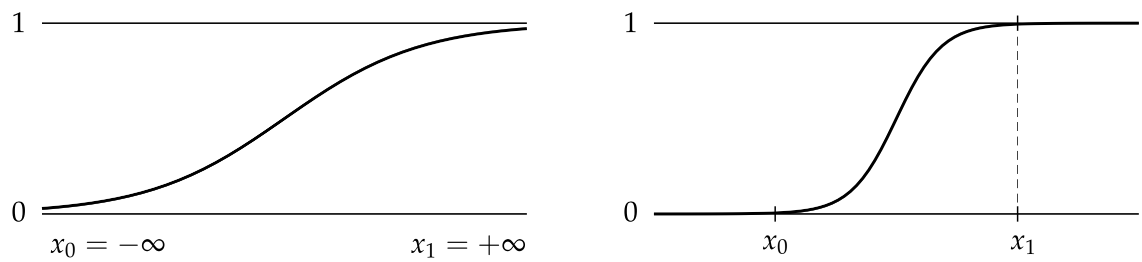

Figure 3.1: Increasing solutions

Proof. Necessity of (3.3). Let $u = u(x)$ be a nondecreasing solution of the BVP (2.1), (3.1) such that $u(0) = s_*$. Since the equation is autonomous this condition is just a normalization of a solution. It follows from (3.1) that

$$-\infty \le x_0 := \inf\{x \in \mathbb{R} : u(x) > 0\} < 0$$

is well defined. By (3.2) and continuity of v we have

$$0 < x_1 := \sup\{x \in \mathbb{R} : v(y) > 0 \text{ for all } y \in (x_0, x)\} \le +\infty\,.$$

Since $d(s) > 0, s \in (0,1)$, it follows from the definition of $v(x)$ that u is a strictly increasing function in (x_0, x_1) and therefore the following limit

$$\bar{u}(x_1) := \lim_{x \to x_1-} u(x)$$

is well defined. If $x_1 = +\infty$ then by the second condition in (3.1) it must be $\bar{u}(x_1) = 1$. On the other hand, if $x_1 < +\infty$, we have $\bar{u}(x_1) = u(x_1)$, $v(x_1) = 0$ and $s_* < u(x_1) \le 1$. We rule out the case $u(x_1) < 1$. Indeed, $v(x_1) = 0$ implies $u'(x_1-) = u'(x_1+) = u'(x_1) = 0$. From $s_* < u(x_1)$ and (2.3) we deduce $v'(x_1) = -g(u(x_1)) < 0$. Therefore, there exists $\delta > 0$ such that for all $x \in (x_1, x_1 + \delta)$ we have $v(x) < 0$ and hence also $u'(x-) < 0$ and $u'(x+) < 0$. This contradicts our assumption that u is nondecreasing.

We proved that $u(x_1) = 1$, i.e., $u = u(x)$ is strictly increasing and maps (x_0, x_1) onto $(0,1)$. Let $\xi_i \in (x_0, x_1)$ be such that

$$u(\xi_i) = s_i\,, \quad i = 1,2,\dots,n, \quad \xi_0 = x_0\,, \quad \xi_{n+1} = x_1\,.$$

Then u is continuous in (x_0, x_1) and piecewise C^1-function in the sense that

$$u\big|_{(\xi_i,\xi_{i+1})} \in C^1(\xi_i, \xi_{i+1}), \quad u'(x) > 0, \quad x \in (\xi_i, \xi_{i+1}), \quad i = 0,1,\dots,n,$$

and the limits $\lim_{x\to\xi_i-} u'(x)$, $\lim_{x\to\xi_i+} u'(x)$, $i = 1,2,\dots,n$, exist finite. Hence there exists continuous strictly increasing inverse function $u^{-1} : (0,1) \to (x_0, x_1)$, $x = u^{-1}(u)$, such that

$$u^{-1}\Big|_{(s_i,s_{i+1})} \in C^1(s_i, s_{i+1}), \quad i = 0,1,\dots,n\,,$$

and the limits

$$\lim_{u\to s_i-} \frac{\mathrm{d}}{\mathrm{d}u} u^{-1}(u), \qquad \lim_{u\to s_i+} \frac{\mathrm{d}}{\mathrm{d}u} u^{-1}(u)$$

exist finite, $i = 1,2,\dots,n$. We employ the change of variables as indicated in [5, p. 174]. Set

$$w(u) = v(u^{-1}(u)), \quad u \in (0,1).$$

Then w is piecewise C^1-function in $(0,1)$,

$$w\big|_{(s_i,s_{i+1})} \in C^1(s_i, s_{i+1}), \quad i = 0,1,\dots,n$$

with finite limits $\lim_{u\to s_i-} w'(u)$, $\lim_{u\to s_i+} w'(u)$, $i = 1,2,\dots,n$. For any $x \in (\xi_i, \xi_{i+1})$ and $u \in (s_i, s_{i+1})$, $i = 0,1,\dots,n$,

$$\frac{\mathrm{d}}{\mathrm{d}x} v(x) = \frac{\mathrm{d}}{\mathrm{d}x} w(u(x)) = \frac{\mathrm{d}w}{\mathrm{d}u}\,(u(x))\,u'(x). \tag{3.4}$$

From $v(x) = d(u(x))(u'(x))^{p-1}$ we deduce

$$u'(x) = \left(\frac{v(x)}{d(u(x))}\right)^{p'-1}, \qquad p' = \frac{p}{p-1}. \tag{3.5}$$

It follows from (3.4), (3.5) that

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}w}{\mathrm{d}u}(u(x))\left(\frac{v(x)}{d(u(x))}\right)^{p'-1} = \frac{\mathrm{d}w}{\mathrm{d}u}(u)\left(\frac{w(u)}{d(u)}\right)^{p'-1}.$$

Therefore, the equation

$$v'(x) + g(u(x)) = 0, \quad x \in (\xi_i, \xi_{i+1}),$$

transforms to

$$\frac{\mathrm{d}w}{\mathrm{d}u}\left(\frac{w(u)}{d(u)}\right)^{p'-1} + g(u) = 0, \quad u \in (s_i, s_{i+1}),$$

$i = 0, 1, \ldots, n$, or equivalently,

$$(w(u))^{p'-1}\frac{\mathrm{d}w}{\mathrm{d}u} + (d(u))^{p'-1}g(u) = 0, \tag{3.6}$$

$$\frac{1}{p'}\frac{\mathrm{d}}{\mathrm{d}u}(w(u))^{p'} + (d(u))^{p'-1}g(u) = 0. \tag{3.7}$$

The last equality holds in $(0, 1)$ except the points $s_1, s_2, \ldots, s_n$ and w is continuous in $(0, 1)$. Set

$$f(s) := -(d(s))^{\frac{1}{p-1}}g(s), \quad s \in (0, 1),$$

$$F(s) := \int_0^s f(\sigma)\,\mathrm{d}\sigma.$$

Integrating (3.7) over the interval $(0, u)$ we arrive at

$$(w(u))^{p'} = p'F(u) + (w(0+))^{p'}, \quad u \in (0, 1).$$

Clearly, $F(0) = 0$, and

$$\lim_{u\to 0+} w(u) = \lim_{x\to x_0+} v(x) = 0 \tag{3.8}$$

by the definition of a solution. Therefore we have

$$w(u) = \left(p'F(u)\right)^{\frac{1}{p'}}, \quad u \in (0, 1). \tag{3.9}$$

By the definition of a solution we must also have

$$\lim_{u\to 1-} w(u) = \lim_{x\to x_1-} v(x) = 0. \tag{3.10}$$

But (3.9) and (3.10) imply $F(1) = 0$, i.e., (3.3) must hold. Therefore, (3.3) is a necessary condition.

Sufficiency of (3.3). Let (3.3) hold. Then $w = w(u)$ given by (3.9) satisfies (3.6)–(3.10) above. For $u \in (0, 1)$ set

$$x(u) = \int_{s_*}^{u}\left(\frac{d(s)}{w(s)}\right)^{\frac{1}{p-1}}\mathrm{d}s.$$

The function $x = x(u)$ is strictly increasing and maps the interval $(0,1)$ onto (x_0, x_1) where $-\infty \le x_0 < 0 < x_1 \le +\infty$. Let $u : (x_0, x_1) \to (0,1)$ be an inverse function. Then $u(0) = s_*$, u is strictly increasing and

$$\lim_{x\to x_0+} u(x) = 0, \qquad \lim_{x\to x_1-} u(x) = 1.$$

Let $x \in (\xi_i, \xi_{i+1})$, $i = 0, 1, \dots, n$, where $u(\xi_i) = s_i$, $i = 0, 1, \dots, n+1$. Then

$$\frac{\mathrm{d}u(x)}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x(u)}{\mathrm{d}u}} = \left(\frac{w(u(x))}{d(u(x))}\right)^{\frac{1}{p-1}}, \quad u(x) \in (s_i, s_{i+1}),$$

i.e., $u \in C^1(\xi_i, \xi_{i+1}), u'(x) > 0$ and

$$d(u(x)) \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{p-1} = w(u(x)) =: v(x), \tag{3.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[d(u(x)) \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{p-1}\right] = \frac{\mathrm{d}}{\mathrm{d}x} w(u(x)) = \frac{\mathrm{d}w}{\mathrm{d}u}\frac{\mathrm{d}u(x)}{\mathrm{d}x}. \tag{3.12}$$

From (3.6), (3.11) we deduce

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}u} &= -\left(w(u)\right)^{-(p'-1)} \left(d(u)\right)^{p'-1} g(u)\\ &= -\left(d(u(x))\right)^{-(p'-1)} \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{-(p-1)(p'-1)} \left(d(u(x))\right)^{p'-1} g(u(x))\\ &= -\left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{-1} g(u(x)).\end{aligned}$$

Substituting this to (3.12), we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[d(u(x)) \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{p-1}\right] = -g(u(x)), \quad x \in (\xi_i, \xi_{i+1}).$$

It follows from (3.8), (3.10) and (3.11) that

$$\lim_{x\to x_0+} d(u(x)) \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{p-1} = \lim_{x\to x_1-} d(u(x)) \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{p-1} = 0$$

and the following one-sided limits are finite

$$\lim_{x\to \xi_i-} d(u(x)) \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{p-1} = \lim_{x\to \xi_i+} d(u(x)) \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{p-1}, \tag{3.13}$$

$i = 1, 2, \dots, n$. Since $u = u(x)$ is monotone increasing function, we have

$$\lim_{x\to\xi_i-} d(u(x)) = \lim_{s\to s_i-} d(s) \quad \text{and} \quad \lim_{x\to\xi_i+} d(u(x)) = \lim_{s\to s_i+} d(s). \tag{3.14}$$

Transition condition (iv) now follows from (3.13), (3.14).

Therefore, if for $x_0 > -\infty$ we set $u(x) = 0, x \in (-\infty, x_0]$ and for $x_1 < +\infty$ we set $u(x) = 1, x \in [x_1, +\infty)$, then $u = u(x), x \in \mathbb{R}$, is a nondecreasing solution of the BVP (2.1), (3.1) and it has the properties listed in the statement of Theorem 3.2. This proves the sufficiency of (3.3). $\square$

Remark 3.3. The condition (3.3) substitutes the balanced bistable nonlinearity condition (1.2) in case of density dependent diffusion. It follows from Theorem 3.2 that it is not only the reaction term but rather mutual interaction between the density dependent diffusion coefficient and reaction which decides about the existence and/or nonexistence of nonconstant stationary solutions of the generalized version of the bistable equation (1.6).

Remark 3.4. Let us replace the boundary conditions (3.1) by "opposite" ones:

$$\lim_{x\to-\infty} u(x) = 1 \quad \text{and} \quad \lim_{x\to+\infty} u(x) = 0. \tag{3.15}$$

If u is a solution of the BVP (2.1), (3.15) then passing to the limit for $y \to +\infty$ in (2.2) we arrive at

$$v(x) - \int_x^{+\infty} g(u(\xi))\,\mathrm{d}\xi = 0 \tag{3.16}$$

for arbitrary $x \in \mathbb{R}$. Modifying the proof of Theorem 3.2 and using (3.16) instead of (3.2), we show that (3.3) is a necessary and sufficient condition for the existence of nonincreasing solution of the BVP (2.1), (3.15). If (3.3) holds then there is a unique solution $u = u(x)$ of (2.1), (3.15) satisfying analogue of (i)–(iv). In particular, it is strictly decreasing in (x_0, x_1), $u(x) = 1$ for $x \in (-\infty, x_0]$ if $x_0 > -\infty$ and $u(x) = 0$ for $x \in [x_1, +\infty)$ if $x_1 < +\infty$, see Figure 3.2.

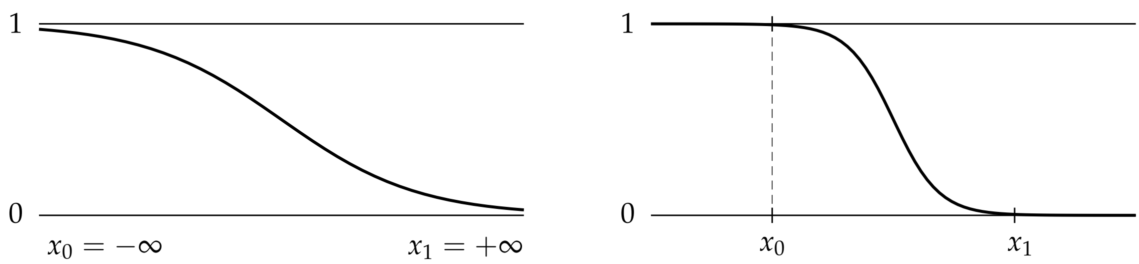

Figure 3.2: Decreasing solutions

Remark 3.5. It follows from the proof of Theorem 3.2 that

$$x_0 = x(0) = \int_{s_*}^{0} \left(\frac{d(s)}{w(s)}\right)^{\frac{1}{p-1}} \mathrm{d}s = \left(\frac{1}{p'}\right)^{\frac{1}{p}} \int_{s_*}^{0} \frac{(d(s))^{\frac{1}{p-1}}}{-\int_0^s (d(\sigma))^{\frac{1}{p-1}} g(\sigma)\,\mathrm{d}\sigma}\,\mathrm{d}s, \tag{3.17}$$

$$x_1 = x(1) = \int_{s_*}^{1} \left(\frac{d(s)}{w(s)}\right)^{\frac{1}{p-1}} \mathrm{d}s = \left(\frac{1}{p'}\right)^{\frac{1}{p}} \int_{s_*}^{1} \frac{(d(s))^{\frac{1}{p-1}}}{-\int_0^s (d(\sigma))^{\frac{1}{p-1}} g(\sigma)\,\mathrm{d}\sigma}\,\mathrm{d}s. \tag{3.18}$$

Therefore, the fact that x_0 and x_1 are finite or infinite depends on the asymptotic behavior of the diffusion coefficient $d = d(s)$ and reaction term $g = g(s)$ near the equilibria 0 and 1. The detailed discussion of different configurations between d and g which lead to x_0 and/or x_1 finite or infinite is presented in the next section.

Remark 3.6. Since the equation (2.1) is autonomous, if $u = u(x)$ is a solution to (2.1), (3.1) then given any $\xi \in \mathbb{R}$ fixed, $\tilde{u} = \tilde{u}(x) := u(x - \xi)$ is also a solution of (2.1), (3.1). Of course, if x_0 and/or x_1 are finite, then corresponding $\tilde{x}_0$ and $\tilde{x}_1$ associated with $\tilde{u}$ satisfy $\tilde{x}_0 = x_0 + \xi$ and $\tilde{x}_1 = x_1 + \xi$. Obviously, the same applies to (2.1), (3.15). If $x_0 = -\infty$ and $x_1 = +\infty$ and (3.3) holds, all possible solutions of (2.1), (3.1) are strictly increasing in $(-\infty, +\infty)$ and satisfy (i)–(iv) of Theorem 3.2, where $u(0) = s_*$ is replaced by $u(\xi) = s_*$, $\xi \in \mathbb{R}$. On the other hand, if $x_0 \in \mathbb{R}$ and/or $x_1 \in \mathbb{R}$, then the set of possible solutions of (2.1), (3.1) is much richer than in the previous case. Indeed, we have plenty of possibilities how to define also a nonmonotone solution of (2.1), (3.1) (or (2.1), (3.15)). For example, if both x_0 and x_1 associated with strictly increasing solution $u = u(x)$ from Theorem 3.2 are finite then the same holds for corresponding $\hat{x}_0$ and $\hat{x}_1$ associated with the strictly decreasing solution from Remark 3.4. Having in mind the translation invariance of solutions mentioned above, we may choose u_1 and $\hat{u}$ such that $x_1 < \hat{x}_0$. If we define $u(x) = 0$, $x \in (-\infty, x_0]$, $u(x) = u_1(x)$, $x \in (x_0, x_1)$, $u(x) = 1$, $x \in [x_1, \hat{x}_0]$, $u(x) = \hat{u}(x)$, $x \in (\hat{x}_0, \hat{x}_1)$, $u(x) = 0$, $x \in [\hat{x}_1, +\infty)$, we get solution of (2.1) satisfying the boundary conditions

$$\lim_{x\to-\infty} u(x) = \lim_{x\to+\infty} u(x) = 0. \tag{3.19}$$

Now, if $\tilde{u}_1 = \tilde{u}_1(x)$ is a translation of u_1 such that $\tilde{x}_0 > \hat{x}_1$, we can extend the previous function u as $u(x) = 0$, $x \in [\hat{x}_1, \tilde{x}_0]$, $u(x) = \tilde{u}(x)$, $x \in (\tilde{x}_0, \tilde{x}_1)$, $u(x) = 1$, $x \in [\tilde{x}_1, +\infty)$ to get a nonmonotone solution of (2.1), (3.1), see Figure 3.3. It is obvious that by suitably modifying the above construction we may construct continuum of solutions not only of (2.1), (3.1) but also of (2.1), (3.19). Of course, the same approach leads to the continuum of solutions of (2.1), (3.15) and of (2.1), (3.20), respectively, where

$$\lim_{x\to-\infty} u(x) = \lim_{x\to+\infty} u(x) = 1. \tag{3.20}$$

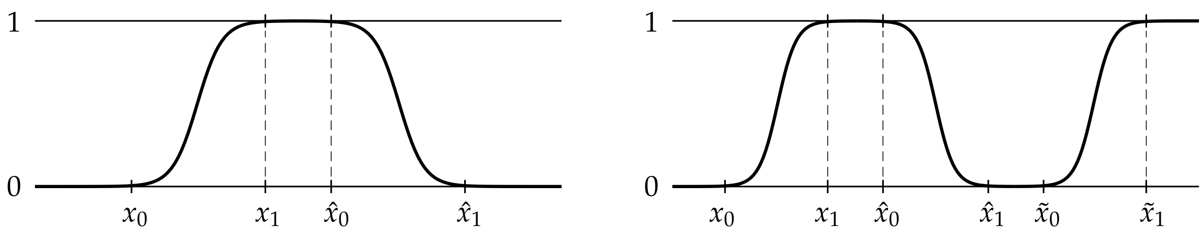

Figure 3.3: Nonmonotone solutions

4 Qualitative properties of solutions

In this section we study the qualitative properties of the solutions from Theorem 3.2. In particular, we focus on two issues. Our primary concern is to provide detailed classification of the *asymptotic behavior* of the stationary solution $u = u(x)$ as $x \to -\infty$ and $x \to +\infty$ and to show how it is affected by the behavior of the diffusion coefficient d and reaction g near the equilibria 0 and 1. However, we also want to study the impact of the *discontinuity* of $d = d(s)$ on the *lack of smoothness of the solution* $u = u(x)$. The role of the transition condition at the points where u assumes values where the discontinuity of d occurs will be illustrated.

In order to simplify the expressions arising throughout this section we will use the following notation:

$$h_1(t) \sim h_2(t) \text{ as } t \to t_0 \quad \text{if and only if} \quad \lim_{t\to t_0} \frac{h_1(t)}{h_2(t)} \in (0, +\infty).$$

We start with the asymptotic analysis of

$$x(u) = x(s_*) + \left(\frac{1}{p'}\right)^{\frac{1}{p}} \int_{s_*}^{u} \frac{(d(s))^{\frac{1}{p-1}}}{\left(-\int_0^s (d(\sigma))^{\frac{1}{p-1}} g(\sigma)\,\mathrm{d}\sigma\right)^{\frac{1}{p}}}\,\mathrm{d}s$$

for $u \to 0+$. Let us assume that $g(s) \sim -s^\alpha$, $d(s) \sim s^\beta$ as $s \to 0+$ for some $\alpha > 0$, $\beta \in \mathbb{R}$. Then formally we get

$$-\int_0^s (d(\sigma))^{\frac{1}{p-1}} g(\sigma)\, \mathrm{d}\sigma \ \sim\ \int_0^s \sigma^{\alpha+\frac{\beta}{p-1}}\, \mathrm{d}\sigma \ \sim\ s^{\alpha+\frac{\beta}{p-1}+1} \quad \text{as} \quad s \to 0+\,.$$

Since we assume that $s \mapsto (d(s))^{\frac{1}{p-1}} g(s)$ is integrable in $(0,1)$, we have to assume

$$\alpha + \frac{\beta}{p-1} > -1\,. \tag{4.1}$$

Then for $u \to 0+$ we can write

$$x(u) \ \sim\ \int_{s_*}^u \frac{(d(s))^{\frac{1}{p-1}}}{\left(-\int_0^s (d(\sigma))^{\frac{1}{p-1}} g(\sigma)\, \mathrm{d}\sigma\right)^{\frac{1}{p}}}\, \mathrm{d}s \ \sim\ \int_{s_*}^u s^{\frac{\beta}{p-1}-\frac{\alpha}{p}-\frac{\beta}{p(p-1)}-\frac{1}{p}}\, \mathrm{d}s = \int_{s_*}^u s^{\frac{\beta-\alpha-1}{p}}\, \mathrm{d}s. \tag{4.2}$$

Convergence or divergence of the integral

$$I := \int_0^{s_*} s^{\frac{\beta-\alpha-1}{p}}\, \mathrm{d}s$$

leads to the following primary distinction between two qualitatively different cases:

Case 1: $I = +\infty$ if $\alpha - \beta \geq p-1$,

Case 2: $I < +\infty$ if $\alpha - \beta < p-1$.

Case 1. Let $\alpha - \beta = p - 1$. Then (4.2) implies that $x(u) \sim \ln u$ as $u \to 0+$ and performing the change of variables yields the asymptotics for $u = u(x)$:

$$u(x) \sim \mathrm{e}^x \to 0+ \qquad \text{for } x \to -\infty\,.$$

For $\alpha - \beta > p - 1$ we have by (4.2) that $x(u) \sim -u^{\frac{\beta-\alpha-1}{p}+1} = -u^{\frac{p-1-(\alpha-\beta)}{p}} \to -\infty$ as $u \to 0+$ and applying the inverse function we obtain

$$u(x) \sim |x|^{\frac{p}{p-1-(\alpha-\beta)}} \to 0+ \qquad \text{for } x \to -\infty\,.$$

In both cases x_0 defined by (3.17) is equal to $-\infty$ and solution $u = u(x)$ approaches zero at either an exponential or power rate.

Remark 4.1. It is interesting to observe that $x_0 = -\infty$ occurs even in the case when the diffusion coefficient degenerates or has a singularity if this fact is compensated by a proper degeneration of the reaction term g.

Possible values of parameters α, β for which Case 1 occurs for different values of p are shown in Figures 4.1, 4.2, 4.3 where condition (4.1) is taken into account.

Case 2. Let $\alpha - \beta < p - 1$. Then $I < +\infty$ and hence from (3.17) we deduce $x(0) = x_0 > -\infty$. Moreover,

$$I \to +\infty \quad \text{as} \quad \frac{\beta - \alpha - 1}{p} \to -1+\,,$$

i.e., we have $x_0 \to -\infty$ as $p - 1 - (\alpha - \beta) \to 0+$. More precisely, for $u \to 0+$ we have

$$x(u) - x_0 \ \sim\ u^{\frac{\beta-\alpha-1}{p}+1} = u^{\frac{p-1-(\alpha-\beta)}{p}}\,.$$

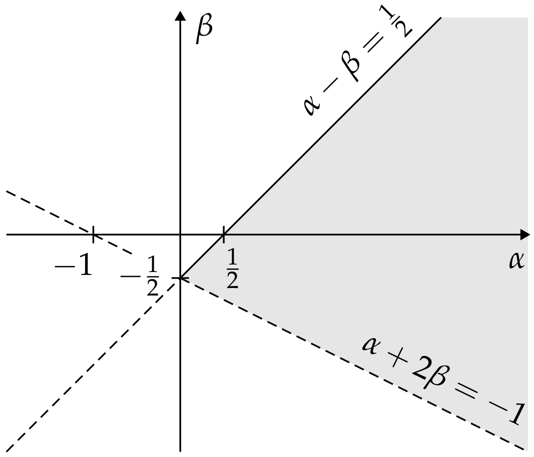

Figure 4.1: $p = \frac{3}{2}$

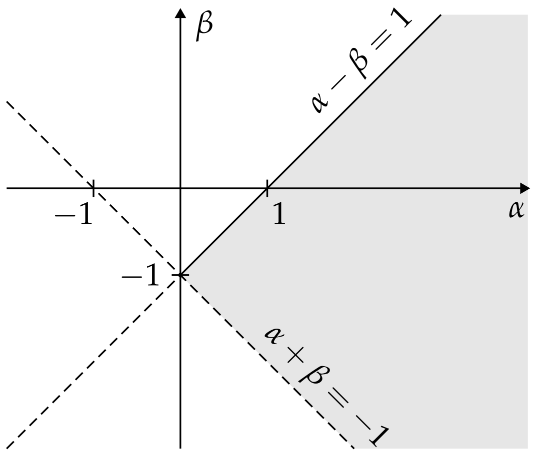

Figure 4.2: $p = 2$

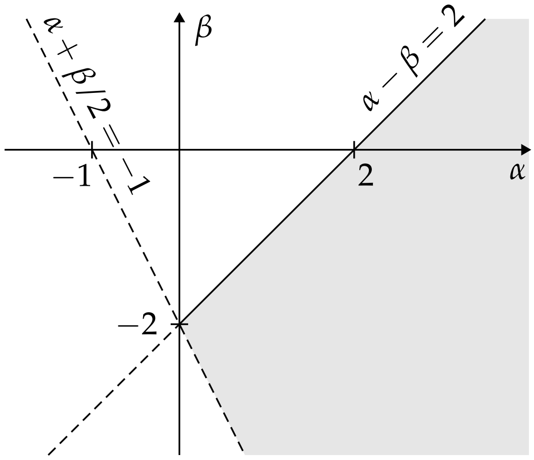

Figure 4.3: $p = 3$

Depending on the shape of $x(u)$ we further distinguish among three cases:

a) $\left.\dfrac{\mathrm{d}x}{\mathrm{d}u}\right|_{u=0+} \sim u^{\frac{\beta-\alpha-1}{p}} \to +\infty \quad \text{for } u \to 0+ \qquad \text{if } \alpha - \beta > -1,$

b) $\left.\dfrac{\mathrm{d}x}{\mathrm{d}u}\right|_{u=0+} \sim u^{0} \to k > 0 \quad \text{for } u \to 0+ \qquad \text{if } \alpha - \beta = -1,$

c) $\left.\dfrac{\mathrm{d}x}{\mathrm{d}u}\right|_{u=0+} \sim u^{\frac{\beta-\alpha-1}{p}} \to 0_{+} \quad \text{for } u \to 0+ \qquad \text{if } \alpha - \beta < -1.$

An inverse point of view gives us the asymptotics of $u = u(x)$ for $x \to x_0$:

$$u(x) \sim (x - x_0)^{\frac{p}{p-1-(\alpha-\beta)}} .$$

As for the derivatives, we have

$$\text{a)} \quad \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+} \sim (x-x_0)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}} \to 0 \quad \text{for } x \to x_0+ \qquad \text{if } \alpha-\beta > -1,$$

$$\text{b)} \quad \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+} \sim (x-x_0)^{0} \to k > 0 \quad \text{for } x \to x_0+ \qquad \text{if } \alpha-\beta = -1,$$

$$\text{c)} \quad \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+} \sim (x-x_0)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}} \to +\infty \quad \text{for } x \to x_0+ \qquad \text{if } \alpha-\beta < -1.$$

Remark 4.2. We observe that only in case a) the solution $u = u(x)$ is smooth in the neighborhood of x_0 since $u(x) = 0$ for $x \in (-\infty, x_0]$. In the other two cases we only get continuous solutions instead of smooth ones as a consequence of allowing for the diffusion term $d = d(s)$ to degenerate as $s \to 0+$. The asymptotic behavior of such solutions near the point x_0 is illustrated in Figures 4.4, 4.5, 4.6.

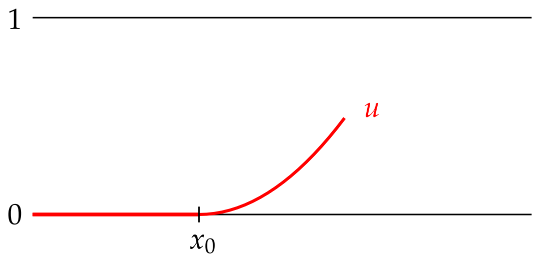

Figure 4.4: Case a)

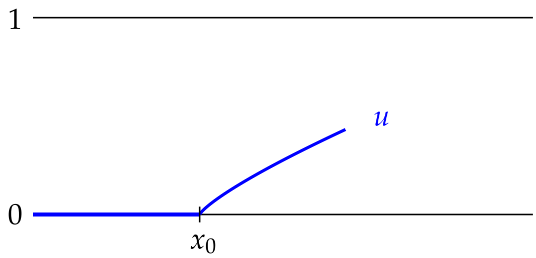

Figure 4.5: Case b)

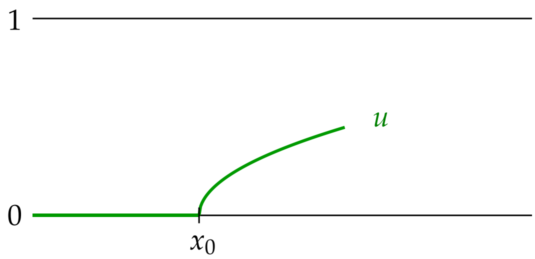

Figure 4.6: Case c)

Values of α, β for which these cases occur are for different values of p depicted in Figures 4.7, 4.8, 4.9. Areas corresponding to cases a) – c) are shown in respective colors as in Figures 4.4, 4.5, 4.6.

Proceeding similarly for $u \to 1-$ and assuming $g(s) \sim (1-s)^{\gamma}$, $d(s) \sim (1-s)^{\delta}$ as $s \to 1-$ for some $\gamma > 0$, $\delta \in \mathbb{R}$ satisfying the analogue of condition (4.1):

$$\gamma + \frac{\delta}{p-1} > -1,$$

we get the following asymptotics:

Case 1: $\gamma - \delta \geq p - 1$. Then $x_1 = +\infty$ by (3.18) and we distinguish between two cases. Either

$$u(x) \sim 1 - \mathrm{e}^{-x} \to 1- \quad \text{for } x \to +\infty$$

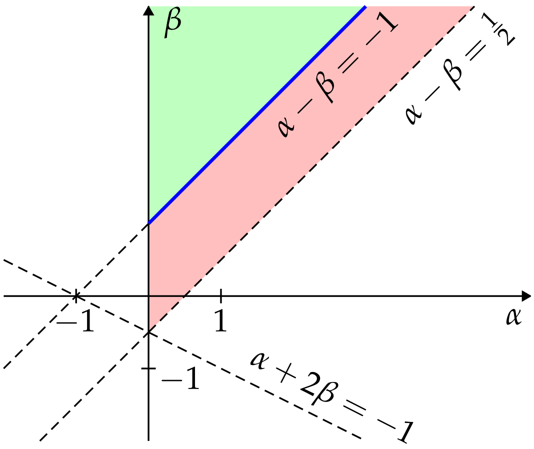

Figure 4.7: $p = \frac{3}{2}$

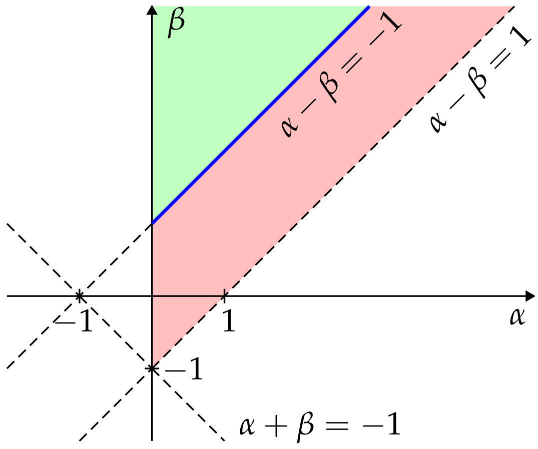

Figure 4.8: $p = 2$

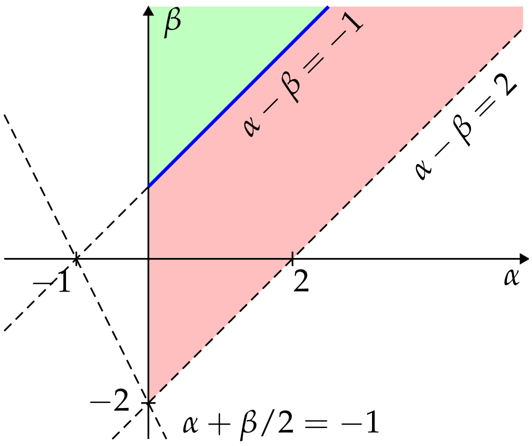

Figure 4.9: $p = 3$

if $\gamma - \delta = p - 1$, or else

$$u(x) \sim 1 - |x|^{\frac{p}{p-1-(\gamma-\delta)}} \to 1-$$

if $\gamma - \delta > p - 1$.

Case 2: $\gamma - \delta < p - 1$. Then $x_1 < +\infty$ by (3.18) and

$$u(x) \sim 1 - (x_1 - x)^{\frac{p}{p-1-(\gamma-\beta)}} \to 1- \quad \text{for } x \to x_1- \,.$$

As for the one-sided derivatives of u at x_1 we have

a) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1-} \sim (x_1 - x)^{\frac{\gamma-\delta+1}{p-1-(\gamma-\delta)}} \to 0 \quad \text{for } x \to x_1- \qquad \text{if } \gamma - \delta > -1,$$

b) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1-} \sim (x_1 - x)^0 \to k > 0 \quad \text{for } x \to x_1- \qquad \text{if } \gamma - \delta = -1,$$

c) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1-} \sim (x_1 - x)^{\frac{\gamma-\delta+1}{p-1-(\gamma-\delta)}} \to +\infty \quad \text{for } x \to x_1- \qquad \text{if } \gamma - \delta < -1.$$

Remark 4.3. While all the illustrative pictures in Section 3 do not reflect the effect of the discontinuity of d, finally, we want to focus on how the solution $u = u(x)$ is affected by discontinuous diffusion coefficient $d = d(s)$. Let us assume for simplicity that d only has one point of discontinuity $s_1 \in (0,1)$ and it is smooth and bounded in $(0,s_1)$ and $(s_1,1)$. Then $M_u = \{\xi_1\}$ and it follows from Theorem 3.2, (iv), that the jump of d at s_1 must be compensated by the proper "opposite" jump of u' at ξ_1, see Figure 4.10, namely

$$|u'(\xi_1-)|^{p-2} u'(\xi_1-) \lim_{s\to s_1-} d(s) = |u'(\xi_1+)|^{p-2} u'(\xi_1+) \lim_{s\to s_1+} d(s).$$

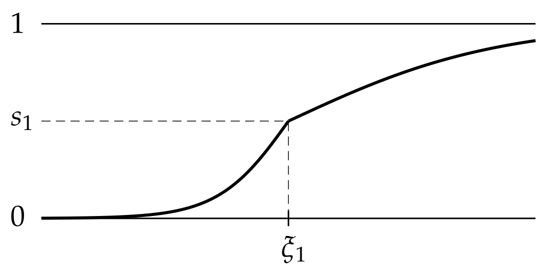

Figure 4.10: Profile of solution $u = u(x)$ for d discontinuous at s_1

5 Final discussions

Let us consider the initial value problem for the quasilinear bistable equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(d(u(x,t)) \left|\dfrac{\partial u}{\partial x}\right|^{p-2} \dfrac{\partial u}{\partial x}\right) + g(u(x,t)), \quad x \in \mathbb{R},\, t > 0, \\ u(x,0) = \varphi(x), \quad x \in \mathbb{R}. \end{cases} \tag{5.1}$$

Here, $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function, d and g are as in Section 2 and (3.3) (balanced bistable condition) holds. If $\varphi = \varphi(x)$ satisfies the hypothesis

$$\limsup_{x\to-\infty} \varphi(x) < s_* \quad \text{and} \quad \liminf_{x\to+\infty} \varphi(x) > s_*$$

then one would expect that there exists $\xi \in \mathbb{R}$ such that the solution $u = u(x,t)$ of (5.1) satisfies

$$\lim_{t\to+\infty} u(x,t) = u(x-\xi), \quad x \in \mathbb{R},$$

where $u = u(x)$ is a solution given by Theorem 3.2, see Figure 5.1.

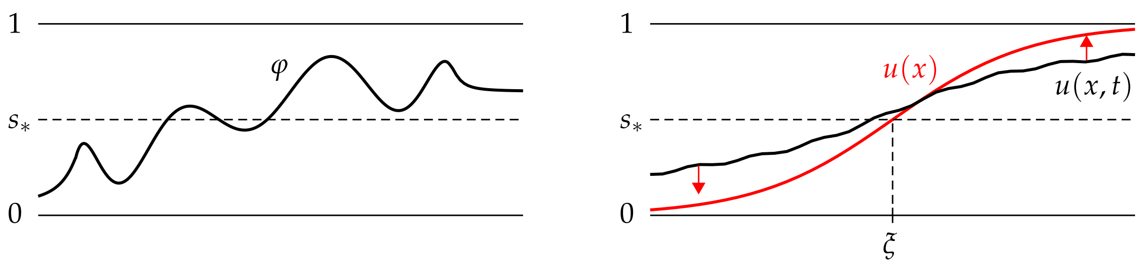

Figure 5.1: Convergence to a stationary solution

It is maybe too ambitious to prove this fact if d is a discontinuous function. However, an affirmative answer to this question, even for d continuous or smooth, would be an interesting result. Even reliable numerical simulation of the asymptotic behavior of the solution $u = u(x,t)$ of the initial value problem (5.1) for $t \to +\infty$ might be of great help.

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TRAVELING WAVES FOR UNBALANCED BISTABLE EQUATIONS WITH DENSITY DEPENDENT DIFFUSION

PAVEL DRÁBEK, MICHAELA ZAHRADNÍKOVÁ

Abstract. We study the existence and qualitative properties of traveling wave solutions for the unbalanced bistable reaction-diffusion equation with a rather general density dependent diffusion coefficient. In particular, it allows for singularities and/or degenerations as well as discontinuities of the first kind at a finite number of points. The reaction term vanishes at equilibria and it is a continuous, possibly non-Lipschitz function. We prove the existence of a unique speed of propagation and a unique traveling wave profile (up to translation) which is a non-smooth function in general. In the case of the power-type behavior of the diffusion and reaction near equilibria we provide detailed asymptotic analysis of the profile.

1. Introduction

We are concerned with the traveling wave solutions of the bistable equation

$$
\begin{gathered}
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(d(u)\big|\frac{\partial u}{\partial x}\big|^{p-2}\frac{\partial u}{\partial x}\Big) + g(u), \quad (x,t) \in \mathbb{R}\times\mathbb{R}_+, \\
u(x,t) = U(x-ct)
\end{gathered} \tag{1.1}
$$

for a speed of propagation $c \in \mathbb{R}$. Here $\mathbb{R}_+ := [0,+\infty)$, $1 < p < +\infty$ and the properties of the density dependent diffusion coefficient $d = d(s)$ as well as the reaction term $g = g(s)$ will be specified later.

If $p = 2$, $d \equiv 1$ and $g : [0,1] \to \mathbb{R}$ is a smooth function such that $g(0) = g(s_*) = g(1) = 0$, $g(s) < 0$ for $s \in (0,s_*)$, $g(s) > 0$ for $s \in (s_*,1)$, equation (1.1) is studied, e.g., in [1, 2]. The authors of these articles explain how the mathematical modeling of diploid individuals (homozygote and heterozygote) in population dynamics leads to the bistable equation. If in addition the reaction term g satisfies the *unbalanced bistable* condition

$$
\int_0^1 g(s)\,\mathrm{d}s > 0, \tag{1.2}
$$

the mathematical model describes the so called *heterozygote inferior* case. If $g(s) \le 0$ instead of $g(s) < 0$ for $s \in (0,s_*)$, the bistable equation models the flame propagation in chemical reactor theory. In contrast with the population dynamics model, where u denotes the relative density of the population of one allele, in the combustion model u represents a normalized temperature and s_* represents

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a critical temperature at which an exothermic reaction starts (see, e.g., [10]). The bistable equation with reaction term like above was also suggested in [4] as a model for a nerve which has been treated with certain toxins. In [11] this equation serves as a model for a bistable active transmission line. Other possible interpretations may be found in [13].

In this paper we focus on more general, in particular non-smooth (even non-Lipschitz) reaction term g as well as on the density dependent diffusion coefficient d which can be singular and/or degenerate near the equilibria 0 and 1 and discontinuous in a finite number of points in the interval $(0,1)$. We show the existence of a traveling wave solution if the classical unbalanced bistable condition (1.2) is replaced by the following more general one:

$$\int_0^1 \big(d(s)\big)^{\frac{1}{p-1}} g(s)\,\mathrm{d}s > 0. \tag{1.3}$$

In our previous work [7] we proved that if equality holds in (1.3) then (1.1) possesses nonconstant stationary solutions (also called *standing wave solutions*) “connecting” the equilibria 0 and 1. We also studied qualitative properties of these stationary solutions, in particular, the lack of smoothness and the asymptotic properties near the equilibria 0 and 1. In contrast with the stationary case, inequality (1.3) renders the time dependent traveling wave solutions of the form

$$u(x,t) = U(x-ct),$$

where $U = U(z)$, $z \in \mathbb{R}$, is the profile of the traveling wave and $c \in \mathbb{R}$ is the speed of its propagation. Density dependent diffusion coefficient which is discontinuous is studied, e.g., in [12] to model the temperature field in a wire of superconducting material carrying an electrical current and immersed in a bath at constant temperature. Convergence of the solution of the initial value problem for equation in (1.1) with $d \equiv 1$ to a traveling wave is studied, e.g., in [9] for C^1 reaction term g or in [6] for reaction term g which is only one-sided Lipschitz continuous.

Our results concerning the existence and uniqueness of the profile U and speed c extend and generalize those from [1, Theorem 4.2], [2, Theorem 4.1], [5, Theorem 3.1]. The asymptotic analysis of U near 0 and 1 extends that from [5, Section 6].

2. Preliminaries

Let $g : [0,1] \to \mathbb{R}$, $g \in C[0,1]$ be such that $g(0) = g(s_*) = g(1) = 0$ for $s_* \in (0,1)$ and

$$g(s) \le 0,\ s \in (0, s_*), \quad g(s) > 0,\ s \in (s_*, 1).$$

The diffusion coefficient $d : [0,1] \to \mathbb{R}$ is supposed to be nonnegative lower semi-continuous and $d > 0$ in $(0,1)$. There exist $0 = s_0 < s_1 < s_2 < \dots < s_n < s_{n+1} = 1$ such that $d\big|_{(s_i,s_{i+1})} \in C(s_i, s_{i+1})$, $i = 0, \dots, n$, and d has discontinuity of the first kind (finite jump) at s_i, $i = 1, \dots, n$.

We introduce the moving coordinate $z = x - ct$ and write $u(x,t) = U(x-ct) = U(z)$. For the sake of simplicity we write $(\cdot)'$ instead of $\frac{\mathrm{d}}{\mathrm{d}z}(\cdot)$. Then (1.1) transforms into

$$\Big(d(U(z))|U'(z)|^{p-2}U'(z)\Big)' + cU'(z) + g(U(z)) = 0\,. \tag{2.1}$$

Let $U : \mathbb{R} \to [0,1]$ be a monotone continuous function. We denote

$$M_U := \{z \in \mathbb{R} : U(z) = s_i,\ i = 1, 2, \dots, n\},$$

$$N_U := \{z \in \mathbb{R} : U(z) = 0 \text{ or } U(z) = 1\}.$$

Then M_U and N_U are closed sets, M_U is a union of a finite number of points or intervals,

$$N_U = (-\infty, z_0] \cup [z_1, +\infty),$$

where $-\infty \le z_0 < z_1 \le +\infty$ and we use the convention $(-\infty, z_0] = \emptyset$ if $z_0 = -\infty$ and $[z_1, +\infty) = \emptyset$ if $z_1 = +\infty$. Below we introduce the definition of a monotone solution of (2.1).

Definition 2.1. A monotone continuous function $U : \mathbb{R} \to [0, 1]$ is called a solution of (2.1) if

(a) For any $z \notin M_U \cup N_U$ the derivative $U'(z)$ exists and it is finite. For $z \in \operatorname{int} M_U \cup \operatorname{int} N_U$ we have $U'(z) = 0$.

(b) For any $z \in \partial M_U$ there exist finite one sided derivatives $U'(z-)$, $U'(z+)$ and

$$L(z) := |U'(z-)|^{p-2}U'(z-) \lim_{\xi \to z-} d(U(\xi)) = |U'(z+)|^{p-2}U'(z+) \lim_{\xi \to z+} d(U(\xi)).$$

(c) Function $v : \mathbb{R} \to \mathbb{R}$ defined by

$$v(z) := \begin{cases} d(U(z))|U'(z)|^{p-2}U'(z), & z \notin M_U \cup N_U, \\ 0, & z \in N_U \cup \operatorname{int} M_U, \\ L(z), & z \in \partial M_U \end{cases}$$

is continuous and for any $z, \hat{z} \in \mathbb{R}$,

$$v(\hat{z}) - v(z) + c\,(U(\hat{z}) - U(z)) + \int_z^{\hat{z}} g(U(\xi))\,\mathrm{d}\xi = 0\,. \tag{2.2}$$

Moreover, $\lim_{z \to \pm\infty} v(z) = 0$ if either
$\lim_{z \to -\infty} U(z) = 1$ and $\lim_{z \to +\infty} U(z) = 0$, or
$\lim_{z \to -\infty} U(z) = 0$ and $\lim_{z \to +\infty} U(z) = 1$.

Remark 2.2. Constant functions $U \equiv k$, where k is such that $g(k) = 0$, are solutions of (2.1). In particular, $U \equiv 0$, $U \equiv 1$ and $U \equiv s_*$ are solutions.

Remark 2.3. Let $z \notin M_U \cup N_U, \hat{z} = z + h, h \neq 0$. Divide (2.2) by h and let $h \to 0$. Then, by Definition 2.1 (a), the derivative $U'(z)$ exists and

$$v'(z) + cU'(z) + g(U(z)) = 0. \tag{2.3}$$

In particular, v is differentiable in z.

Remark 2.4. Let U be a solution of (2.1) in the sense of Definition 2.1 such that either $U(z) \to 1$ as $z \to -\infty$ and $U(z) \to 0$ as $z \to +\infty$ or $U(z) \to 0$ as $z \to -\infty$ and $U(z) \to 1$ as $z \to +\infty$. If d is not continuous in $(0, 1)$ then $M_U \neq \emptyset$ and either $M_U = \partial M_U$ (i.e., $\operatorname{int} M_U = \emptyset$) or else $\operatorname{int} M_U \neq \emptyset$. In the former case $M_U = \{\xi_1, \dots, \xi_n\}$, where $U(\xi_i) = s_i$, $i = 1, \dots, n$. In the latter case there exist $1 \le k \le n$ and $1 \le i_1 < i_2 < \dots < i_k \le n$ such that $U \equiv s_{i_j}$ on some interval $[a_{i_j}, b_{i_j}]$, $a_{i_j} < b_{i_j}$, $j = 1, \dots, k$, and $\operatorname{int} M_U = \cup_{j=1}^n (a_{i_j}, b_{i_j})$. The equation (2.1) is satisfied pointwise in $\operatorname{int} M_U$ and it follows from the continuity of v that if $a_{i_j} > -\infty$ or $b_{i_j} < +\infty$ we have $U'(a_{i_j}) = 0$ or $U'(b_{i_j}) = 0$, $j = 1, \dots, k$, respectively, because $d(s) > 0$, $s \in (0, 1)$.

If $z_0 > -\infty$ then $U'(z_0-) = 0$ and $U'(z_0+)$ exists finite or infinite. Similarly, if $z_1 < +\infty$ then $U'(z_1+) = 0$ and $U'(z_1-)$ exists finite or infinite.

Remark 2.5. Let $p = 2$, $d \equiv 1$ and $g \in C^1[0,1]$. Let $U = U(z)$ be a solution in the sense of Definition 2.1. Then $M_U = \emptyset$, $N_U = \emptyset$, and (2.1) holds pointwise, i.e., $U \in C^2(\mathbb{R})$ and it is a classical solution. For more general d we have to employ the first integral (2.2) because of the lack of differentiability of a solution U.

3. Equivalent first order ODE

We will look for monotone traveling waves $U = U(z)$ satisfying boundary conditions

$$\lim_{z\to-\infty} U(z) = 1 \quad \text{and} \quad \lim_{z\to+\infty} U(z) = 0. \tag{3.1}$$

Let $U : \mathbb{R} \to [0,1]$ be a monotone nonincreasing solution of the BVP (2.1), (3.1) such that U is strictly decreasing at any point $z \in \mathbb{R}$ where $U(z) \in (0,1)$. Then there exist $-\infty \le z_0 < z_1 \le +\infty$ such that $U(z) = 1$, $z \in (-\infty, z_0]$ and $U(z) = 0, z \in [z_1, +\infty)$. Moreover, $M_U = \{\xi_1, \xi_2, \dots, \xi_n\}$ where $U(\xi_i) = s_i$, $i = 1, 2, \dots, n$. In particular, $\operatorname{int} M_U = \emptyset$ and $M_U = \partial M_U$, see Remark 2.4. For all $z \notin M_U \cup N_U$ we have $U'(z) < 0$ and for all $z \in M_U$ we have $U'(z-) < 0$ and $U'(z+) < 0$. The function U is continuous and piecewise C^1 in the sense that $U\big|_{(\xi_i,\xi_{i+1})} \in C^1(\xi_i, \xi_{i+1})$. Therefore, there exists strictly decreasing inverse function $U^{-1} : (0,1) \to (z_0, z_1)$, $z = U^{-1}(U)$, such that $U^{-1}\big|_{(s_i,s_{i+1})} \in C^1(s_i, s_{i+1})$, $i = 0, 1, \dots, n$ and the limits

$$\lim_{U\to s_i-} \frac{\mathrm{d}}{\mathrm{d}U} U^{-1}(U), \quad \lim_{U\to s_i+} \frac{\mathrm{d}}{\mathrm{d}U} U^{-1}(U)$$

exist and are finite for $i = 1, 2, \dots, n$. Hence, we make the change of variables (cf. [7, 8])

$$w(U) = v(U^{-1}(U)), \quad U \in (0,1). \tag{3.2}$$

It follows from Remark 2.3 that $w = w(U)$ is a piecewise C^1-function in $(0,1)$,

$$w\big|_{(s_i,s_{i+1})} \in C^1(s_i, s_{i+1}), \quad i = 0, 1, \dots, n,$$

with finite limits $\lim_{U\to s_i-} w'(U)$, $\lim_{U\to s_i+} w'(U)$, $i = 1, 2, \dots, n$. Therefore, for any $z \in (\xi_i, \xi_{i+1})$ and $U \in (s_i, s_{i+1})$, $i = 0, 1, \dots, n$, we have

$$\frac{\mathrm{d}}{\mathrm{d}z} v(z) = \frac{\mathrm{d}}{\mathrm{d}z} w(U(z)) = \frac{\mathrm{d}w}{\mathrm{d}U}\,(U(z))\,U'(z). \tag{3.3}$$

From $v(z) = -d(U(z))|U'(z)|^{p-1}$ we deduce that

$$U'(z) = -\Big|\frac{v(z)}{d(U(z))}\Big|^{p'-1}, \quad p' = \frac{p}{p-1}\,. \tag{3.4}$$

From (3.2), (3.3) and (3.4),

$$\frac{\mathrm{d}v}{\mathrm{d}z} = -\frac{\mathrm{d}w}{\mathrm{d}U}\,(U(z))\,\Big|\frac{v(z)}{d(U(z))}\Big|^{p'-1} = -\frac{\mathrm{d}w}{\mathrm{d}U}\Big|\frac{w(U)}{d(U)}\Big|^{p'-1}.$$

Therefore, equation (2.3), namely

$$v'(z) + cU'(z) + g(U(z)) = 0, \quad z \in (\xi_i, \xi_{i+1}),$$

becomes

$$-\frac{\mathrm{d}w}{\mathrm{d}U}\Big|\frac{w(U)}{d(U)}\Big|^{p'-1} - c\Big|\frac{w(U)}{d(U)}\Big|^{p'-1} + g(U) = 0, \quad U \in (s_i, s_{i+1}),$$

$i = 0, 1, \dots, n$. This is equivalent to

$$|w|^{p'-1}\frac{\mathrm{d}w}{\mathrm{d}U} = -c|w|^{p'-1} + (d(U))^{p'-1}\, g(U), \tag{3.5}$$

or

$$\frac{1}{p'}\frac{\mathrm{d}}{\mathrm{d}U}|w|^{p'} = c|w|^{p'-1} - (d(U))^{p'-1}\, g(U). \tag{3.6}$$

Set $f(U) = (d(U))^{\frac{1}{p-1}}\, g(U)$ and write t instead of U, and $y(t) = |w(t)|^{p'}$. Then (3.6) becomes

$$y'(t) = p'\big[c\,(y(t))^{1/p} - f(t)\big], \quad t \in (0,1) \setminus \cup_{i=1}^{n}\{s_i\}. \tag{3.7}$$

From (3.1) and Definition 2.1(c) we deduce that $v(z) \to 0$ as $z \to z_0+$ or $z \to z_1-$ which is equivalent to $\lim_{U\to 0+} w(U) = \lim_{U\to 1-} w(U) = 0$. Therefore, $y = y(t)$ satisfies the boundary conditions

$$y(0) = y(1) = 0. \tag{3.8}$$

On the other hand, let us suppose that $y = y(t)$ is a positive solution of (3.7), (3.8). Set $w(s) := -(y(s))^{1/p'}$. Then w satisfies (3.5) and (3.6). For $U \in (0,1)$ set

$$z(U) = z(s_*) - \int_{s_*}^{U} \Big|\frac{d(s)}{w(s)}\Big|^{\frac{1}{p-1}}\,\mathrm{d}s. \tag{3.9}$$

Then the function $z = z(U)$ is continuous strictly decreasing and maps the interval $(0,1)$ onto (z_0, z_1), where $-\infty \le z_0 < z_1 \le +\infty$. Let us denote by $U : (z_0, z_1) \to (0,1)$ the inverse function to $z = z(U)$. Then $U(z(s_*)) = s_*$, U is continuous strictly decreasing,

$$\lim_{z\to z_0+} U(z) = 1 \quad \text{and} \quad \lim_{z\to z_1-} U(z) = 0.$$

Let $z \in (\xi_i, \xi_{i+1})$, $i = 0, 1, \dots, n$, where $U(\xi_i) = s_i$, $i = 0, 1, \dots, n, n+1$. Then from (3.9) we deduce

$$\frac{\mathrm{d}U}{\mathrm{d}z} = \frac{1}{\frac{\mathrm{d}z(U)}{\mathrm{d}U}} = -\Big|\frac{w(U)}{d(U)}\Big|^{\frac{1}{p-1}}, \quad U \in (s_i, s_{i+1}),$$

i.e., $U \in C^1(\xi_i, \xi_{i+1})$, $U'(z) < 0$ and

$$-\, d(U(z))\Big|\frac{\mathrm{d}U(z)}{\mathrm{d}z}\Big|^{p-1} = w(U(z)) =: v(z), \tag{3.10}$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}z}\Big[d(U(z))\Big|\frac{\mathrm{d}U}{\mathrm{d}z}\Big|^{p-2}\frac{\mathrm{d}U}{\mathrm{d}z}\Big] = \frac{\mathrm{d}}{\mathrm{d}z}w(U(z)) = \frac{\mathrm{d}w}{\mathrm{d}U}\,\frac{\mathrm{d}U(z)}{\mathrm{d}z}\,. \tag{3.11}$$

From (3.5), (3.10) we deduce that

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}U} &= -|w(U)|^{-(p'-1)}\Big(-c|w(U)|^{p'-1} + (d(U))^{p'-1}\, g(U)\Big)\\
&= -c + |w(U)|^{-(p'-1)}\,(d(U))^{p'-1}\, g(U)\\
&= -c + (d(U(z)))^{-(p'-1)}\,\Big|\frac{\mathrm{d}U(z)}{\mathrm{d}z}\Big|^{-(p-1)(p'-1)}\,(d(U(z)))^{p'-1}\, g(U(z))\\
&= -c + \Big|\frac{\mathrm{d}U(z)}{\mathrm{d}z}\Big|^{-1} g(U(z)).
\end{aligned}$$

Let us substitute this into (3.11):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\big[d(U(z))\big|\frac{\mathrm{d}U}{\mathrm{d}z}\big|^{p-2}\frac{\mathrm{d}U}{\mathrm{d}z}\big] &= \big[-c+\big|\frac{\mathrm{d}U(z)}{\mathrm{d}z}\big|^{-1}g(U(z))\big]\frac{\mathrm{d}U(z)}{\mathrm{d}z}\\ &= -c\frac{\mathrm{d}U(z)}{\mathrm{d}z}-g(U(z)),\end{aligned}$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}z}\big[d(U(z))\big|\frac{\mathrm{d}U}{\mathrm{d}z}\big|^{p-2}\frac{\mathrm{d}U}{\mathrm{d}z}\big]+c\frac{\mathrm{d}U(z)}{\mathrm{d}z}+g(U(z))=0,\quad z\in(\xi_i,\xi_{i+1}),$$

$i=0,1,\dots,n$. It follows from (3.10) and the continuity of U that

$$\lim_{z\to z_0+}d(U(z))|U'(z)|^{p-2}U'(z)=\lim_{z\to z_1-}d(U(z))|U'(z)|^{p-2}U'(z)=0$$

and the following one sided limits are finite

$$\lim_{z\to\xi_i-}d(U(z))|U'(z)|^{p-2}U'(z)=\lim_{z\to\xi_i+}d(U(z))|U'(z)|^{p-2}U'(z),$$

$i=1,2,\dots,n$. Since U is monotone decreasing in (z_0,z_1), we have

$$\lim_{z\to\xi_i-}d(U(z))=\lim_{s\to s_i+}d(s)\quad\text{and}\quad\lim_{z\to\xi_i+}d(U(z))=\lim_{s\to s_i-}d(s).$$

Therefore, U satisfies the transition condition

$$|U'(\xi_i-)|^{p-2}\,U'(\xi_i-)\lim_{s\to s_i+}d(s)=|U'(\xi_i+)|^{p-2}\,U'(\xi_i+)\lim_{s\to s_i-}d(s).$$

We summarize the above reasoning in the following equivalence.

Proposition 3.1. *A function $U:\mathbb{R}\to[0,1]$ is a monotone non-increasing solution of* (2.1), (3.1) *which is strictly decreasing at any point $z\in\mathbb{R}$ where $U(z)\in(0,1)$ if and only if $y:[0,1]\to\mathbb{R}$ is a positive solution of* (3.7), (3.8).

Thanks to this proposition we can study the first order problem (3.7), (3.8) and derive the existence result for (2.1), (3.1). Let us recall that there are "two unknowns" in the first order problem. Indeed, besides the positive solution $y=y(t)$ we also look for unknown speed of propagation $c\in\mathbb{R}$.

Lemma 3.2. *Let us assume that* (1.3) *holds and BVP* (3.7), (3.8) *has a positive solution. Then $c>0$.*

Proof. Let $y(t)>0, t\in(0,1)$ be a positive solution of (3.7), (3.8). Integrating (3.7) and using (3.8) we obtain

$$0=y(1)-y(0)=\int_0^1 y'(t)\,\mathrm{d}t=p'\Big[c\int_0^1 y(t)\,\mathrm{d}t-\int_0^1 f(t)\,\mathrm{d}t\Big].$$

Hence

$$c=\frac{\int_0^1 (d(t))^{\frac{1}{p-1}}\,g(t)\,\mathrm{d}t}{\int_0^1 y(t)\,\mathrm{d}t}>0. \qquad \square$$

Remark 3.3. Let d and g be as in Section 2 and the following balanced condition holds

$$\int_0^1 (d(s))^{\frac{1}{p-1}}\,g(s)\,\mathrm{d}s=0.$$

Then

$$y(t)=-p'\int_0^t (d(s))^{\frac{1}{p-1}}\,g(s)\,\mathrm{d}s,\quad t\in(0,1) \tag{3.12}$$

is a unique positive solution of (3.7), (3.8) with $c = 0$. The solution given by (3.12) leads to the standing wave. Its profile $U = U(x)$ satisfies

$$\big(d(U(x))|U'(x)|^{p-2}U'(x)\big)' + g(U(x)) = 0, \quad x \in \mathbb{R}.$$

Detailed analysis of these solutions for g satisfying $g < 0$ in $(0, s_*)$ is given in [7].

4. Existence result

In this section we first concentrate on the existence result for the first order BVP (3.7), (3.8). More precisely, we prove that under appropriate assumptions on $f = f(t)$ there exists a unique real number $c > 0$ and an absolutely continuous function $y = y(t)$ such that $y(t) > 0, t \in (0,1)$, (3.8) holds and the equation (3.7) is satisfied in the sense of Carathéodory (see [3, Chapter 2]).

The following result is interesting on its own. In combination with Proposition 3.1 it is a tool to prove the existence and uniqueness of the traveling speed c and monotone decreasing traveling wave profile U.

Theorem 4.1. *Let $f \in L^1(0,1)$, $f(t) \le 0$, $t \in (0, s_*)$, $f(t) > 0$, $t \in (s_*, 1)$, and*

$$\int_0^1 f(s)\,\mathrm{d}s > 0. \tag{4.1}$$

Then there is a unique number $c > 0$ and an absolutely continuous function $y = y(t)$, $t \in [0,1]$, such that $y(0) = y(1) = 0$, $y(t) > 0, t \in (0,1)$, and

$$y'(t) = p'\Big[c\left(y^+(t)\right)^{1/p} - f(t)\Big]$$

for a.a. $t \in (0,1)$. Here $y^+ = \max\{y, 0\}$ denotes the positive part of y.

We prove Theorem 4.1 using the concept of solution of the first order ODE in the sense of Carathéodory. For $(t, y, c) \in [0,1] \times \mathbb{R}^2$ and $f = f(t)$ we set

$$h(t, y, c) := p'\Big[c\left(y^+\right)^{1/p} - f(t)\Big]$$

and consider the following two initial value problems which depend on a parameter $c \in \mathbb{R}$:

$$y'(t) = h(t, y(t), c), \quad y(0) = 0 \tag{4.2}$$

and

$$y'(t) = h(t, y(t), c), \quad y(1) = 0. \tag{4.3}$$

In both cases we look for a solution $y = y(t)$, $t \in [0,1]$. Therefore, (4.2) is referred to as a *forward initial value problem*, while (4.3) is referred to as a *backward initial value problem*. Note that $f \in L^1(0,1)$ implies that $h = h(t, y, c)$ satisfies Carathéodory's conditions, i.e., for a.e. $t \in [0,1]$ fixed, $h(t, \cdot, \cdot)$ is continuous with respect to y and c and for every $y \in \mathbb{R}$ and $c \in \mathbb{R}$ fixed, $h(\cdot, y, c)$ is measurable with respect to t. In what follows, for any fixed $c \in \mathbb{R}$, $y_c = y_c(t)$ denotes the solution in the sense of Carathéodory of the forward and backward initial value problem (4.2) and (4.3), respectively. In particular, y_c is absolutely continuous in $[0,1]$ and the equation holds a.e. in $[0,1]$. Let us mention the following global existence result.

Lemma 4.2. *Let $f \in L^1(0,1)$, $c \in \mathbb{R}$. Then there exists at least one global solution $y_c = y_c(t)$ of both (4.2) and (4.3) defined on the entire interval $[0,1]$.*

Proof. For any fixed $K > 0$ and $c \in \mathbb{R}$ there exists $m_{c,K} \in L^1(0,1)$ such that for any $y \in [-K, K]$ we have $|h(t, y, c)| \le m_{c,K}(t)$ for a.a. $t \in [0, 1]$. This fact follows from the definition of h. But then according to [14, Theorem 10.XX] there exist solutions $y_c = y_c(t)$ of both (4.2) and (4.3) which are defined for all $t \in [0, 1]$. □

Remark 4.3. The uniqueness of the solution in the above lemma does not hold in general due to the fact that the function $y \mapsto c\,(y^+)^{1/p}$, $y \in \mathbb{R}$, does not satisfy the Lipschitz condition at 0. However, it is nondecreasing for $c > 0$ and non-increasing for $c < 0$. Therefore, it satisfies one-sided Lipschitz condition in either case and we have the following uniqueness results separately for the forward and backward initial value problems, depending on the sign of c.

Lemma 4.4. *Let $f \in L^1(0, 1)$. If $c \le 0$ then* (4.2) *has exactly one solution $y_c = y_c(t), t \in [0, 1]$. If $c \ge 0$ then* (4.3) *has exactly one solution $y_c = y_c(t), t \in [0, 1]$.*

The proof of the above lemma follows from combination of Theorems 9.X and 10.XX in [14]. Thanks to the uniqueness result we also have continuous dependence of solutions on the parameter c.

Lemma 4.5. *Let $f \in L^1(0, 1)$, $c_0 \ge 0$. Then $c \to c_0 \ne 0$ ($c \to 0_+$ if $c_0 = 0$) implies that solutions $y_c = y_c(t)$ of the backward initial value problem* (4.3) *converge uniformly in $[0, 1]$ (i.e., in the topology of $C[0, 1]$) to y_{c_0}. Similar result holds for $c_0 \le 0$ and the forward initial value problem* (4.2).

The proof of the above lemma follows from the uniqueness result in Lemma 4.4 and [3, Theorems 4.1 and 4.2].

As we already observed in the proof of Lemma 3.2, the assumption (4.1) yields $c > 0$. For this reason, we further focus on parameters $c \in [0, +\infty)$ and the backward initial value problem (4.3). We know that for any $c \in [0, +\infty)$ there is a unique solution of (4.3), $y_c = y_c(t), t \in [0, 1]$. Our goal is to show that there is unique $c_* > 0$ such that $y_{c_*}(t) > 0$, $t \in (0, 1)$, $y_{c_*}(0) = 0$. To this end we have to investigate in more detail the dependence of the solution $y_c = y_c(t)$ of the backward initial value problem (4.3) on the parameter c.

Let us introduce the notion of the *defect* $P_c\varphi$ of a function $\varphi = \varphi(t)$ with respect to the differential equation $y' = h(t, y, c)$, see e.g. [14, §9.II]:

$$P_c\varphi := \varphi' - h(t, \varphi, c).$$

The following comparison argument is one of our basic tools.

Lemma 4.6. *Let $f \in L^1(\varrho, 1)$,$0 \le \varrho < 1$, $c \ge 0$, $\varphi(1) \le \psi(1)$, $P_c\varphi \ge P_c\psi$ a.e. in $[\varrho, 1]$. Then either $\varphi < \psi$ in $[\varrho, 1]$ or there exists $\xi \in [\varrho, 1]$ such that $\varphi = \psi$ in $[\xi, 1]$ and $\varphi < \psi$ in $(\varrho, \xi]$. In particular, $\varphi \le \psi$ in $[\varrho, 1]$.*

The proof of the above lemma follows from combination of Theorems 9.X and 10.XXI in [14].

Corollary 4.7. *Let f be as in Theorem 4.1, and $0 \le c_1 < c_2$. Then*

$$y_{c_1}(t) > y_{c_2}(t), \quad t \in (0, 1).$$

In particular, we have the weak comparison at the terminal value 0: $y_{c_1}(0) \ge y_{c_2}(0)$.

Proof. We have

$$P_{c_2}y_{c_1} = y'_{c_1} - h(t, y_{c_1}, c_2) = \underbrace{y'_{c_1} - h(t, y_{c_1}, c_1)}_{=0} + h(t, y_{c_1}, c_1) - h(t, y_{c_1}, c_2)$$

$$= p'(c_1 - c_2)\left(y_{c_1}^+\right)^{1/p} \le 0 = y'_{c_2} - h(t, y_{c_2}, c_2) = P_{c_2} y_{c_2}\,.$$

Then Lemma 4.6 with $\varrho = 0$ yields that either $y_{c_1} > y_{c_2}$ in $(0,1)$ or there exists $\xi \in [0,1]$ such that $y_{c_1} = y_{c_2}$ in $[\xi, 1]$ and $y_{c_1} > y_{c_2}$ in $(0,\xi)$. Since both y_{c_1} and y_{c_2} solve the backward initial value problem, we subtract the equation in (4.3) for $c = c_2$ from that for $c = c_1$ and obtain

$$p'(c_1 - c_2)\left(y_{c_1}^+\right)^{1/p} = 0 \quad \text{in } [\xi, 1],$$

i.e., $y_{c_1} = y_{c_2} \le 0$ in $[\xi, 1]$. But from (4.1) and (4.3) we then obtain

$$y_{c_1}(\xi) = y_{c_2}(\xi) = p' \int_\xi^1 f(\sigma)\,\mathrm{d}\sigma > 0$$

if $\xi < 1$. Therefore, $\xi = 1$ and $y_{c_1} > y_{c_2}$ in $(0,1)$. Using Lemma 4.2 and extending y_{c_1}, y_{c_2} continuously to 0, we end up with $y_{c_1}(0) \ge y_{c_2}(0)$. □

Remark 4.8. Unfortunately, the comparison argument above does not allow us to conclude the strict inequality $y_{c_1}(0) > y_{c_2}(0)$ in general. For this reason the proof of uniqueness of c_* is more involved and requires more detailed analysis of the solution of the backward initial value problem (4.3) at the terminal value 0.

Corollary 4.9. *Let $f \in L^1(0,1)$, $f(t) \le 0$, $t \in (0, s_*)$, $f(t) > 0$, $t \in (s_*, 1)$ and $\tilde f(t) = 0$, $t \in (0, s_*)$, $\tilde f(t) = f(t)$, $t \in (s_*, 1)$. Let $c \ge 0$ and $\tilde y_c = \tilde y(t)$, $t \in [0,1]$ be a solution of* (4.3) *with f replaced by $\tilde f$. Then $y_c \le \tilde y_c$ in* $(0,1]$.

Proof. Set $\tilde h(t,y,c) := p'\Big[c(y^+(t)^{1/p}) - \tilde f(t)\Big]$. Then $\tilde h \le h$ and so

$$\begin{aligned} P_c \tilde y_c = \tilde y'_c - h(t, \tilde y_c, c) &= \underbrace{\tilde y'_c - \tilde h(t, \tilde y_c, c)}_{=0} + \tilde h(t, \tilde y_c, c) - h(t, \tilde y_c, c) \\ &\le 0 = y'_c - h(t, y_c, c) = P_c y_c \quad \text{a.e. in} \quad (0,1). \end{aligned}$$

It then follows from Lemma 4.6 with $\varrho = 0$ that $y_c \le \tilde y_c$ in $[0,1]$. □

Corollary 4.10. *Let $f \in L^1(0,1)$, $f(t) \le 0$, $t \in (0, s_*)$, $f(t) > 0$, $t \in (s_*, 1)$. Then there exists $c_\# > 0$ such that $y_{c_\#}(0) < 0$.*

Proof. Let $\tilde f$ be as in Corollary 4.9. Since $\tilde f = 0$ in $(0, s_*)$, we have

$$\tilde y'_c = p' c\left(\tilde y_c^+(t)\right)^{1/p} \quad \text{a.e. in } (0, s_*). \tag{4.4}$$

Assume that there exist $c_n \to +\infty$ such that $\tilde y_{c_n} \ge 0$ in $[0,1]$. Then $\tilde y_c^+ = \tilde y_c$ and separating variables in (4.4) yields

$$\left(\tilde y_{c_n}(t)\right)^{1/p'} = \left(\tilde y_{c_n}(s_*)\right)^{1/p'} + c_n(t - s_*), \quad t \in [0, s_*), \tag{4.5}$$

and

$$\tilde y'_{c_n}(t) = p'\big[c_n\left(\tilde y_{c_n}(t)\right)^{1/p} - f(t)\big], \quad t \in [s_*, 1),$$

with $\tilde y_{c_n}(1) = 0$. Therefore

$$\tilde y_{c_n}(s_*) \le p' \int_{s_*}^1 f(\sigma)\,\mathrm{d}\sigma < +\infty. \tag{4.6}$$

Then we conclude from (4.5), (4.6) that there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we must have $\tilde y_{c_n}(0) < 0$. But Corollary 4.9 yields that $y_{c_n}(0) \le \tilde y_{c_n}(0) < 0$. Therefore, we may set $c_\# = c_n$, $n \ge n_0$. □

Corollary 4.11. *Let $f \in L^1(0,1)$, $f(t) \le 0$, $t \in (0, s_*)$, $f(t) > 0$, $t \in (s_*, 1)$ and $y_c = y_c(t)$ be a solution of (4.3) with $c > 0$. If $y_c(0) \ge 0$ then $y_c(t) > 0$ for $t \in (0,1)$.*

Proof. We have

$$P_c 0 = 0 - h(t,0,c) = p' f(t) \ge 0 = y_c' - h(t, y_c, c) = P_c y_c \quad \text{a.e. in } [s_*, 1].$$

Then by Lemma 4.6 with $\varrho = s_*$ either $y_c > 0$ in $[s_*, 1)$ or there exists $\xi \in [s_*, 1]$ such that $y_c = 0$ in $[\xi, 1]$ and $y_c > 0$ in $[s_*, \xi)$. In the latter case

$$0 = y_c(\xi) = p' \Big[c \int_1^\xi \big(y_c^+(t) \big)^{1/p} \, \mathrm{d}t - \int_1^\xi f(t) \, \mathrm{d}t \Big] = p' \int_\xi^1 f(t) \, \mathrm{d}t$$

forces $\xi = 1$ (note that $f > 0$ in $(s_*, 1)$). Therefore, $y_c > 0$ in $[s_*, 1)$. Assume that y_c vanishes at $(0,1)$ and $\eta \in (0, s_*)$ be its largest zero. Then

$$y_c'(t) = p' \left[c (y_c^+(t))^{1/p} - f(t) \right] \ge p' c (y_c^+(t))^{1/p}, \quad \text{for a.e. } t \in [0, \eta].$$

Separating variables and integrating over $[t, \eta]$,

$$(y_c^+(t))^{1/p'} \le -c(\eta - t), \quad t \in [0, \eta].$$

In particular, we have

$$\big(y_c^+(0) \big)^{1/p'} \le -c\eta < 0,$$

a contradiction with $y_c(0) \ge 0$. □

Proof of Theorem 4.1. It follows from the assumptions on f that

$$y_0(t) = p' \int_t^1 f(\sigma) \, \mathrm{d}\sigma > 0$$

for all $t \in [0,1)$. In particular, $y_0(0) > 0$. On the other hand, from Corollary 4.10 there exists $c_\# > 0$ such that $y_{c_\#}(0) < 0$. The continuous dependence on parameter c in Lemma 4.5, intermediate value theorem and the monotonicity of function $\mathcal{S} : c \mapsto y_c(0)$ in Corollary 4.7 imply that there exist $0 < c_1 \le c_2 < c_\#$ such that $\mathcal{S}(c) = 0$ for all $c \in [c_1, c_2]$, $\mathcal{S}(c) > 0$, $c < c_1$ and $\mathcal{S}(c) < 0$, $c > c_2$. Below we derive the strong comparison argument which shows that, actually, $c_1 = c_2$. Indeed, let $c_1 < c_2$. By Corollaries 4.7 and 4.11 we have

$$y_{c_1}(t) > y_{c_2}(t) > 0 \quad \text{for } t \in (0,1). \tag{4.7}$$

Notice that for $c \in (c_1, c_2)$ we also have

$$y_{c_1}'(t) = p' \big[c_1 \, (y_{c_1}(t))^{1/p} - f(t) \big] \le p' \big[c \, (y_{c_1}(t))^{1/p} - f(t) \big], \tag{4.8}$$

$$y_{c_2}'(t) = p' \big[c_2 \, (y_{c_2}(t))^{1/p} - f(t) \big] \ge p' \big[c \, (y_{c_2}(t))^{1/p} - f(t) \big], \tag{4.9}$$

for a.e. $t \in (0,1)$. Set $z_1 = (y_{c_1})^{1/p'} > 0$, $z_2 = (y_{c_2})^{1/p'} > 0$. Then $z_1 > z_2$ in $(0,1)$ and it follows from (4.8) and (4.9) that

$$z_1'(t) \le c - \frac{f(t)}{(z_1(t))^{\frac{1}{p-1}}} \, , \tag{4.10}$$

$$z_2'(t) \ge c - \frac{f(t)}{(z_2(t))^{\frac{1}{p-1}}} \, , \tag{4.11}$$

for a.e. $t \in (0,1)$. Let us subtract (4.11) from (4.10) and restrict on the interval $(0, s_*)$. Then

$$(z_1(t) - z_2(t))' \leq -f(t)\Big(\frac{1}{(z_1(t))^{\frac{1}{p-1}}} - \frac{1}{(z_2(t))^{\frac{1}{p-1}}}\Big)$$

and

$$\begin{aligned}&(z_1(t) - z_2(t))\,(z_1(t) - z_2(t))' \\ &\leq -f(t)\Big(\frac{1}{(z_1(t))^{\frac{1}{p-1}}} - \frac{1}{(z_2(t))^{\frac{1}{p-1}}}\Big)\,(z_1(t) - z_2(t)) \leq 0\end{aligned}$$

for a.e. $t \in (0, s_*)$ (notice that $f(t) \leq 0$ in $(0, s_*)$). Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,(z_1(t) - z_2(t))^2 \leq 0 \quad \text{for a.e. } t \in (0, s_*). \tag{4.12}$$

Since $z_1(0) = z_2(0) = 0$, it follows from (4.12) that $z_1(t) = z_2(t)$, $t \in (0, s_*)$, i.e., $y_{c_1}(t) = y_{c_2}(t)$, $t \in (0, s_*)$. However, this contradicts (4.7). Therefore, $c_1 = c_2$ and $c_* = c_1 = c_2$ is the unique value of c for which $y_c(0) = 0$. As mentioned above, $y_{c_*}(t) > 0$, $t \in (0,1)$, and y_{c_*} is the unique solution of the backward initial value problem (4.3). The proof is complete. □

Theorem 4.12. *Let d and g be as in Section 2 and* (1.3) *holds. Then there is a unique value of $c = c_*$ and unique non-increasing traveling wave profile $U = U(z)$, $z \in \mathbb{R}$, such that U solves the BVP* (2.1), (3.1). *Furthermore, $c_* > 0$ and*

(i) *there exist $-\infty \leq z_0 < 0 < z_1 \leq +\infty$ such that $U(z) = 1$ for $z \in (-\infty, z_0]$, $U(z) = 0$ for $z \in [z_1, +\infty)$;*
(ii) *U is strictly decreasing in (z_0, z_1), $U(0) = s_*$;*
(iii) *for $i = 0, 1, 2, \dots, n, n+1$ let $\xi_i \in [z_0, z_1]$ be such that $U(\xi_i) = s_i$, then U is a piecewise C^1- function in the sense that U is continuous,*
$$U\big|_{(\xi_i,\xi_{i+1})} \in C^1(\xi_i, \xi_{i+1}), \quad i = 0, 1, \dots, n,$$
and the limits $U'(\xi_i-) := \lim_{z\to\xi_i-} U'(z)$, $U'(\xi_i+) := \lim_{z\to\xi_i+} U'(z)$ exist are finite for all $i = 1, 2, \dots, n$;
(iv) *for any $i = 1, 2, \dots, n$ the following transition condition holds:*
$$|U'(\xi_i-)|^{p-2}\,U'(\xi_i-) \lim_{s\to s_i+} d(s) = |U'(\xi_i+)|^{p-2}\,U'(\xi_i+) \lim_{s\to s_i-} d(s).$$

Proof. The existence and uniqueness of c_* and U follow directly from Theorem 4.1 and Proposition 3.1. The properties of U are derived in the reasoning preceding the statement of Proposition 3.1. □

Remark 4.13. The inequality ">" in (1.3) was motivated by modeling heterzygote inferior case. The assumption

$$\int_0^1 (d(s))^{\frac{1}{p-1}}\,g(s)\,\mathrm{d}s < 0 \tag{4.13}$$

leads to negative traveling speed of propagation $c_* < 0$ and it can be treated in a similar way. However, in this case the main tool is a shooting argument applied to the forward initial value problem and the strong comparison argument must be derived at the terminal value 1.

We can also prove similar results for increasing traveling wave profile U satisfying

$$\lim_{z\to-\infty} U(z) = 0 \quad \text{and} \quad \lim_{z\to+\infty} U(z) = 1.$$

In this case the assumption (1.3) leads to $c_* < 0$ while (4.13) leads to $c_* > 0$, respectively.

Remark 4.14. Notice that condition $U(0) = s_*$ has just a normalizing character. Indeed, since the equation (2.1) is autonomous then given any $\xi \in \mathbb{R}$ the translation $V(z) = U(z-\xi)$, $z \in \mathbb{R}$, is also a solution of (2.1) which satisfies $V(\xi) = s_*$.

5. Asymptotic analysis of the traveling wave profile

In this section we focus on the asymptotic behavior of the traveling wave profile $U = U(z)$ as $z \to \pm\infty$. Similar asymptotic analysis for standing waves ($c = 0$) was done in [7]. Even though the main idea is the same also in the case $c \neq 0$, the analysis is much more involved and not so precise because the solution of equation (3.7) for $c \neq 0$ cannot be obtained in a closed form by simple integration of $f = f(t)$ as in the stationary case ($c = 0$), cf. [7].

For the sake of brevity, for $t_0 \in \mathbb{R}$ we write

$$h_1(t) \sim h_2(t) \text{ as } t \to t_0 \quad \text{if and only if} \quad \lim_{t\to t_0} \frac{h_1(t)}{h_2(t)} \in (0, +\infty).$$

5.1. **Asymptotics near 1.** Let us assume that $g(t) \sim (1-t)^\gamma$ and $d(t) \sim (1-t)^\delta$ as $t \to 1-$ for some $\gamma > 0$ and $\delta \in \mathbb{R}$. Then, formally,

$$f(t) = (d(t))^{\frac{1}{p-1}}\, g(t) \sim (1-t)^{\gamma+\frac{\delta}{p-1}} \quad \text{as } t \to 1- \ .$$

The fact that $f \in L^1(0,1)$ then implies the following necessary condition for parameters γ and δ:

$$\gamma + \frac{\delta}{p-1} > -1. \tag{5.1}$$

It follows from (3.9) that the inverse function to a profile $U = U(z)$ corresponding to the speed $c > 0$ and normalized by $U(0) = s_*$ is given by

$$z(U) = -\int_{s_*}^{U} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\, \mathrm{d}t, \quad U \in (0,1), \tag{5.2}$$

where $y_{c_*} = y_{c_*}(t)$ is the unique positive solution of (3.7), (3.8). In order to find the asymptotic behavior of $z = z(U)$ as $U \to 1-$ from (5.2) we need to establish the asymptotics of $y_{c_*} = y_{c_*}(t)$ as $t \to 1-$. The asymptotics of $U = U(z)$ as $z \to -\infty$ then would follow applying the inverse point of view to $z = z(U)$ for $U \to 1-$.

From our assumptions it follows that there exists $\theta > 0$ (small enough) such that both d and g are continuous in $(1-\theta, 1)$. Therefore, $f = f(t)$ is also continuous in $(1-\theta, 1)$ and hence $f(t) \sim (1-t)^{\gamma+\frac{\delta}{p-1}}$ is equivalent to

$$f(t) = \eta(t)(1-t)^{\gamma+\frac{\delta}{p-1}}, \quad t \in (1-\theta, 1),$$

where $\eta = \eta(t)$ is a continuous function in $(1-\theta, 1)$, $\lim_{t\to1-} \eta(t) \in (0, +\infty)$.

In what follows we discuss different cases with respect to parameters γ, δ and p.

A. Let $\gamma+\frac{\delta}{p-1}\leq\frac{1}{p-1}$. Then for $\kappa>0$ we set $y_\kappa(t)=\kappa(1-t)^{\gamma+\frac{\delta}{p-1}+1}$, $t\in(1-\theta,1)$ and calculate the defect

$$\begin{aligned}
P_{c_*}y_\kappa &= y_\kappa' - p'[c_*\,(y_\kappa)^{\frac{1}{p}} - f(t)]\\
&= -\kappa\Big(\gamma+\frac{\delta}{p-1}+1\Big)(1-t)^{\gamma+\frac{\delta}{p-1}}\\
&\quad - p'\Big[c_*\kappa^{\frac{1}{p}}(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}} - \eta(t)(1-t)^{\gamma+\frac{\delta}{p-1}}\Big]\\
&= (1-t)^{\gamma+\frac{\delta}{p-1}}\Big[-\kappa\Big(\gamma+\frac{\delta}{p-1}+1\Big)+p'\eta(t)\Big]\\
&\quad -(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}}p'c_*\kappa^{\frac{1}{p}},
\end{aligned}\tag{5.3}$$

$t\in(1-\theta,1)$. Our assumption $\gamma+\frac{\delta}{p-1}\leq\frac{1}{p-1}$ implies

$$\gamma+\frac{\delta}{p-1}\leq\frac{\gamma+\frac{\delta}{p-1}+1}{p},$$

and therefore the power $(1-t)^{\gamma+\frac{\delta}{p-1}}$ dominates the power $(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}}$ near 1. It then follows from (5.3) that we may distinguish between two cases:

- A1. There exists $\underline{\kappa}\ll 1$ so small that $P_{c_*}y_{\underline{\kappa}}>0=P_{c_*}y_{c_*}$ a.e. in $(1-\theta,1)$.
- A2. There exists $\bar\kappa\gg 1$ so large that $P_{c_*}y_{\bar\kappa}<0=P_{c_*}y_{c_*}$ a.e. in $(1-\theta,1)$.

Case A1. Let $\frac{\gamma-\delta+1}{p}<1$. It follows from Lemma 4.6 with $\varrho=1-\theta$ that

$$y_{c_*}(t)\geq y_{\underline{\kappa}}(t)\quad\text{in }(1-\theta,1).\tag{5.4}$$

From (5.2) and (5.4) we conclude that there exists $c_1>0$ such that

$$\begin{aligned}
z_0=\lim_{U\to1-}z(U) &= -\int_{s_*}^1\frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t\\
&\geq -\int_{s_*}^1\frac{(d(t))^{\frac{1}{p-1}}}{(y_{\underline{\kappa}}(t))^{\frac{1}{p}}}\,\mathrm{d}t\\
&\geq -c_1\int_{s_*}^1\frac{(1-t)^{\frac{\delta}{p-1}}}{(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}}}\,\mathrm{d}t\\
&= -c_1\int_{s_*}^1\frac{\mathrm{d}t}{(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}-\frac{\delta}{p-1}}}\\
&= -c_1\int_{s_*}^1\frac{\mathrm{d}t}{(1-t)^{\frac{\gamma-\delta+1}{p}}}>-\infty.
\end{aligned}$$

Case A2. Let $\frac{\gamma-\delta+1}{p}\geq 1$. It follows from Lemma 4.6 with $\varrho=1-\theta$ that

$$y_{c_*}(t)\leq y_{\bar\kappa}(t)\quad\text{in }(1-\theta,1).\tag{5.5}$$

From (5.2) and (5.5) we conclude that there exists $c_2>0$ such that

$$z_0=-\int_{s_*}^1\frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t\leq-\int_{s_*}^1\frac{(d(t))^{\frac{1}{p-1}}}{(y_{\bar\kappa}(t))^{\frac{1}{p}}}\,\mathrm{d}t$$

$$
\begin{aligned}
&\le -c_2 \int_{s_*}^{1} \frac{(1-t)^{\frac{\delta}{p-1}}}{(1-t)^{\frac{\gamma+\frac{\delta}{p-1}+1}{p}}}\,\mathrm{d}t \\
&= -c_2 \int_{s_*}^{1} \frac{\mathrm{d}t}{(1-t)^{\frac{\gamma-\delta+1}{p}}} = -\infty.
\end{aligned}
$$

Taking into account (5.1), we may summarize these two cases as follows.

Theorem 5.1. *Let us assume* $\gamma > 0$,

$$
-1 < \gamma + \frac{\delta}{p-1} \le \frac{1}{p-1}, \tag{5.6}
$$

$$
\frac{\gamma-\delta+1}{p} < 1, \tag{5.7}
$$

Then $z_0 > -\infty$*. Let us assume* $\gamma > 0$, (5.6)*, and*

$$
\frac{\gamma-\delta+1}{p} \ge 1. \tag{5.8}
$$

Then $z_0 = -\infty$*.*

B. Let $\gamma + \frac{\delta}{p-1} > \frac{1}{p-1}$. Then for $\kappa > 0$ we set $y_\kappa(t) = \kappa(1-t)^{p\left(\gamma+\frac{\delta}{p-1}\right)}$, $t \in (1-\theta, 1)$ and calculate

$$
\begin{aligned}
P_{c_*} y_\kappa &= y_\kappa' - p'[c_* (y_\kappa)^{\frac{1}{p}} - f(t)] \\
&= -\kappa p\Big(\gamma + \frac{\delta}{p-1}\Big)(1-t)^{p\left(\gamma+\frac{\delta}{p-1}\right)-1} \\
&\quad - p'\big[c_* \kappa^{\frac{1}{p}} (1-t)^{\gamma+\frac{\delta}{p-1}} - \eta(t)(1-t)^{\gamma+\frac{\delta}{p-1}}\big] \\
&= -\kappa p\Big(\gamma + \frac{\delta}{p-1}\Big)(1-t)^{p\left(\gamma+\frac{\delta}{p-1}\right)-1} \\
&\quad - p'\big[c_* \kappa^{\frac{1}{p}} - \eta(t)\big](1-t)^{\gamma+\frac{\delta}{p-1}},
\end{aligned} \tag{5.9}
$$

for $t \in (1-\theta, 1)$. Our assumption $\gamma(p-1) + \delta > 1$ implies

$$
\gamma + \frac{\delta}{p-1} < p\left(\gamma + \frac{\delta}{p-1}\right) - 1,
$$

and the power $(1-t)t^{\gamma+\frac{\delta}{p-1}}$ dominates the power $(1-t)^{p\left(\gamma+\frac{\delta}{p-1}\right)-1}$ near 1. It follows from (5.9) that we may distinguish between two cases:

B1. There exists $\underline{\kappa} \ll 1$ so small that $P_{c_*} y_{\underline{\kappa}} > 0 = P_{c_*} y_{c_*}$ a.e. in $(1-\theta, 1)$.

B2. There exists $\bar{\kappa} \gg 1$ so large that $P_{c_*} y_{\bar{\kappa}} < 0 = P_{c_*} y_{c_*}$ a.e. in $(1-\theta, 1)$.

Case B1. Let $\gamma < 1$. From Lemma 4.6 with $\varrho = 1-\theta$ we obtain

$$
y_{c_*}(t) \ge y_{\underline{\kappa}}(t) \quad \text{in } (1-\theta, 1)
$$

and therefore, similarly as in Case A1 we conclude that there exists $c_3 > 0$ such that

$$
z_0 \ge -c_3 \int_{s_*}^{1} \frac{(1-t)^{\frac{\delta}{p-1}}}{(1-t)^{\gamma+\frac{\delta}{p-1}}}\,\mathrm{d}t = -c_3 \int_{s_*}^{1} \frac{\mathrm{d}t}{(1-t)^{\gamma}} > -\infty.
$$

Case B2. Let $\gamma \ge 1$. From Lemma 4.6 with $\varrho = 1-\theta$ we obtain

$$
y_{c_*}(t) \le y_{\bar{\kappa}}(t) \quad \text{in } (1-\theta, 1)
$$

and as in Case A2 we conclude that there exists $c_4 > 0$ such that

$$z_0 \le -c_4 \int_{s_*}^1 \frac{(1-t)^{\frac{\delta}{p-1}}}{(1-t)^{\gamma+\frac{\delta}{p-1}}}\,\mathrm{d}t = -c_4 \int_{s_*}^1 \frac{\mathrm{d}t}{(1-t)^{\gamma}} = -\infty.$$

We summarize these two cases as follows.

Theorem 5.2. *Let us assume* $\gamma > 0$,

$$\gamma + \frac{\delta}{p-1} > \frac{1}{p-1}, \tag{5.10}$$

$$\gamma < 1. \tag{5.11}$$

Then $z_0 > -\infty$. *Let us assume* $\gamma > 0$, (5.10) *and*

$$\gamma \ge 1. \tag{5.12}$$

Then $z_0 = -\infty$.

Remark 5.3. To visualize conditions (5.6)–(5.8) and (5.10)–(5.12), we introduce the following sets:

$$\mathcal{M}_1^1 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, -1 < \gamma + \frac{\delta}{p-1} \le \frac{1}{p-1}, \gamma - \delta + 1 < p\},$$
$$\mathcal{M}_1^2 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, -1 < \gamma + \frac{\delta}{p-1} \le \frac{1}{p-1}, \gamma - \delta + 1 \ge p\},$$
$$\mathcal{M}_1^3 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, \gamma + \frac{\delta}{p-1} > \frac{1}{p-1}, \gamma < 1\},$$
$$\mathcal{M}_1^4 := \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, \gamma + \frac{\delta}{p-1} > \frac{1}{p-1}, \gamma \ge 1\}.$$

Then $z_0 > -\infty$ if and only if $(\gamma,\delta) \in \mathcal{M}_1^1 \cup \mathcal{M}_1^3$ and $z_0 = -\infty$ if and only if $(\gamma,\delta) \in \mathcal{M}_1^2 \cup \mathcal{M}_1^4$. See Figure 1 for geometric interpretation. Our results generalize those from [5, Section 6].

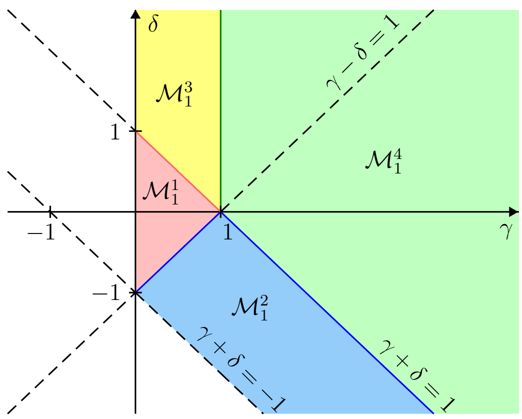

FIGURE 1. Visualization of the sets $\mathcal{M}_1^1$, $\mathcal{M}_1^2$, $\mathcal{M}_1^3$ and $\mathcal{M}_1^4$ for $p = 2$

5.2. **Asymptotics near 0.** Let us assume that $g(t) \sim -t^{\alpha}$ and $d(t) \sim t^{\beta}$ as $t \to 0+$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Then, formally, $f(t) \sim -t^{\alpha+\frac{\beta}{p-1}}$ as $t \to 0+$. The assumption $f \in L^1(0,1)$ yields necessary condition for parameters α and β:

$$\alpha + \frac{\beta}{p-1} > -1. \tag{5.13}$$

The main idea to find the asymptotics of $U = U(z)$ as $z \to +\infty$ is now based on the investigation of the asymptotics of its inverse $z = z(U)$ as $U \to 0+$. For this purpose we employ the formula (5.2) and, in particular, its limit for $U \to 0+$:

$$z_1 = \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \,\mathrm{d}t. \tag{5.14}$$

The main difference between this and previous case (asymptotics near 1) consists in the fact that we cannot use the comparison argument based on Lemma 4.6 due to the lack of uniqueness for the forward initial value problem (4.2). However, special form of our equation allows for the uniqueness result for this problem if we restrict on the set of positive solutions in a neighbourhood of 0. We will explain this idea below.

Lemma 5.4. *Let f be as in Theorem 4.1. Then the forward initial value problem* (4.2) *with $c > 0$ has a unique positive solution in* $(0, s_*)$.

Proof. Let $y = y(t)$, $t \in (0, s_*)$, be a solution of the forward initial value problem (4.2) with $c > 0$, cf. Lemma 4.2. Then

$$y'(t) = p'\big[c\,(y^+(t))^{\frac{1}{p}} - f(t)\big] \ge 0, \quad t \in (0, s_*)$$

and therefore

$$y(t) = y(0) + \int_0^t y'(\sigma)\,\mathrm{d}\sigma \ge 0, \quad t \in (0, s_*).$$

Assume that there are two positive solutions $y_1 = y_1(t)$, $y_2 = y_2(t)$, $t \in (0, s_*)$ of (4.2). Then $z_1 = (y_1)^{1/p'} > 0$, $z_2 = (y_2)^{1/p'} > 0$ solve the forward initial value problem

$$z_i'(t) = c - \frac{f(t)}{(z_i(t))^{\frac{1}{p-1}}} \quad \text{for a.e. } t \in (0, s_*),$$
$$z_i(0) = 0$$

for $i = 1, 2$. It then follows that

$$(z_1(t) - z_2(t))' = -f(t)\Big(\frac{1}{(z_1(t))^{\frac{1}{p-1}}} - \frac{1}{(z_2(t))^{\frac{1}{p-1}}}\Big),$$
$$(z_1(t) - z_2(t))^+ (z_1(t) - z_2(t))' = -f(t)\Big(\frac{1}{(z_1(t))^{\frac{1}{p-1}}} - \frac{1}{(z_2(t))^{\frac{1}{p-1}}}\Big)(z_1(t) - z_2(t))^+$$

for a.e. $t \in (0, s_*)$. Since $f(t) \le 0$, $t \in (0, s_*)$, it follows from here that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[\,(z_1(t) - z_2(t))^+\,\big]^2 \le 0, \quad \text{a.e. in } (0, s_*). \tag{5.15}$$

But $z_1(0) = z_2(0) = 0$ and (5.15) imply $z_1(t) \le z_2(t)$. Similarly, we prove that $z_2(t) \le z_1(t)$. Therefore, $y_1(t) = y_2(t)$ for $t \in (0, s_*)$. □

Remark 5.5. It follows from Lemma 5.4 that the restriction of the unique positive solution $y_{c_*} = y_{c_*}(t)$, $t \in [0,1]$, of the boundary value problem (3.7), (3.8) to the interval $(0, s_*)$ is also the unique solution of the forward initial value problem (4.2) with $c = c_*$ on $(0, s_*)$.

With the uniqueness result from Lemma 5.4 in hand, we can use the following comparison argument which is our tool for the asymptotic analysis near 0.

Lemma 5.6. *Let $f \in L^1(0,1)$ be as in Theorem 4.1, $0 < \theta < s_*$, $\varphi = \varphi(t)$, $\psi = \psi(t)$, $t \in [0,\theta]$ satisfy $\varphi(0) = \psi(0) = 0$, $\varphi'(t) \le h(t, \varphi(t), c_*)$, $\psi'(t) \ge h(t, \psi(t), c_*)$ for a.e. $t \in [0,\theta]$, and let $y_{c_*} = y_{c_*}(t)$, $t \in [0,1]$, be the unique solution of* (3.7), (3.8). *Then*

$$\varphi(t) \le y_{c_*}(t) \le \psi(t), \quad t \in [0,\theta].$$

Proof. The proof follows directly from [14, §10.XXII] combined with the uniqueness result in Lemma 5.4 and Remark 5.5. $\square$

The assumptions on d and g imply that for θ such that $0 < \theta < \min\{s_*, s_1\}$ the function $f = f(t)$ is continuous in $(0,\theta)$ and $f(t) \sim -t^{\alpha + \frac{\beta}{p-1}}$ is equivalent to

$$f(t) = -\eta(t)(1-t)^{\alpha + \frac{\beta}{p-1}}, \quad t \in (0,\theta),$$

where $\eta = \eta(t)$ is a continuous function in $(0,\theta)$, $\lim_{t\to 0+} \eta(t) \in (0, +\infty)$.

In what follows we discuss different cases with respect to parameters α, β and p.

A. Let $\alpha + \frac{\beta}{p-1} \le \frac{1}{p-1}$. For $\kappa > 0$ we set $y_\kappa(t) = \kappa t^{\alpha + \frac{\beta}{p-1} + 1}$, $t \in [0,\theta]$. Then

$$\begin{aligned}
&y'_\kappa - p'[c_* (y_\kappa)^{\frac{1}{p}} - f(t)] \\
&= \kappa\Big(\alpha + \frac{\beta}{p-1} + 1\Big) t^{\alpha + \frac{\beta}{p-1}} - p'\big[c_* \kappa^{\frac{1}{p}} t^{\frac{\alpha + \frac{\beta}{p-1} + 1}{p}} + \eta(t) t^{\alpha + \frac{\beta}{p-1}}\big] \\
&= t^{\alpha + \frac{\beta}{p-1}} \big[\kappa\Big(\alpha + \frac{\beta}{p-1} + 1\Big) - p'\eta(t)\big] - t^{\frac{\alpha + \frac{\beta}{p-1} + 1}{p}} p' c_* \kappa^{\frac{1}{p}},
\end{aligned} \tag{5.16}$$

for a.e. $t \in [0,\theta]$. The assumption $\alpha + \frac{\beta}{p-1} \le \frac{1}{p-1}$ implies

$$\alpha + \frac{\beta}{p-1} \le \frac{\alpha + \frac{\beta}{p-1} + 1}{p}$$

and therefore the power $t^{\alpha + \frac{\beta}{p-1}}$ dominates the power $t^{\frac{\alpha + \frac{\beta}{p-1} + 1}{p}}$ near 0.

- A1. There exists $\underline{\kappa} \ll 1$ so small that $y'_{\underline{\kappa}}(t) \le p'[c_* (y_{\underline{\kappa}}(t)^{\frac{1}{p}}) - f(t)]$ for a.e. $t \in [0,\theta]$.
- A2. There exists $\bar\kappa \gg 1$ so large that $y'_{\bar\kappa}(t) \ge p'[c_* (y_{\bar\kappa}(t)^{\frac{1}{p}}) - f(t)]$ for a.e. $t \in [0,\theta]$.

It follows from Lemma 5.6 that solution $y_{c_*} = y_{c_*}(t)$ of the BVP (3.7), (3.8) must satisfy

$$y_{\underline{\kappa}}(t) \le y_{c_*}(t) \le y_{\bar\kappa}(t), \quad t \in [0,\theta].$$

Case A1. Let $\frac{\alpha - \beta + 1}{p} < 1$. Then there exists $c_1 > 0$ such that

$$z_1 = \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}} \,\mathrm{d}t \le \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\underline{\kappa}}(t))^{\frac{1}{p}}} \,\mathrm{d}t$$

$$\le c_1 \int_0^{s_*} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{\alpha+\frac{\beta}{p-1}+1}{p}}}\,\mathrm{d}t = c_1 \int_0^{s_*} \frac{\mathrm{d}t}{t^{\frac{\alpha-\beta+1}{p}}} < +\infty.$$

Case A2. Let $\frac{\alpha-\beta+1}{p} \ge 1$. Then there exists $c_2 > 0$ such that

$$\begin{aligned} z_1 &= \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t \ge \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\bar\kappa}(t))^{\frac{1}{p}}}\,\mathrm{d}t \\ &\ge c_2 \int_0^{s_*} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{\alpha+\frac{\beta}{p-1}+1}{p}}}\,\mathrm{d}t = c_2 \int_0^{s_*} \frac{\mathrm{d}t}{t^{\frac{\alpha-\beta+1}{p}}} = +\infty. \end{aligned}$$

We can summarize these two cases as follows.

Theorem 5.7. *Let us assume $\alpha > 0$,*

$$-1 < \alpha + \frac{\beta}{p-1} \le \frac{1}{p-1}, \tag{5.17}$$

$$\frac{\alpha-\beta+1}{p} < 1. \tag{5.18}$$

Then $z_1 < +\infty$. Let us assume $\alpha > 0$, (5.17) and

$$\frac{\alpha-\beta+1}{p} \ge 1. \tag{5.19}$$

Then $z_1 = +\infty$.

B. Let $\alpha + \frac{\beta}{p-1} > \frac{1}{p-1}$. For $\kappa > 0$ we set $y_\kappa(t) = \kappa t^{p'}$, $t \in [0,\theta]$. Then

$$\begin{aligned} y_\kappa' - p'[c_* (y_\kappa)^{\frac{1}{p}} - f(t)] &= \kappa p' t^{p'-1} - p'[c_* \kappa^{\frac{1}{p}} t^{\frac{p'}{p}} + \eta(t) t^{\alpha+\frac{\beta}{p-1}}] \\ &= \big(\kappa p' - p' c_* \kappa^{\frac{1}{p}}\big) t^{\frac{1}{p-1}} - p'\eta(t) t^{\alpha+\frac{\beta}{p-1}}, \end{aligned} \tag{5.20}$$

for a.e. $t \in [0,\theta]$. The assumption $\alpha + \frac{\beta}{p-1} > \frac{1}{p-1}$ implies that the power $t^{\frac{1}{p-1}}$ dominates $t^{\alpha+\frac{\beta}{p-1}}$ near 0.

- B1. There exists $\underline{\kappa} \ll 1$ so small that $y_{\underline{\kappa}}'(t) \le p'[c_*(y_{\underline{\kappa}}(t)^{\frac{1}{p}}) - f(t)]$ for a.e. $t \in [0,\theta]$.
- B2. There exists $\bar\kappa \gg 1$ so large that $y_{\bar\kappa}'(t) \ge p'[c_*(y_{\bar\kappa}(t)^{\frac{1}{p}}) - f(t)]$ for a.e. $t \in [0,\theta]$.

From Lemma 5.6 we conclude that solution $y_{c_*} = y_{c_*}(t)$ of the BVP (3.7), (3.8) must satisfy

$$y_{\underline{\kappa}}(t) \le y_{c_*}(t) \le y_{\bar\kappa}(t), \quad t \in [0,\theta].$$

Case B1. Let $\beta > p-2$. Then there exists $c_3 > 0$ such that

$$z_1 = \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t \le \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\underline{\kappa}}(t))^{\frac{1}{p}}}\,\mathrm{d}t \le c_3 \int_0^{s_*} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{p'}{p}}}\,\mathrm{d}t = c_3 \int_0^{s_*} \frac{\mathrm{d}t}{t^{\frac{1-\beta}{p-1}}} < +\infty.$$

Case B2. Let $\beta \le p-2$. Then there exists $c_4 > 0$ such that

$$z_1 = \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{c_*}(t))^{\frac{1}{p}}}\,\mathrm{d}t \ge \int_0^{s_*} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\bar\kappa}(t))^{\frac{1}{p}}}\,\mathrm{d}t \ge c_4 \int_0^{s_*} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{p'}{p}}}\,\mathrm{d}t = c_4 \int_0^{s_*} \frac{\mathrm{d}t}{t^{\frac{1-\beta}{p-1}}} = +\infty.$$

We summarize these two cases as follows.

Theorem 5.8. *Let us assume* $\alpha > 0$,

$$\alpha + \frac{\beta}{p-1} > \frac{1}{p-1}, \tag{5.21}$$

$$\beta > 2 - p. \tag{5.22}$$

Then $z_1 < +\infty$. *Let us assume* $\alpha > 0$, (5.21) *and*

$$\beta \leq 2 - p. \tag{5.23}$$

Then $z_1 = +\infty$.

Remark 5.9. To visualize conditions (5.17)–(5.19) and (5.21)–(5.23), we introduce the sets:

$$\mathcal{M}_0^1 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, -1 < \alpha + \frac{\beta}{p-1} \leq \frac{1}{p-1}, \alpha - \beta + 1 < p\},$$

$$\mathcal{M}_0^2 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, -1 < \alpha + \frac{\beta}{p-1} \leq \frac{1}{p-1}, \alpha - \beta + 1 \geq p\},$$

$$\mathcal{M}_0^3 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, \alpha + \frac{\beta}{p-1} > \frac{1}{p-1}, \beta > 2 - p\},$$

$$\mathcal{M}_0^4 := \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > 0, \alpha + \frac{\beta}{p-1} > \frac{1}{p-1}, \beta \leq 2 - p\}.$$

Then $z_1 < +\infty$ if and only if $(\alpha,\beta) \in \mathcal{M}_0^1 \cup \mathcal{M}_0^3$ and $z_1 = +\infty$ if and only if $(\alpha,\beta) \in \mathcal{M}_0^2 \cup \mathcal{M}_0^4$. The reader is invited to see Figure 2 for geometric interpretation and compare the sets $\mathcal{M}_0^1$, $\mathcal{M}_0^2$, $\mathcal{M}_0^3$, $\mathcal{M}_0^4$ and $\mathcal{M}_1^1$, $\mathcal{M}_1^2$, $\mathcal{M}_1^3$, $\mathcal{M}_1^4$.

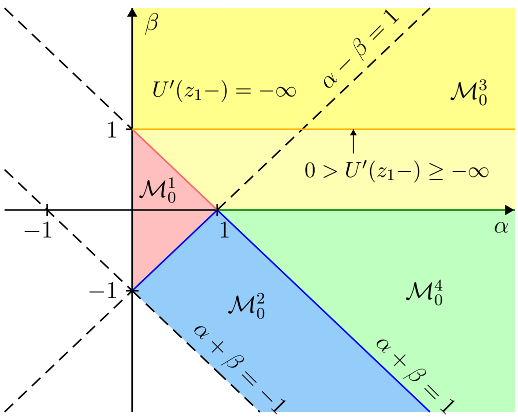

FIGURE 2. Visualization of the sets $\mathcal{M}_0^1$, $\mathcal{M}_0^2$, $\mathcal{M}_0^3$ and $\mathcal{M}_0^4$ for $p = 2$

Remark 5.10. Let us assume $z_0 > -\infty$, i.e., $(\gamma,\delta) \in \mathcal{M}_1^1 \cup \mathcal{M}_1^3$. Then $U'(z_0-) = 0$ and it follows from Definition 2.1 that $U'(z_0+)$ exists finite or infinite, see Remark 2.4. Since U is a monotone decreasing function, we have $-\infty \leq U'(z_0+) \leq 0$. If $z_1 < +\infty$, i.e., $(\alpha,\beta) \in \mathcal{M}_0^1 \cup \mathcal{M}_0^3$ then by similar reasons $U'(z_1+) = 0$ and $-\infty \leq U'(z_1-) \leq 0$.

In the case $(\alpha,\beta)\in\mathcal{M}_0^3$ our one-sided estimates on z_1 allow for more precise information about the smoothness of U at z_1. Indeed, in this case we have

$$\begin{aligned} 0 &\geq z'(0+) = \lim_{U\to 0+} z'(U) \\ &= -\lim_{U\to 0+}\frac{(d(U))^{\frac{1}{p-1}}}{(y_{c_*}(U))^{\frac{1}{p}}} \geq -\lim_{U\to 0+}\frac{(d(U))^{\frac{1}{p-1}}}{\big(y_{\underline{K}}(U)\big)^{\frac{1}{p}}} \\ &\geq -c_3\lim_{U\to 0+} U^{\frac{\beta-1}{p-1}}. \end{aligned} \tag{5.24}$$

We distinguish the following two cases:

1. If $\beta>1$ then from (5.24) we obtain $z'(0+)=0$ and therefore $U'(z_1-)=-\infty$.
2. If $\beta=1$ then we deduce from (5.24) that $0\geq z'(0+)\geq -c_3$ and therefore $0>U'(z_1-)\geq -\infty$.

In either case the traveling wave profile U is "sharp" in the sense that U' has a jump at z_1 (finite or infinite).

In other cases $(\alpha,\beta)\in\mathcal{M}_0^1$ and $(\gamma,\delta)\in\mathcal{M}_1^1\cup\mathcal{M}_1^3$ our one-sided estimates on z_1 and z_0, respectively, do not provide analogous information as above. This is a big difference between the traveling wave and standing wave, see [7].

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Pavel Drábek
Department of Mathematics, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 8, 30100 Plzeň, Czech Republic
Email address: pdrabek@kma.zcu.cz

Michaela Zahradníková
Department of Mathematics, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 8, 30100 Plzeň, Czech Republic
Email address: mzahrad@kma.zcu.cz