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Quasigraphs and skeletal partitions

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ABSTRACT

We give a new proof of the Skeletal Lemma, which is the main technical tool in our paper on Hamilton cycles in line graphs (Kaiser and Vrána, 2012). It generalises results on disjoint spanning trees in graphs to the context of 3-hypergraphs. The lemma is proved in a slightly stronger version that is more suitable for applications. The proof is simplified and formulated in a more accessible way.

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1. Introduction

The main tool used in our work on Hamilton cycles in line graphs [1] is a result called ‘Skeletal Lemma’ [1, Lemma 17]. It deals with quasigraphs in 3-hypergraphs (see below for definitions) and is related to Tutte’s and Nash-Williams’ characterisation of graphs with two disjoint spanning trees.

In our recent paper [2], we need to use the lemma in a slightly stronger form that unfortunately does not follow from the formulation given in the paper. Instead of pointing out the necessary modifications to the long and complicated proof, we decided to use this opportunity to rewrite the proof completely, trying to formulate it in a way as conceptually simple as we can. That is the purpose of the present paper which is a companion paper to [2]. In addition, the present paper aims to give the full proof in detail, even in parts where the argument in [1] is somewhat sketchy.

The structure of the paper is as follows. In Section 2, we review the basic notions related to quasigraphs, the structures forming a central concept of our proof. In Section 3, we develop the basic properties of the notion of connectivity and especially anticonnectivity of a quasigraph on a set of vertices. This allows us to define, for any quasigraph, a sequence of successively more and more refined partitions of the vertex set that serves as a measure of ‘quality’ of the quasigraph. This is done in Section 4. Section 5 gives the proof of the main result, a stronger version of the Skeletal Lemma (Theorem 6). Finally, in Section 6, we infer the result we need for the above mentioned application in [2] (Theorem 16).

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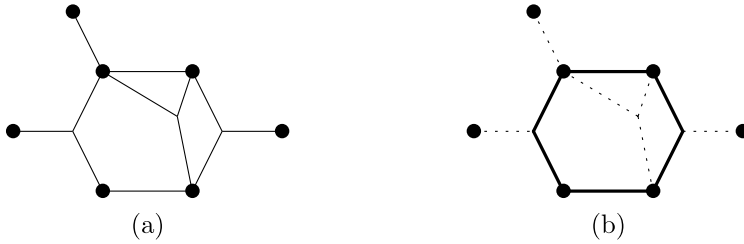


Fig. 1. (a) A 3-hypergraph H . Hyperedges of size 3 are depicted as three lines meeting at a point without a vertex mark. (b) A quasigraph π in H . For each hyperedge e used by π , the pair $\pi(e)$ is shown using one or two bold lines, depending on the size of e . The other lines are shown dotted for greater contrast.

2. Quasigraphs

A 3-hypergraph is a hypergraph whose hyperedges have size 2 or 3. Throughout this paper, let H be a 3-hypergraph. A quasigraph in H is a mapping π that assigns to each hyperedge e of H either a subset of e of size 2, or the empty set. The hyperedges e with $\pi(e) \neq \emptyset$ are said to be used by π . (See Fig. 1 for an illustration.) A quasigraph π in H restricts to a quasigraph in any subhypergraph H' of H ; to avoid excessive notation, we will use π to denote the corresponding quasigraph in H' as well.

Given a quasigraph π in H , we let π^* denote the graph on $V(H)$, obtained by considering the pairs $\pi(e')$ as edges whenever $\pi(e') \neq \emptyset$ ($e' \in E(H)$). If π^* is a forest, then π is acyclic. If π^* is the union of a cycle and a set of isolated vertices, then π is a quasicycle. A 3-hypergraph H is acyclic if there exists no quasicycle in H .

If e is a hyperedge of H , then we define $\pi - e$ as the quasigraph which satisfies $(\pi - e)(e) = \emptyset$, and coincides with π on all hyperedges other than e . If e is a hyperedge not used by π , and if $u, v \in e$, then $\pi + (uv)_e$ is the quasigraph that coincides with π except that its value on e is uv rather than \emptyset .

The complement $\bar{\pi}$ of π is the subhypergraph of H (on the same vertex set) consisting of the hyperedges not used by π .

Let \mathcal{P} be a partition of a set $X \subseteq V(H)$. We say that \mathcal{P} is nontrivial if $\mathcal{P} \neq \{X\}$. For $Y \subseteq X$, the partition $\mathcal{P}[Y]$ of Y induced by \mathcal{P} has all nonempty intersections $P \cap Y$, where $P \in \mathcal{P}$, as its classes. If a hyperedge e of H intersects at least two classes of \mathcal{P} , then it is said to be \mathcal{P} -crossing.

Assume now that $X = V(H)$, i.e., \mathcal{P} is a partition of $V(H)$. If $e \in E(H)$, then e/\mathcal{P} is defined as the set of all classes of \mathcal{P} intersected by e . The hypergraph H/\mathcal{P} has vertex set \mathcal{P} and its hyperedges are all the sets of the form e/\mathcal{P} , where e is a \mathcal{P} -crossing hyperedge of H . Thus, H/\mathcal{P} is a 3-hypergraph. A quasigraph π/\mathcal{P} in this hypergraph is defined by setting, for every \mathcal{P} -crossing hyperedge e of H ,

$$(\pi/\mathcal{P})(e/\mathcal{P}) = \begin{cases} \pi(e)/\mathcal{P} & \text{if } \pi(e) \text{ is } \mathcal{P}\text{-crossing,} \\ \emptyset & \text{otherwise.} \end{cases}$$

We extend the above notation and write, e.g., uv/\mathcal{P} for the set of classes of \mathcal{P} intersecting $\{u, v\}$, where $u, v \in V(H)$.

By the above definitions, the complement of π/\mathcal{P} is the subhypergraph of H/\mathcal{P} consisting of all the hyperedges e/\mathcal{P} such that $\pi(e)$ is contained in some class of \mathcal{P} (including $\pi(e) = \emptyset$). We often consider quasigraphs γ in $\bar{\pi}/\mathcal{P}$ (typically, such a γ is a quasicycle). An example is given in Fig. 2.

3. (Anti)connectivity

In this section, we define and explore the notions of components and anticomponents of a quasigraph (on a set of vertices) that are completely essential for our arguments. We refer to Fig. 3 for an illustration of these notions.

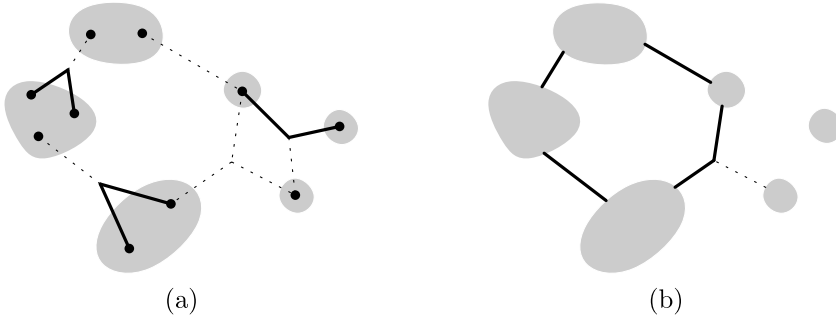


Fig. 2. (a) A quasigraph π in a 3-hypergraph H and a partition \mathcal{P} of $V(H)$. The classes of the partition are shown in grey. (b) A quasicycle γ in the complement of π/\mathcal{P} in the hypergraph H/\mathcal{P} . Note that the vertex set of this hypergraph is \mathcal{P} .

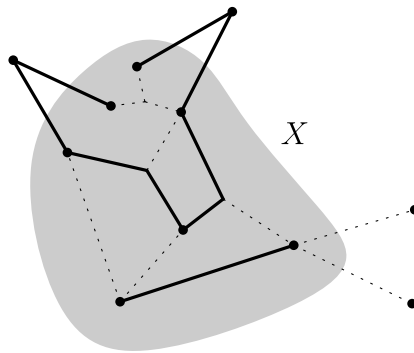


Fig. 3. A quasigraph π in H and a set $X \subseteq V(H)$ (shown grey). The quasigraph π is anticonnected on X and has four components on X .

Recall that H denotes a 3-hypergraph. Let π be a quasigraph in H and $X \subseteq V(H)$. We say that π is *connected on X* if the induced subgraph of π^* on X is connected. The *components* of π on X are defined as the vertex sets of the connected components of the induced subgraph of π^* on X .

We say that π is *anticonnected on X* if for each nontrivial partition \mathcal{R} of X , there is an \mathcal{R} -crossing hyperedge f of H such that $\pi(f)$ is a subset of one of the classes of \mathcal{R} (possibly $\pi(f) = \emptyset$). If we need to refer to the hypergraph H , we say that π is *anticonnected on X in H* .

Observe that π is both connected and anticonnected on any set consisting of a single vertex.

Lemma 1. *Let π be a quasigraph in (a 3-hypergraph) H and X, Y subsets of $V(H)$ such that π is anticonnected on X and Y . Then π is anticonnected on $X \cup Y$ whenever one of the following holds:*

- (i) X and Y intersect, or
- (ii) *there is a hyperedge h of H intersecting both X and Y , such that $\pi(h)$ is a subset of X or Y (possibly $\pi(h) = \emptyset$).*

Proof. Let \mathcal{R} be a nontrivial partition of $X \cup Y$. We find for \mathcal{R} the hyperedge whose existence is required by the definition of anticonnectedness.

Suppose first that $\mathcal{R}[X]$ is nontrivial. Since π is anticonnected on X , there is a hyperedge f of H such that f intersects at least two classes of $\mathcal{R}[X]$ and one of them contains $\pi(f)$. Thus, f intersects at least two classes of \mathcal{R} and one of them contains $\pi(f)$.

We can thus assume, by symmetry, that both $\mathcal{R}[X]$ and $\mathcal{R}[Y]$ are trivial. This implies that $\mathcal{R} = \{X, Y\}$, so X and Y are disjoint. In this case, the hyperedge h from (ii) has the required property. \square

By Lemma 1, the maximal sets $Y \subseteq X$ such that π is anticonnected on Y partition X . We call them the *anticonponents* of π on X .

Lemma 2. *Let π and ρ be quasigraphs in H and Y be a subset of $V(H)$ such that $\pi(e) = \rho(e)$ for every hyperedge e of H with $|e \cap Y| \geq 2$. Then π is anticonnected on Y if and only if ρ is anticonnected on Y .*

Proof. Suppose that π is anticonnected on Y and let \mathcal{R} be a nontrivial partition of Y . Consider a hyperedge f of H such that f intersects two classes of \mathcal{R} and one of them contains $\pi(f)$. By the assumption, $\rho(f) = \pi(f)$, so the same holds for ρ in place of π . Since \mathcal{R} is arbitrary, ρ is anticonnected on Y . The lemma follows by symmetry. \square

We prove several further lemmas that describe some of the basic properties of (anti)connectivity of quasigraphs.

Lemma 3. *Let π be a quasigraph in H , $X \subseteq V(H)$ and e a hyperedge of H with $|e \cap X| \leq 1$. If π is anticonnected on X in H , then π is anticonnected on X in $H - e$.*

Proof. Let \mathcal{R} be a nontrivial partition of X . Since π is anticonnected on X in H , there is a hyperedge f of H intersecting at least two classes of \mathcal{R} , one of which contains $\pi(f)$. The hyperedge f is distinct from e as $|e \cap X| \leq 1$. Thus, $f \in E(H - e)$. Since \mathcal{R} is arbitrary, π is anticonnected on X in $H - e$. \square

Lemma 4. *Let π be a quasigraph in H and $Y \subseteq X$ subsets of $V(H)$. Suppose that e is a hyperedge of H not used by π and containing vertices $u, v \in Y$. Define ρ as the quasigraph $\pi + (uv)_e$. The following holds:*

- (i) if π is anticonnected on X and ρ is anticonnected on Y , then ρ is anticonnected on X ,
- (ii) if π is connected on X , then so is ρ .

Proof. We prove (i). Consider an arbitrary partition \mathcal{R} of X . We aim to show that there is a hyperedge f of H such that f intersects two classes of \mathcal{R} and $\rho(f)$ is contained in one of them. This is certainly true if $\mathcal{R}[Y]$ is nontrivial, since ρ is assumed to be anticonnected on Y . Thus, we may assume that Y is contained in a class of \mathcal{R} .

Since π is anticonnected on X , there is a hyperedge h of H such that h intersects two classes of \mathcal{R} and $\pi(h)$ is contained in one of them. We set $f := h$. If $h \neq e$, this choice works because $\rho(h) = \pi(h)$. If $h = e$, then $\rho(h)$ is contained in Y and therefore in a class of \mathcal{R} . This concludes the proof of (i).

Part (ii) follows directly from the fact that π^* is a subgraph of ρ^* , and therefore the induced subgraph of π^* on X is a subgraph of the induced subgraph of ρ^* on X . \square

Lemma 5. *Let π be a quasigraph in H and $Y \subseteq X$ subsets of $V(H)$. Suppose that e is a hyperedge of H with $\pi(e) \subseteq Y$. Define σ as the quasigraph $\pi - e$. It holds that*

- (i) if π is anticonnected on X , then so is σ ,
- (ii) if π is connected on X and σ is connected on Y , then σ is connected on X .

Proof. We prove (i). Suppose, for contradiction, that σ is not anticonnected on X . By the definition, there is a partition \mathcal{S} of X such that for all hyperedges f of H intersecting at least two classes of \mathcal{S} , $\sigma(f)$ intersects two classes of \mathcal{S} as well. On the other hand, since π is anticonnected on X and π has the same value as σ on every hyperedge other than e , it must be that the hyperedge e intersects two classes of \mathcal{S} (and $\pi(e)$ is contained in one class). Since $\sigma(e) = \emptyset$, we obtain a contradiction.

Next, we prove (ii). Note that σ^* is a subgraph of π^* . Since σ is connected on Y , so is π . We show that σ is connected on X . Let π_X^* be the induced subgraph of π^* on X , and let $\pi(e) = \{u, v\}$.

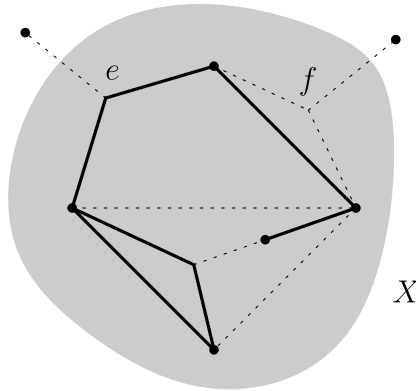


Fig. 4. A quasigraph π in a 3-hypergraph H and a set $X \subseteq V(H)$ (shown grey) such that π is both connected and anticonnected on X . The hyperedge e is an X -bridge with respect to π , while f is an X -antibrIDGE with respect to π .

We need to prove that any two vertices in X are joined by a walk in the induced subgraph of σ^* on X , which equals $\pi_X^* - uv$. This is easy from the fact that π_X^* is connected, and that the edge uv may be replaced in any walk by a path from u to v in the induced subgraph of σ^* on Y (which is connected). \square

Let us now define two notions that will play a role when we introduce the sequence of a quasigraph in Section 4. (See Fig. 4 for an illustration.) Suppose that $X \subseteq V(H)$ such that the quasigraph π is both connected and anticonnected on X . Let e be a hyperedge with $|e \cap X| = 2$.

We say that e is an X -bridge (with respect to π) if e is used by π , $\pi(e) \subseteq X$, and $\pi - e$ is not connected on X in $H - e$. Similarly, e is an X -antibrIDGE (with respect to π) if e is not used by π and π is not anticonnected on X in $H - e$.

4. The plane sequence of a quasigraph

Let H be a 3-hypergraph. Throughout our arguments, we will work with partitions of $V(H)$. Given two such partitions \mathcal{P} and \mathcal{Q} , we say that \mathcal{P} refines \mathcal{Q} (and write $\mathcal{P} \leq \mathcal{Q}$) if each class of \mathcal{P} is a subset of some class of \mathcal{Q} . If \mathcal{P} refines \mathcal{Q} and $\mathcal{P} \neq \mathcal{Q}$, we write $\mathcal{P} < \mathcal{Q}$.

Let π be a quasigraph in H . In [1], we associate with π a sequence of partitions of $V(H)$. In the present paper, we proceed similarly, but for technical reasons, we need to extend the original definition to involve a two-dimensional analogue of a sequence. A plane sequence is a family $(\mathcal{P}_{i,j})_{i,j \geq 0}$ of partitions of $V(H)$.

It will be convenient to consider the lexicographic order \leq on pairs of nonnegative integers: $(i, j) \leq (i', j')$ if either $i < i'$, or $i = i'$ and $j \leq j'$. This is extended in the natural way to the set

$$\mathcal{T} = \{(i, j) : 0 \leq i < \infty, 0 \leq j \leq \infty\} \cup \{(\infty, \infty)\}.$$

For instance, $(1, \infty) < (2, 0) < (\infty, \infty)$. This is a well-ordering on the set \mathcal{T} , which allows us to perform transfinite induction over \mathcal{T} (cf. [3]).

The (plane) sequence of π , denoted by \mathbb{P}^π , consists of partitions $\mathcal{P}_{i,j}^\pi$ of $V(H)$, where $(i, j) \in \mathcal{T}$. We let $\mathcal{P}_{0,0}^\pi$ be the trivial partition $\{V(H)\}$. If $j \geq 1$ and $\mathcal{P}_{i,j-1}^\pi$ is defined, then we let

$$\mathcal{P}_{i,j}^\pi = \begin{cases} \{K : K \text{ is a component of } \pi \text{ on some } X \in \mathcal{P}_{i,j-1}^\pi\} & \text{if } j \text{ is odd,} \\ \{K : K \text{ is an anticomponent of } \pi \text{ on some } X \in \mathcal{P}_{i,j-1}^\pi\} & \text{if } j \text{ is even.} \end{cases}$$

See Fig. 5 for an example.

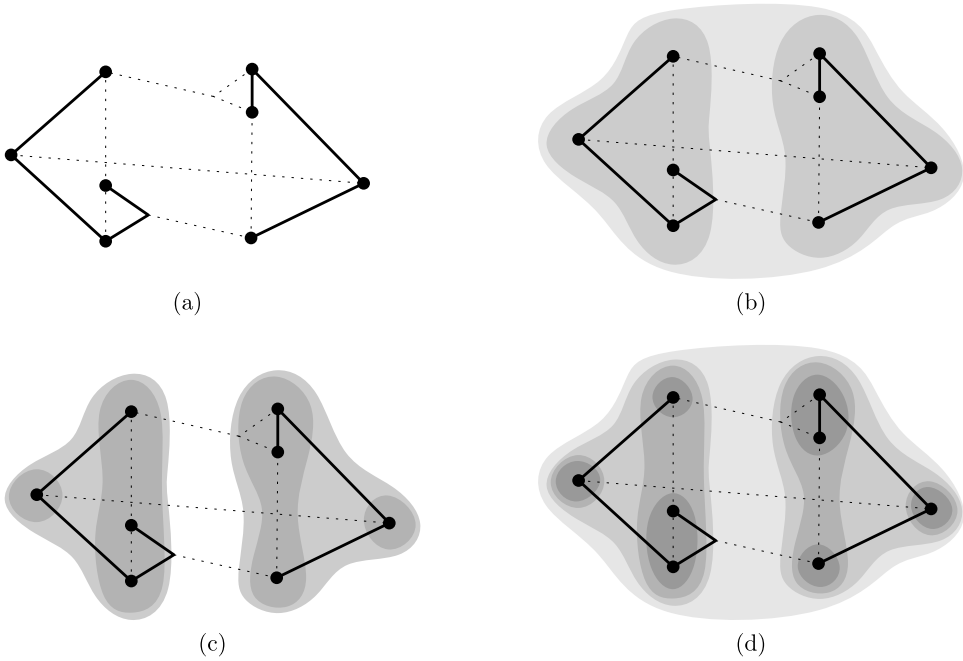


Fig. 5. (a) A quasigraph π (bold) in a 3-hypergraph H . (b) The single class of $\mathcal{P}_{0,0}^\pi$ (lighter grey) and the classes of $\mathcal{P}_{0,1}^\pi$ (darker grey). (c) The classes of $\mathcal{P}_{0,1}^\pi$ (lighter) and $\mathcal{P}_{0,2}^\pi$ (darker). (d) All the partitions $\mathcal{P}_{0,0}^\pi, \dots, \mathcal{P}_{0,3}^\pi$ (lightest to darkest grey). In this case, $\mathcal{P}_{0,3}^\pi = \mathcal{P}_{0,\infty}^\pi$.

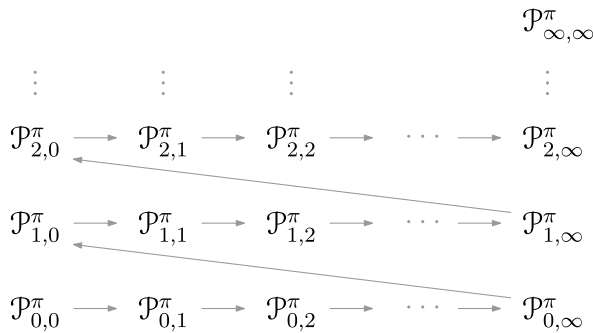


Fig. 6. The order of partitions in the construction of a plane sequence of a quasigraph.

So far, this yields the partitions $\mathcal{P}_{0,0}^\pi, \mathcal{P}_{0,1}^\pi, \dots$. We first notice that since H is finite, there is some j_0 such that

$$\mathcal{P}_{0,j_0}^\pi = \mathcal{P}_{0,j_0+1}^\pi = \dots,$$

and we set $\mathcal{P}_{0,\infty}^\pi$ equal to \mathcal{P}_{0,j_0}^π . We will use an analogous definition to construct $\mathcal{P}_{i,\infty}^\pi$ for $i > 0$ when $\mathcal{P}_{i,0}^\pi, \mathcal{P}_{i,1}^\pi, \dots$ will have been defined. (See Fig. 6 for a schematic illustration.)

By the construction, $\mathcal{P}_{0,\infty}^\pi$ has the property that π is both connected and anticonnected on each of its classes. We call any such partition of $V(H)$ π -solid.

The definition of the plane sequence of π will be completed once we define $\mathcal{P}_{i,0}^\pi$ for all $i \geq 1$. Thus, let $i \geq 1$ be fixed, and suppose that the partition $\mathcal{P} := \mathcal{P}_{i-1,\infty}^\pi$ is already defined.

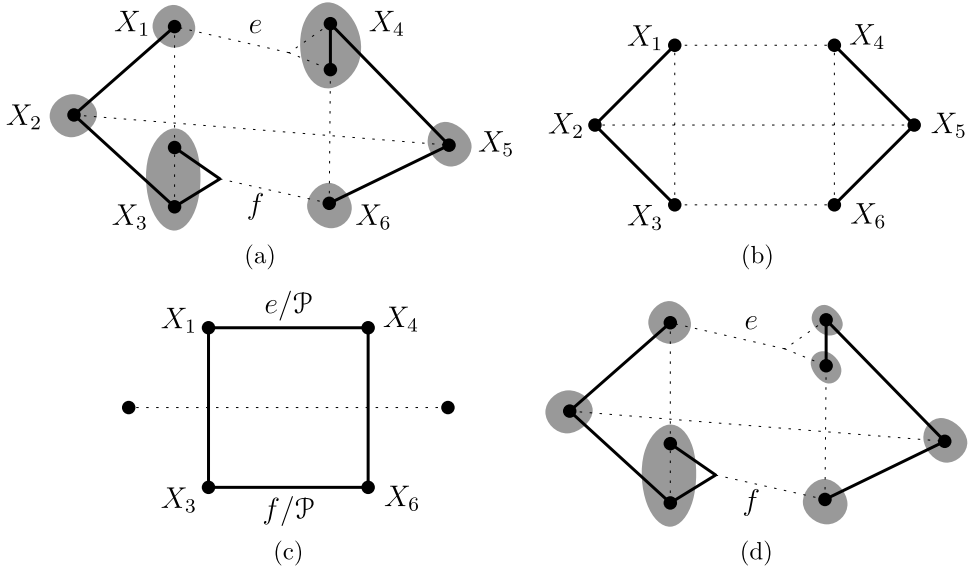


Fig. 7. Constructing the partition $\mathcal{P}_{1,0}^\pi$. (a) A labelling of the classes of $\mathcal{P} := \mathcal{P}_{0,\infty}^\pi$ and two hyperedges in the example from Fig. 5. (b) The quasigraph π/P (bold) in H/P . (c) A quasicycle γ (bold) in π/P . The leading hyperedges of γ are e and f . (d) The partition $\mathcal{P}_{1,0}^\pi$ (assuming that $f \leq_e e$). Note that e is a \mathcal{P} -antibridge.

Let $A, B \in \mathcal{P}$. The *exposure step* of the pair AB is the least (s, t) (with respect to the ordering defined above) such that A and B are contained (as subsets) in different classes of $\mathcal{P}_{s,t}^\pi$. Similarly, for a pair of vertices u, v of H contained in different classes of \mathcal{P} , the exposure step of the pair uv is the least (s, t) such that u and v are contained in different classes of $\mathcal{P}_{s,t}^\pi$.

Suppose that γ is a quasicycle in H/P . The *exposure step* of γ is the least exposure step of $\gamma(e/P)$, where e ranges over all hyperedges of H such that e/P is used by γ . If the exposure step of γ is (s, t) , we also say that γ is *exposed* at (step) (s, t) . We say that a hyperedge e of H is a *leading hyperedge* of γ if e/P is used by γ and the exposure step of $\gamma(e/P)$ equals that of γ .

These definitions apply in particular to a quasicycle γ in π/P since the latter is a subhypergraph of H/P . In addition, they also apply to any cycle in the graph $(\pi/P)^*$, by viewing it as a quasicycle in H/P . Later in this section, we will generalise these notions to the situation where the plane sequence has already been completely defined.

We extend the notions of X -bridge and X -antibridge defined in Section 3 as follows: given a π -solid partition \mathcal{R} of $V(H)$, e is an \mathcal{R} -bridge (\mathcal{R} -antibridge) if there is $X \in \mathcal{R}$ such that e is an X -bridge (X -antibridge, respectively).

We say that a hyperedge e of H crossing \mathcal{R} is *redundant* (with respect to π and \mathcal{R}) if e is not used by π and e is not an \mathcal{R} -antibridge. Note that a hyperedge e unused by π is redundant if $|e| = 2$, or more generally, if each of its vertices is in a different class of \mathcal{R} .

Furthermore, a hyperedge e of H is *weakly redundant* (with respect to π and \mathcal{R}) if either it is redundant, or it is used by π and is not an \mathcal{R} -bridge.

We are now ready to define the partition $\mathcal{P}_{i,0}^\pi$ (see Fig. 7 for an illustration). We will say that $\mathcal{P}_{i,0}^\pi$ is obtained from \mathcal{P} by the *limit step* $(i - 1, \infty)$. (Recall that \mathcal{P} denotes the partition $\mathcal{P}_{i-1,\infty}^\pi$.) At the same time, we will define the *decisive hyperedge* at $(i - 1, \infty)$, d_{i-1}^π , for the current limit step. This will be a hyperedge of H ; for technical reasons, we also allow two extra values, TERMINATE and STOP.

If the complement of π/P in H/P is acyclic, we define $\mathcal{P}_{i,0}^\pi = \mathcal{P}$ and say that π *terminates* at $(i - 1, \infty)$. We set $d_{i-1}^\pi = \text{TERMINATE}$.

Otherwise, let L be the set of hyperedges f of H for which there exists a quasicycle γ in $\overline{\pi/\mathcal{P}}$ such that f is a leading hyperedge of γ . We define $\mathcal{P}_{i,0}^\pi = \mathcal{P}$ if L contains a weakly redundant hyperedge f (with respect to π and \mathcal{P}). In this case, we say that π stops at $(i - 1, \infty)$; we set $d_{i-1}^\pi = \text{STOP}$.

If no weakly redundant hyperedge exists in L , choose the maximum hyperedge e in L according to a fixed linear ordering \leq_E of all hyperedges in H . This case is illustrated in Fig. 7. (For the purposes of this and the following section, the choice of \leq_E is not important; it will be discussed in more detail in Section 6.) Set d_{i-1}^π equal to e . We say that π continues at $(i - 1, \infty)$. Moreover, in this case, any quasicycle in $\overline{\pi/\mathcal{P}}$ whose leading hyperedge is e will be referred to as a decisive quasicycle at $(i - 1, \infty)$.

Since e is not weakly redundant, we can distinguish the following two cases for the definition of $\mathcal{P}_{i,0}^\pi$:

- if e is a \mathcal{P} -antibrIDGE, then the classes of $\mathcal{P}_{i,0}^\pi$ are all the anticomponents of π on X in $H - e$, where X ranges over all classes of \mathcal{P} ,
- if e is a \mathcal{P} -bridge, then the classes of $\mathcal{P}_{i,0}^\pi$ are all the components of $\pi - e$ on X in H , where X ranges over all classes of \mathcal{P} .

Note that in the first of these cases, e is not used by π , while in the second case, it is used by π .

The subsequent partitions $\mathcal{P}_{i,1}^\pi, \mathcal{P}_{i,2}^\pi, \dots$ are then defined as described above, and the partition $\mathcal{P}_{i,\infty}^\pi$ is defined analogously to $\mathcal{P}_{0,\infty}^\pi$. Iterating, we obtain the whole plane sequence $\tilde{\mathbb{P}}^\pi$ and the partitions $\mathcal{P}_{0,\infty}^\pi, \mathcal{P}_{1,\infty}^\pi, \dots$. By the finiteness of H , there is some i_0 such that $\mathcal{P}_{i_0,\infty}^\pi = \mathcal{P}_{i_0+1,\infty}^\pi$, and we define $\mathcal{P}_{\infty,\infty}^\pi$ as $\mathcal{P}_{i_0,\infty}^\pi$.

Observe that in the cases where π terminates or stops at $(i - 1, \infty)$ and we define $\mathcal{P}_{i,0}^\pi$ as $\mathcal{P}_{i-1,\infty}^\pi$, this partition will in fact equal $\mathcal{P}_{\infty,\infty}^\pi$ since none of the subsequent steps in the construction of the plane sequence of π will lead to any modifications.

Now that the sequence of a quasigraph π has been completely defined, let us revisit the definitions of the terms ‘exposure step’ and ‘leading hyperedge’. Although these are defined relative to a partition $\mathcal{P}_{i-1,\infty}^\pi$ (for some $i \geq 1$), this only affects the scope of the definitions: for instance, if vertices u, v are contained in different classes of $\mathcal{P}_{\ell,\infty}^\pi$, where $\ell \geq i$, then the exposure step of uv is the same whether we use $\mathcal{P}_{i-1,\infty}^\pi$ or $\mathcal{P}_{\ell,\infty}^\pi$ for the definition.

In particular, if we let $\mathcal{Q} = \mathcal{P}_{\infty,\infty}^\pi$, then it makes sense to speak of leading hyperedges of any quasicycle in $\overline{\pi/\mathcal{Q}}$ or the exposure step of a pair of vertices contained in different classes of \mathcal{Q} .

We now define a partial order on the set of all quasigraphs in H that is crucial for our argument. First, we define the signature \mathbb{S}^π of a quasigraph π as the sequence

$$\mathbb{S}^\pi = (\mathcal{P}_{0,0}^\pi, \mathcal{P}_{0,1}^\pi, \dots, \mathcal{P}_{0,\infty}^\pi, d_0^\pi, \mathcal{P}_{1,0}^\pi, \dots, \mathcal{P}_{1,\infty}^\pi, d_1^\pi, \dots, \mathcal{P}_{\ell,\infty}^\pi, d_\ell^\pi),$$

where ℓ is minimum such that $d_\ell^\pi \in \{\text{TERMINATE}, \text{STOP}\}$.

We extend the chosen linear ordering \leq_E on $E(H)$ to the union of $E(H)$ with $\{\text{TERMINATE}, \text{STOP}\}$ by making TERMINATE the least and STOP the greatest element, respectively. With this extension, we are able to compare signatures in a lexicographic manner.

We derive from this an order \sqsubseteq on quasigraphs in H , setting $\pi \sqsubseteq \rho$ if \mathbb{S}^π is smaller than or equal to \mathbb{S}^ρ in the lexicographic order on the set of signatures of quasigraphs.

It will be convenient to define several related notions to facilitate the comparison of quasigraphs. Let $(i, j) \in \mathcal{T}$. We define the (i, j) -prefix $\mathbb{S}_{(i,j)}^\pi$ of \mathbb{S}^π as follows:

- if $j < \infty$, then $\mathbb{S}_{(i,j)}^\pi$ is the initial segment of \mathbb{S}^π ending with (and including) $\mathcal{P}_{i,j}^\pi$,
- if $j = \infty$, then $\mathbb{S}_{(i,j)}^\pi$ is the initial segment of \mathbb{S}^π ending with (and including) d_i^π .

We let $\pi \sqsubseteq_{(i,j)} \rho$ if $\mathbb{S}_{(i,j)}^\pi$ is lexicographically smaller or equal to $\mathbb{S}_{(i,j)}^\rho$. Furthermore, we define

$$\begin{aligned} \pi \equiv \rho & \text{ if } \pi \sqsubseteq \rho \text{ and } \rho \sqsubseteq \pi, \\ \pi \equiv_{(i,j)} \rho & \text{ if } \pi \sqsubseteq_{(i,j)} \rho \text{ and } \rho \sqsubseteq_{(i,j)} \pi. \end{aligned}$$

Lastly, the notation $\pi \sqsubset \rho$ means $\pi \sqsubseteq \rho$ and $\pi \not\equiv \rho$.

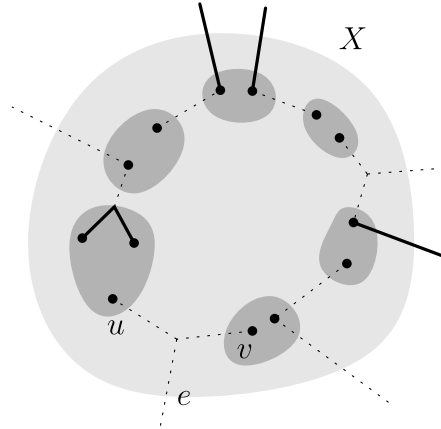


Fig. 8. The situation in Lemma 8: the quasigraph π (bold) and the partition Ω (dark grey). Only some of the hyperedges and vertices are shown; in particular, Ω is assumed to be π -solid.

5. The main result: a variant of the Skeletal Lemma

In this section, we are finally in a position to state and prove the main result of this paper that is essentially a more specific version of [1, Lemma 17]. Before we state it, we need one more definition.

Let H be a 3-hypergraph and π an acyclic quasigraph in H . A partition \mathcal{P} of $V(H)$ is π -skeletal if both of the following conditions hold:

- (1) for each $X \in \mathcal{P}$, π is both connected on X and anticonnected on X (i.e., \mathcal{P} is π -solid),
- (2) the complement of π/\mathcal{P} in H/\mathcal{P} is acyclic.

Theorem 6 (Skeletal Lemma, Stronger Version). *Let π be a quasigraph in a 3-hypergraph H . If $\mathcal{P}_{\infty, \infty}^\pi$ is not π -skeletal or π is not acyclic, then there is a quasigraph ρ in H such that either $\pi \sqsubset \rho$, or $\rho \equiv \pi$ and ρ uses fewer hyperedges than π .*

An obvious corollary of Theorem 6 (which will be further strengthened in Section 6) is the following:

Corollary 7. *For any 3-hypergraph H , there exists an acyclic quasigraph π such that $\mathcal{P}_{\infty, \infty}^\pi$ is π -skeletal.*

Before proving Theorem 6, we need to establish the following crucial lemma. The situation is illustrated in Fig. 8.

Lemma 8. *Let π be a quasigraph in a 3-hypergraph H and let Ω be a π -solid partition of $V(H)$. Suppose that X is a subset of $V(H)$ such that π is anticonnected on X and Ω refines $\{X, V(H) - X\}$. Suppose further that γ is a quasicycle in π/Ω all of whose vertices are subsets of X (as classes of Ω).*

If γ has a redundant leading hyperedge e (with respect to π and Ω), then there are vertices $u, v \in e$ such that each of u and v is contained in a different class in $\gamma(e)$, and the quasigraph $\pi + (uv)_e$ is anticonnected on X .

Proof. Let the vertices of γ^* be $Q_1, \dots, Q_k \in \Omega$ in order, such that $\gamma(e) = \{Q_k, Q_1\}$.

Claim 1. *The quasigraph π is anticonnected on $Q_1 \cup \dots \cup Q_k$ in $H - e$.*

Since e is redundant, it is not a Ω -antibridge, so π is anticonnected on each Q_i in $H - e$, where $i = 1, \dots, k$. We prove, by induction on j , that π is anticonnected on $Q_1 \cup \dots \cup Q_j$ in $H - e$, where $1 \leq j \leq k$. The case $j = 1$ is clear. Supposing that $j > 1$ and the statement is valid for $j - 1$, we prove

it for j . Consider two consecutive vertices Q_{j-1}, Q_j of γ^* and the edge f of γ^* joining them. Since γ is a quasigraph in $\overline{\pi/\Omega}$, f corresponds to a hyperedge $h \neq e$ of H intersecting both Q_{j-1} and Q_j , and such that $\pi(h)$ is contained in Q_{j-1} or Q_j (including the case $\pi(h) = \emptyset$). By the induction hypothesis and Lemma 1, π is anticonnected on $(Q_1 \cup \dots \cup Q_{j-1}) \cup Q_j$ in $H - e$.

Let $u \in Q_k \cap e$ and $v \in Q_1 \cap e$. Observe that u and v are contained in different classes of $\gamma(e)$ as stated in the lemma. By Claim 1, u and v are contained in the same anticomponent of π on X in $H - e$. It follows that u and v are contained in the same anticomponent A of $\rho := \pi + (uv)_e$ on X in H . In fact, the following holds:

Claim 2. *The quasigraph ρ is anticonnected on X in H .*

Suppose the contrary and consider a partition \mathcal{R} of X such that for each hyperedge f of H crossing \mathcal{R} , $\rho(f)$ is not contained in any class of \mathcal{R} . Then e must cross \mathcal{R} , since otherwise \mathcal{R} would demonstrate that π is not anticonnected on X , contrary to the assumption of the lemma. Thus, $\rho(e)$ is not contained in any class of \mathcal{R} , and since $u, v \in X$, u and v are contained in distinct classes of \mathcal{R} . The partition $\mathcal{R}[A]$ of A (the anticomponent of ρ defined above) induced by \mathcal{R} is therefore nontrivial. Since ρ is anticonnected on A , there is a hyperedge h of H such that h crosses $\mathcal{R}[A]$ and $\rho(h)$ is contained in a class of $\mathcal{R}[A]$. But then h crosses \mathcal{R} while $\rho(h)$ is contained in a class of \mathcal{R} , a contradiction with the choice of \mathcal{R} which proves the claim.

We have shown that the present choice of u and v satisfies all requirements of the lemma. This concludes the proof. \square

The following lemma is essential to relating the decisive quasicycles involved in the construction of the plane sequences of quasigraphs π and $\pi - e$, where e is a hyperedge. Note that it can equally well be used to quasigraphs π and $\pi + (uv)_e$, where $u, v \in e$.

Lemma 9. *Suppose that \mathcal{R} is a partition of $V(H)$. If e is a hyperedge of H with $\pi(e)$ contained in a class of \mathcal{R} , then*

$$\pi/\mathcal{R} = (\pi - e)/\mathcal{R}.$$

In particular, the complements of π/\mathcal{R} and of $(\pi - e)/\mathcal{R}$ coincide.

Proof. Both π/\mathcal{R} and $(\pi - e)/\mathcal{R}$ are quasigraphs in H/\mathcal{R} . Their values on f/\mathcal{R} , where $f \neq e$ is any \mathcal{R} -crossing hyperedge, are clearly the same. It thus suffices to compare the values of π/\mathcal{R} and $(\pi - e)/\mathcal{R}$ on e/\mathcal{R} under the assumption that e is \mathcal{R} -crossing. The latter value is \emptyset since e is not used by $\pi - e$. The former one is also \emptyset since $\pi(e)$ is not \mathcal{R} -crossing. \square

Given $(i, j) \in \mathcal{T}$ with $i, j < \infty$ and $(i, j) \neq (0, 0)$, the predecessor of the partition $\mathcal{P}_{i,j}^\pi$ is the partition $\mathcal{P}_{i,j-1}^\pi$ if $j > 0$, and $\mathcal{P}_{i-1,\infty}^\pi$ if $j = 0$ and $i > 0$. The predecessors of the other partitions in the sequence for π are undefined.

Observation 10. *Let Ω be a partition of $V(H)$. If a quasicycle γ in $\overline{\pi/\Omega}$ is exposed at (i, j) with respect to π , then $i, j < \infty$ and $(i, j) \neq (0, 0)$; in particular, the predecessor of $\mathcal{P}_{i,j}^\pi$ exists. In addition, if $j = 1$ and $i \geq 1$, then d_{i-1}^π is a hyperedge not used by π .*

Proof. Clearly, $(i, j) \neq (0, 0)$ since $\mathcal{P}_{0,0}^\pi = \{V(H)\}$. By the definition of the sequence of π , for any $r \geq 0$, $\mathcal{P}_{r,\infty}^\pi$ is equal to one (actually, infinitely many) of the partitions $\mathcal{P}_{r,s}^\pi$, where $0 \leq s < \infty$, and hence it cannot be the exposing partition for γ . A similar argument applies to $\mathcal{P}_{\infty,\infty}^\pi$. As for the last statement, suppose that the exposing partition is $\mathcal{P}_{i,1}^\pi$. Clearly, $d_{i-1}^\pi \notin \{\text{TERMINATE}, \text{STOP}\}$, so it is a hyperedge. If it were used by π , it would be a $\mathcal{P}_{i-1,\infty}^\pi$ -bridge, and the classes of $\mathcal{P}_{i,0}^\pi$ would be the components of $\pi - d_{i-1}^\pi$ on the classes of $\mathcal{P}_{i-1,\infty}^\pi$. By the definition of the sequence of π , $\mathcal{P}_{i,1}^\pi = \mathcal{P}_{i,0}^\pi$ and so $\mathcal{P}_{i,1}^\pi$ cannot be the exposing partition for γ . \square

Observation 11. *Suppose that π is a quasigraph in H , $X \subseteq V(H)$ and e is a hyperedge such that $e \cap X = \{u, v\}$. If $\pi + (uv)_e$ is anticonnected on X , then e is not an X -antibridge with respect to π .*

Proof. Assume the contrary. Then there is a nontrivial partition \mathcal{R} of X such that for each \mathcal{R} -crossing hyperedge f of $H - e$, $\pi(f)$ is not contained in any class of \mathcal{R} . Since there is no such partition for $\pi + (uv)_e$ in H , it must be that e crosses \mathcal{R} and $\{u, v\}$ is a subset of a class of \mathcal{R} . That is impossible since $e \cap X = \{u, v\}$. \square

Lemma 12. *Let π be a quasigraph in H . Let e be a hyperedge not used by π such that vertices $u, v \in e$ are contained in different classes of a partition $\mathcal{P}_{i,j}^\pi$ (where $0 \leq i, j < \infty$), but both of them are contained in the same class X of its predecessor. If the quasigraph $\pi + (uv)_e$ is anticonnected on X , then $\pi \sqsubset \pi + (uv)_e$.*

Proof. Let $\rho = \pi + (uv)_e$. Suppose that $\pi \not\sqsubset \rho$. We begin by proving the following claim:

$$\pi \sqsubseteq_{(s,t)} \rho \text{ for all } (s, t) \in \mathcal{T} \text{ with } (s, t) \leq (i, j). \tag{1}$$

We proceed by (transfinite) induction on (s, t) , assuming the claim for all smaller pairs in \mathcal{T} . The claim (1) holds for $(s, t) = (0, 0)$; assume therefore that $(s, t) > (0, 0)$. Suppose first that $0 < t < \infty$ and the statement holds for $(s, t - 1)$. If t is odd, then any class A of $\mathcal{P}_{s,t}^\pi$ is a component of π on a class of $\mathcal{P}_{s,t-1}^\pi$. We have either $A \supseteq X$, or $A \cap X = \emptyset$. In both cases, ρ is clearly connected on A (by Lemma 4(ii) in the former case). Thus, $\mathcal{P}_{s,t}^\pi \leq \mathcal{P}_{s,t}^\rho$ and $\pi \sqsubseteq_{(s,t)} \rho$.

Next, if t is even (and nonzero), we proceed similarly: if A is an anticomponent of π on a class of $\mathcal{P}_{s,t-1}^\pi$ and $A \supseteq X$, then ρ is anticonnected on A by Lemma 4(i), while if $A \cap X = \emptyset$, the same is true for trivial reasons. Thus, again, $\pi \sqsubseteq_{(s,t)} \rho$.

The next case is $t = \infty$. By the induction hypothesis, $\mathcal{P}_{s,\infty}^\pi \leq \mathcal{P}_{s,\infty}^\rho$; without loss of generality, we may assume that $\mathcal{P}_{s,\infty}^\pi = \mathcal{P}_{s,\infty}^\rho$. Let $\mathcal{S} = \mathcal{P}_{s,\infty}^\pi$. We need to show that $d_s^\pi \leq_E d_s^\rho$. By Lemma 9, $\pi/\mathcal{S} = \bar{\rho}/\mathcal{S}$. In particular, the two hypergraphs have the same quasicycles, and these quasicycles have the same leading hyperedges. It follows that $d_s^\pi \leq_E d_s^\rho$ or $d_s^\pi = \text{stop}$ – but the latter does not hold since π cannot stop (or terminate) at (s, ∞) as $\mathcal{P}_{i,j}^\pi$ differs from its predecessor and $(s, \infty) < (i, j)$.

It remains to consider the case $t = 0$. Here, we have $s > 0$ and the induction hypothesis implies that $\pi \sqsubseteq_{(s-1,\infty)} \rho$. Let us assume that $\pi \equiv_{(s-1,\infty)} \rho$ and denote $\mathcal{P}_{s-1,\infty}^\pi$ by \mathcal{S} . We want to show that $\mathcal{P}_{s,0}^\pi \leq \mathcal{P}_{s,0}^\rho$.

Let $f := d_{s-1}^\pi$. For the same reason as above, π continues at $(s - 1, \infty)$, so $f \notin \{\text{TERMINATE}, \text{STOP}\}$. This means that f is an \mathcal{S} -bridge or an \mathcal{S} -antibridge with respect to π . In addition, $d_{s-1}^\rho = f$ by the assumption that $\pi \equiv_{(s-1,\infty)} \rho$. Hence, f is an \mathcal{S} -bridge or an \mathcal{S} -antibridge with respect to ρ as well.

Consider the set X from the statement of the lemma. Let S be the class of \mathcal{S} containing X . Since the anticomponents of $\pi - f$ and $\rho - f$ on S are clearly the same if $|f \cap S| \leq 1$, we may assume that $|f \cap S| \geq 2$ – indeed, since f crosses \mathcal{S} , we must have equality.

We distinguish three cases:

- (a) $f = e$,
- (b) $f \neq e$ and f is not used by π ,
- (c) $f \neq e$ and f is used by π .

In case (a), $f (= e)$ is not used by π , so it is an \mathcal{S} -antibridge with respect to π . Since it intersects S in two vertices, these two vertices are u and v , and each of them is contained in a different anticomponent of π on S in $H - f$. Moreover, it must be that $i = s, j = 0$ and $X = S$ (since u, v are contained in distinct classes of $\mathcal{P}_{s,0}^\pi$ but in one class S of its predecessor). However, an assumption of the lemma is that ρ is anticonnected on X , a contradiction with Observation 11. In other words, case (a) cannot occur.

In case (b), which is illustrated in Fig. 9, f is also an \mathcal{S} -antibridge with respect to π (and ρ). To prove that $\mathcal{P}_{s,0}^\pi \leq \mathcal{P}_{s,0}^\rho$, it is enough to show that ρ is anticonnected on each anticomponent of π on S in $H - f$. Let A be such an anticomponent. We may assume that $|e \cap A| \geq 2$, otherwise ρ is clearly anticonnected on A . Thus, $X \subseteq A$. By Lemma 4(i), ρ is anticonnected on A as claimed. The discussion of case (b) is complete.

Lastly, in case (c), f is an \mathcal{S} -bridge with respect to π and ρ . Trivially, the quasigraph $\rho - f$ is connected on each component of $\pi - f$ on S in H , so $\mathcal{P}_{s,0}^\pi \leq \mathcal{P}_{s,0}^\rho$.

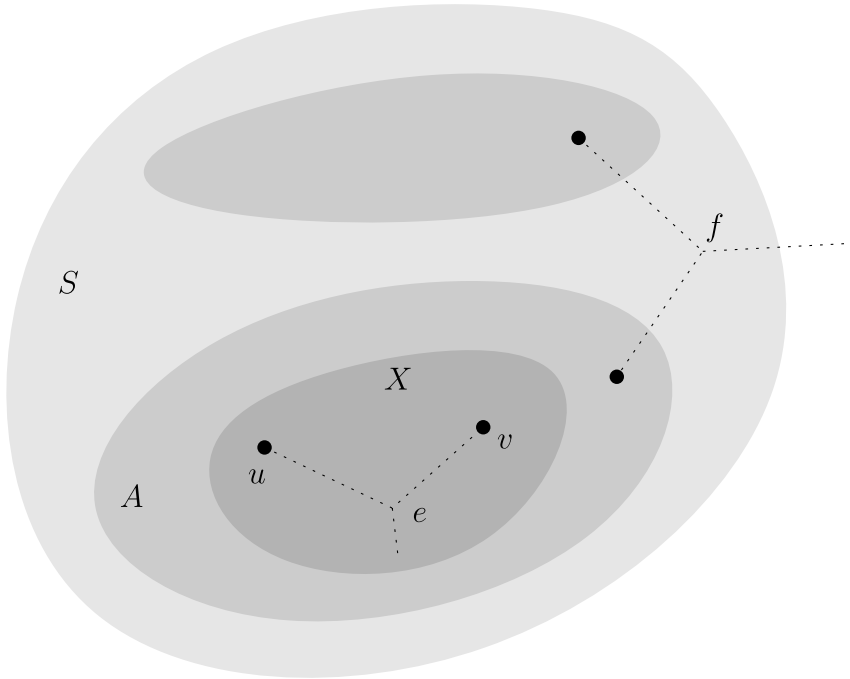


Fig. 9. Case (b) in the proof of Lemma 12. A class S of the partition \mathcal{S} is shown in lightest grey. Hyperedges e and f are shown dotted. Medium grey regions represent the anticompnents of π on S in $H - f$. The set X (darkest grey) is contained in an anticompnent A .

To summarise, each of the cases (a)–(c) leads either to a contradiction, or to the sought conclusion $\mathcal{P}_{s,0}^\pi \leq \mathcal{P}_{s,0}^\rho$. This concludes the proof of (1).

It remains to show that $\pi \sqsubset \rho$. Since $j < \infty$, there are three cases to distinguish based on the value of j . If j is odd, then the classes of $\mathcal{P}_{i,j}^\pi \setminus \mathcal{P}_{i,j-1}^\pi$ are the components of π on X . Since u and v are in different classes of $\mathcal{P}_{i,j}^\pi$, the replacement of π with ρ has the effect of adding the edge uv to π^* , joining the two components into one. Therefore, $\mathcal{P}_{i,j}^\pi < \mathcal{P}_{i,j}^\rho$, and by (1), $\pi \sqsubset \rho$.

If j is even and $j > 0$, the classes of $\mathcal{P}_{i,j}^\pi$ are the anticompnents of π on X . Let A_1 and A_2 be such anticompnents containing u and v , respectively. By Lemma 1, π is anticompnted on $A_1 \cup A_2$ since e intersects both of these sets and is not used by π . This contradiction means that the present case is not possible.

Finally, if $j = 0$, uv is exposed at $(i, 0)$ with respect to π . Let $f = d_{i-1}^\pi$ be the corresponding decisive hyperedge. Since π continues at $(i - 1, \infty)$, f is an X -bridge or an X -antibridge with respect to π . This shows that $f \neq e$ because e is not an X -antibridge by Observation 11, and is not an X -bridge because it is not used by π . Furthermore, similarly to the preceding case, it cannot be that f is an X -antibridge, for then e would intersect two anticompnents of π on X in $H - f$, which is impossible by Lemma 1.

Thus, f is an X -bridge with respect to π , and the classes of $\mathcal{P}_{i,0}^\pi$ are the components of $\pi - f$ on X in H . Since u, v are in different components, the quasigraph $\rho - f$ is connected on X in H , and consequently ρ stops at $\mathcal{P}_{i-1,\infty}^\rho$ and $\pi \sqsubset \rho$. This concludes the proof. \square

Lemma 13. Let π be a quasigraph in H . Let e be a hyperedge of H used by π and $X \subseteq V(H)$ such that $\pi(e) \subseteq X$, $\pi - e$ is connected on X , and one of the following conditions is satisfied:

- (a) the vertices of $\pi(e)$ are contained in different classes of \mathcal{P}_{ij}^π ($0 \leq i, j < \infty$) and X is a class of its predecessor,
- (b) $X \in \mathcal{P}_{\infty, \infty}^\pi$ and $\mathcal{P}_{\infty, \infty}^\pi$ is π -skeletal.

Then $\pi \sqsubseteq \pi - e$. In addition, if (a) is satisfied, then $\pi \sqsubset \pi - e$.

Proof. Suppose that condition (a) is satisfied. We prove, by (transfinite) induction on (s, t) , that

$$\pi \sqsubseteq_{(s,t)} \pi - e \text{ for all } (s, t) \in \mathcal{T} \text{ with } (s, t) \leq (i, j). \tag{2}$$

The statement holds if $(s, t) = (0, 0)$. Consider $(s, t) > (0, 0)$. If $0 < t < \infty$, then we may suppose that $\pi \equiv_{(s,t-1)} \pi - e$; by Lemma 5, $\mathcal{P}_{s,t}^\pi \leq \mathcal{P}_{s,t}^{\pi-e}$ and therefore $\pi \sqsubseteq_{(s,t)} \pi - e$ as desired.

Suppose that $t = \infty$. Let $\mathcal{P} = \mathcal{P}_{s,\infty}^\pi$. Without loss of generality, $\mathcal{P} = \mathcal{P}_{s,\infty}^{\pi-e}$. Since X is a subset of a class of \mathcal{P} , Lemma 9 implies that the complement of π/\mathcal{P} is the same as the complement of $(\pi - e)/\mathcal{P}$. In particular, the quasicycles in these hypergraphs, as well as the sets of their leading hyperedges, are the same.

We state the following simple observation as a claim for easier reference later in the proof:

Claim 1. No leading hyperedge of any quasicycle in the complement of π/\mathcal{P} is weakly redundant.

Indeed, if the claim did not hold, then π would have stopped at (s, ∞) , but condition (a) implies that \mathcal{P}_{ij}^π differs from its predecessor. Since $(s, \infty) < (i, j)$, this would be a contradiction.

By Claim 1, if $\pi - e$ stops at (s, ∞) , then $\pi \sqsubset \pi - e$. We may therefore assume that $\pi - e$ continues at (s, ∞) , in which case the decisive hyperedges at (s, ∞) for π and $\pi - e$ coincide. We conclude that $\pi \sqsubseteq_{(s,\infty)} \pi - e$ in this case.

It remains to consider the case $t = 0$. It suffices to show that $\mathcal{P}_{s,0}^\pi \leq \mathcal{P}_{s,0}^{\pi-e}$ assuming that $\pi \equiv_{(s-1,\infty)} \pi - e$. Let $f := d_{s-1}^{\pi-e}$ be the decisive hyperedge at $(s - 1, \infty)$ with respect to π . The assumption implies that $f = d_{s-1}^{\pi-e}$. Moreover, $f \notin \{\text{TERMINATE}, \text{STOP}\}$ because \mathcal{P}_{ij}^π differs from its predecessor, so π has to continue at $(s - 1, \infty)$ as $(i, j) \geq (s, 0)$.

If there is a class $A \in \mathcal{P}_{s,0}^\pi$ with $\pi(e) \subseteq A$, then we have $X \subseteq A$ (where X is the set from the lemma). It follows from Lemma 5 that $\pi - e$ is connected on A whenever π is, and similarly for anticonnectivity. Consequently, $\mathcal{P}_{s,0}^\pi \leq \mathcal{P}_{s,0}^{\pi-e}$.

We may therefore assume that

$$\pi(e) \text{ intersects two classes } Y, Y' \text{ of } \mathcal{P}_{s,0}^\pi. \tag{3}$$

The situation is illustrated in Fig. 10. By condition (a), $(i, j) = (s, 0)$ and $X \in \mathcal{P}_{s-1,\infty}^\pi$.

By the construction of $\mathcal{P}_{s,0}^\pi$, there are two possibilities: either (i) f is an X -antibrige with respect to π and Y, Y' are the anticomponents of π on X in $H - f$, or (ii) f is an X -bridge with respect to π and Y, Y' are the components of $\pi - f$ on X in H .

Note first that $f \neq e$: otherwise, f would necessarily be an X -bridge (since e is used by π), but $\pi - e$ is assumed to be connected on X .

We next rule out possibility (ii). Assume that Y, Y' are the components of $\pi - f$ on X in H . Now $\pi(e)$ intersects Y and Y' , so $\pi - f$ is connected on $Y \cup Y'$ thanks to e , a contradiction.

Let us therefore restrict ourselves to possibility (i). Let γ denote a quasicycle in the complement of $\pi/\mathcal{P}_{s-1,\infty}^\pi$ such that f is a leading hyperedge of γ . By Lemma 9, γ is a quasicycle in the complement of $(\pi - e)/\mathcal{P}_{s-1,\infty}^\pi$ as well.

Since e intersects both Y and Y' , (i) implies that $\pi - e$ is anticonnected on $Y \cup Y'$ in $H - f$. Consequently, the leading hyperedge f of γ is redundant with respect to $\pi - e$ and $\mathcal{P}_{s-1,\infty}^\pi$. This contradicts our assumption that $\pi \equiv_{(s-1,\infty)} \pi - e$ since π continues at $(s - 1, \infty)$ while $\pi - e$ stops there. The contradiction means that if $\pi(e)$ intersects two classes of $\mathcal{P}_{s,0}^\pi$, then actually $\pi \sqsubset_{(s-1,\infty)} \pi - e$. The case $t = 0$ is settled.

Having proved (2), let us now show that

$$\pi \sqsubset_{(i,j)} \pi - e,$$

where i, j are as in the statement of the lemma. By assumption, $\pi(e)$ intersects two classes of \mathcal{P}_{ij}^π . We note that $0 \leq j < \infty$ and consider two possibilities: $j = 0$ and $j > 0$ (j finite). If $j = 0$, then

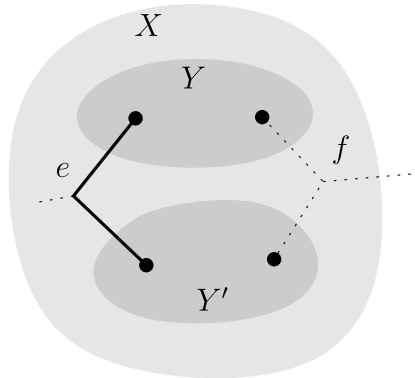


Fig. 10. A situation in the proof of Lemma 13 when $\pi(e)$ intersects two classes Y, Y' of $\mathcal{P}_{s,0}^\pi$ (shown in darker grey). Light grey represents a class X of $\mathcal{P}_{s-1,\infty}^\pi$. Observe that $\pi - e$ is anticonnected on $Y \cup Y'$ in $H - f$.

we have seen in the above paragraph that if $\pi(e)$ intersects two classes of $\mathcal{P}_{s,0}^\pi$ (for any $s \leq i$), then $s = i$ and $\pi \sqsubset_{(s-1,\infty)} \pi - e$. In particular, $\pi \sqsubset_{(i,j)} \pi - e$.

Suppose now that $j > 0$ is finite. Without loss of generality, $\pi \equiv_{(i,j-1)} \pi - e$. Since $\pi(e)$ intersects two classes of $\mathcal{P}_{i,j}^\pi$, we see from the definition of the sequence of π that j is even. Hence, the vertices of $\pi(e)$ lie in different anticomponents of π on X , and $\pi - e$ is anticonnected on the union of these anticomponents by Lemma 1. Consequently, $\mathcal{P}_{i,j}^\pi < \mathcal{P}_{i,j}^{\pi-e}$ and $\pi \sqsubset_{(i,j)} \pi - e$. This concludes the proof for condition (a).

A very similar inductive proof to that used to prove (2) works when condition (b) is satisfied. The main difference is that in the $t = \infty$ case, Claim 1 now holds for a different reason, namely that $\mathcal{P}_{\infty,\infty}^\pi$ is π -skeletal, which means that π never stops – consequently, there is no weakly redundant leading hyperedge of a quasicycle in $\overline{\pi/\mathcal{P}_{s-1,\infty}^\pi}$.

Another difference is that condition (3) cannot occur if (b) is satisfied, which makes the proof for condition (b) somewhat shorter. \square

We can now proceed to the proof of Theorem 6.

Proof of Theorem 6. Let $\mathcal{Q} = \mathcal{P}_{\infty,\infty}^\pi$. We distinguish the following cases.

Case 1. \mathcal{Q} is not π -skeletal.

By the construction, it is clear that \mathcal{Q} is π -solid. Thus, $\overline{\pi/\mathcal{Q}}$ contains a quasicycle. Consider the least s such that $\mathcal{P}_{s,\infty}^\pi = \mathcal{Q}$. Since π stops at $\mathcal{P}_{s,\infty}^\pi$, there is a quasicycle γ in $\overline{\pi/\mathcal{Q}}$ and a leading hyperedge e of γ such that e is weakly redundant. Let the exposure step for γ be (i, j) , where $0 \leq i, j < \infty$. We put $\mathcal{P} = \mathcal{P}_{i,j}^\pi$, and let X be the class of the predecessor of $\mathcal{P}_{i,j}^\pi$ containing both vertices of $\gamma(e)$. Furthermore, let $Q_1, Q_2 \in \mathcal{Q}$ be the two vertices of $\gamma(e)$, and let $P_i \in \mathcal{P}$ be such that $P_i \supseteq Q_i$ ($i = 1, 2$).

Subcase 1.1. e is not used by π .

First, j is odd or $j = 0$; otherwise, P_1, P_2 would be anticomponents of π on X , but Lemma 1 shows that π is anticonnected on $P_1 \cup P_2$, which would be a contradiction.

Consider first the case that j is odd, so P_1 and P_2 are components of π on X . Moreover, suppose for now that $j > 1$. By the construction of the sequence for π , π is anticonnected on X . Applying Lemma 8 (with the current values of X, Q, γ and e), we obtain vertices u, v such that $u \in Q_1, v \in Q_2$ and $\pi + (uv)_e$ is anticonnected on X . Lemma 12 then implies that $\pi \sqsubset \pi + (u_1u_2)_e$ and we are done.

Suppose that $j = 1$. If $i = 0$ or the decisive hyperedge d_{i-1}^π at $(i - 1, \infty)$ is not used by π , then the above argument works, since in this case π is anticonnected on X . The other case ($i > 0$ and

\mathcal{P}_{i-1}^π is used by π) is excluded by **Observation 10**. This settles the case $j = 1$ and more broadly the case that j is odd.

It remains to consider the possibility that $j = 0$. Clearly, $i > 0$ as $\mathcal{P}_{0,0}^\pi = \{V(H)\}$. The predecessor of $\mathcal{P}_{i,0}^\pi$ is $\mathcal{P}_{i-1,\infty}^\pi$, which is π -solid. Thus, π is anticonnected on X and the argument used for odd $j > 1$ applies.

Subcase 1.2. e is used by π .

Since e is a leading hyperedge of a quasicycle in $\overline{\pi/Q}$, $\pi(e)$ is a subset of some $Q \in \Omega$. Since e is weakly redundant with respect to Ω , it is not a Q -bridge – in other words, $\pi - e$ is connected on Q . **Lemma 13** implies that $\pi \sqsubseteq \pi - e$. Since $\pi - e$ uses fewer hyperedges than π , it has the desired properties.

Case 2. π is not acyclic.

Suppose that there is a cycle C in the graph π^* . That means that either π/Q is not acyclic, or there is a cycle in the induced subgraph of π^* on some $Q \in \Omega$.

Subcase 2.1. The quasigraph π/Q is not acyclic.

Let C be a cycle in $(\pi/Q)^*$ and suppose its exposure step is (i, j) , where $0 \leq i, j < \infty$. Let e be a leading hyperedge of C . Note that each of the vertices of $\pi(e)$ is in a different class of the partition $\mathcal{P}_{i,j}^\pi$, but both are contained in the same class X of its predecessor.

Since e is contained in the cycle C all of whose vertices are subsets of X , $\pi - e$ is connected on X . Thus, the assumptions of **Lemma 13** are satisfied (we use the current values of e and X). It follows that $\pi \sqsubseteq \pi - e$; since $\pi - e$ uses fewer hyperedges than π , $\pi - e$ has the desired properties.

Subcase 2.2. There is $Q \in \Omega$ such that the induced subgraph of π^* on Q contains a cycle.

Let C be a cycle in the induced subgraph of π^* on Q , and let e be a hyperedge such that $\pi(e)$ is an edge of C . Clearly, $\pi - e$ is connected on Q . By **Lemma 13**, $\pi \sqsubseteq \pi - e$. Since $\pi - e$ uses fewer hyperedges than π does, we are done. \square

6. Removing bad leaves

For the purposes of the application of the Skeletal Lemma in [2], we have to prove the lemma in a stronger form (**Theorem 6**) allowing us to deal with a certain configuration that is problematic for the analysis in [2], namely a ‘bad leaf’ in a quasigraph. As a result, we will be able to exclude this configuration in **Theorem 16** (at the cost of some local modifications to the hypergraph).

Let H be a 3-hypergraph and let π be an acyclic quasigraph in H . In each component of the graph π^* , we choose an arbitrary root and orient all the edges of π^* towards the root. A hyperedge e of H is associated with a vertex u if it is used by π and u is the tail of $\pi(e)$ in the resulting oriented graph. Thus, every vertex has at most one associated hyperedge, and conversely, each hyperedge is associated with at most one vertex.

A vertex u of H is a *bad leaf* for π if all of the following hold:

- (i) u is a leaf of π^* ,
- (ii) u is incident with exactly three hyperedges, exactly one of which has size 3 (say, e), and
- (iii) e is associated with u .

To eliminate a bad leaf u , we use a *switch* operation illustrated in **Fig. 11**. Suppose that u is incident with hyperedges ua, ub and ucd , where ucd is associated with u , and $\pi(ucd) = uc$. We remove from H the hyperedges ua, ub and ucd and add the hyperedges uab, uc and ud ; the resulting hypergraph is denoted by $H^{(u)}$. We say that a hypergraph \tilde{H} is *related* to H if it can be obtained from H by a finite series of switch operations.

With π as above, a quasigraph $\pi^{(u)}$ in $H^{(u)}$ is obtained by setting $\pi^{(u)}(uc) = uc$, and leaving both ud and uab unused. Observe that $(\pi^{(u)})^* = \pi^*$, and $\pi^{(u)}$ has fewer bad leaves than π .

A problem we have to address is that a partition \mathcal{P} which is π -skeletal in H need no longer be $\pi^{(u)}$ -skeletal in $H^{(u)}$, since the switch may create an unwanted cycle in $\overline{\pi^{(u)}/\mathcal{P}}$. This is illustrated in

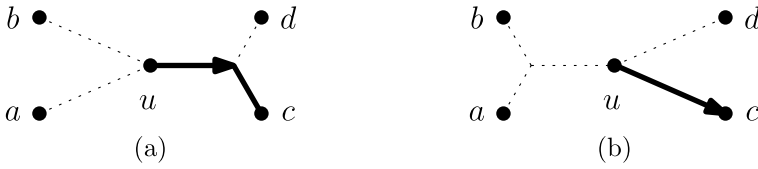


Fig. 11. (a) A bad leaf u . (b) The result of the switch at u .

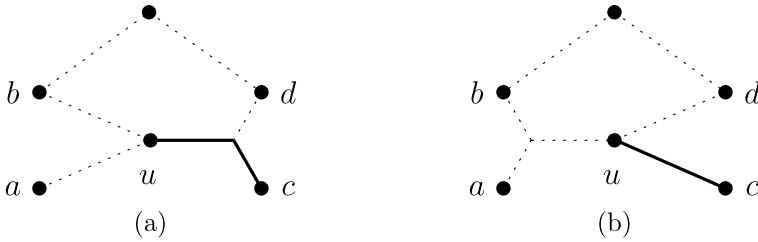


Fig. 12. (a) A quasigraph π in H such that the complement of π is acyclic. (b) The quasigraph obtained by a switch at the vertex u no longer has acyclic complement.

Fig. 12. The following paragraphs describe the steps taken to resolve this problem. First, we extend the order \sqsubseteq defined on quasigraphs in H to the set of all quasigraphs in hypergraphs related to H . Since all such hypergraphs have the same vertex set, we can readily compare the partitions of their vertex sets.

We have to be more careful, however, in the definition of the sequence of π , where a linear ordering \leq_E of hyperedges of H is used: this ordering should involve all hyperedges of hypergraphs related to H . We define \leq_E as follows. We fix a linear ordering \leq of $V(H)$. On the set of 3-hyperedges of hypergraphs related to H , \leq_E is the associated lexicographic ordering, and the same holds for the set of 2-hyperedges of hypergraphs related to H . We make each 2-hyperedge greater than any 3-hyperedge with respect to \leq_E . Finally, we add the elements TERMINATE and STOP to the ordered set as the least and greatest element of \leq_E , respectively.

This allows for a definition of the sequence of π consistent with the switch operation. Furthermore, the definition of the ordering \sqsubseteq as given in Section 4 is well suited for our purpose, and remains without change.

Let us mention that although the incorporation of the decisive hyperedges in the signature of a quasigraph may have seemed unnecessary (mainly thanks to Lemma 9), the present section is the reason why we chose this definition. In fact, the only situation in our arguments when the comparison of the decisive hyperedges is relevant is immediately after a switch, as in Lemma 15 below.

We first prove that switching a bad leaf of π does not affect the (anti)connectivity of π on a set of vertices.

Lemma 14. *Let π be an acyclic quasigraph in H and $X \subseteq V(H)$. Suppose that π has a bad leaf u and σ is obtained from π by switching at u . The following holds:*

- (i) if π is anticonnected on X , then so is σ ,
- (ii) if π is connected on X , then so is σ .

Proof. We prove (i). Suppose that π is anticonnected on X , but the quasigraph σ (in a hypergraph \tilde{H} related to H) is not. The definition implies that there is a partition \mathcal{P} of X such that for every \mathcal{P} -crossing hyperedge f of \tilde{H} , $\sigma(f)$ is not contained in any class of \mathcal{P} . At the same time, there is a \mathcal{P} -crossing hyperedge e of H such that $\pi(e)$ is contained in some class of \mathcal{P} . Clearly, e must be

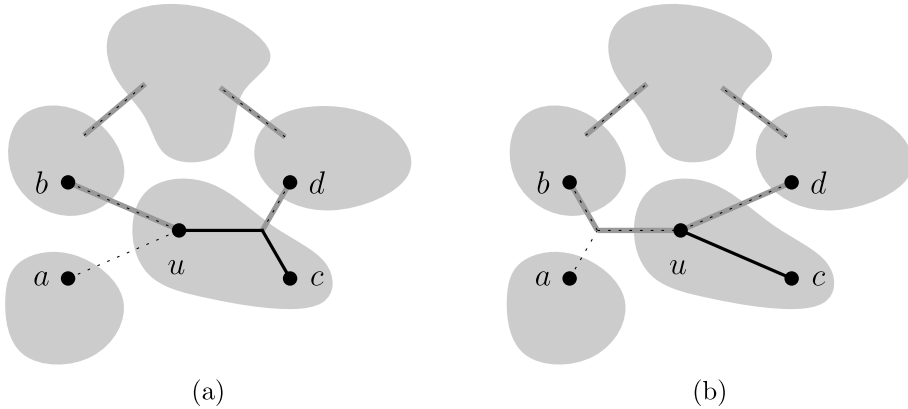


Fig. 13. Corresponding quasicycles in the complement of π/\mathcal{P} (in H/\mathcal{P}) and in the complement of $\pi^{(u)}/\mathcal{P}$ (in \tilde{H}/\mathcal{P}). The quasigraph π is represented by bold lines, the partition \mathcal{P} is shown in grey. (a) A quasicycle γ in the complement of π/\mathcal{P} (shown with darker grey bars). (b) The corresponding quasicycle γ' in $\pi^{(u)}/\mathcal{P}$ (shown in the same way).

incident with u (since the other hyperedges exist both in H and \tilde{H} , and the values of π and σ coincide).

Let the neighbours of u in \tilde{H} be labelled as in Fig. 11(b). Let A be the class of \mathcal{P} containing u ; by the above property of σ , we can easily see that $a \in A$ if $a \in X$, and similarly for b and d . This implies that $c \in X \setminus A$ and $e = ucd$, but then $\pi(ucd)$ crosses \mathcal{P} , a contradiction.

Part (ii) is immediate from the fact that $\pi^* = \sigma^*$. \square

Lemma 15. *Let π be an acyclic quasigraph in H such that $\mathcal{P}_{\infty, \infty}^\pi$ is π -skeletal. If π has a bad leaf u and the quasigraph σ (in a hypergraph related to H) is obtained from π by a switch at u , then $\pi \sqsubseteq \sigma$.*

Proof. We show that

$$\pi \sqsubseteq_{(i,j)} \sigma \text{ for all } (i, j) \in \mathcal{I}. \tag{4}$$

We proceed by transfinite induction on (i, j) . The claim is trivial for $(i, j) = (0, 0)$. Suppose now that $j > 0$ is finite. We may assume that $\pi \equiv_{(i,j-1)} \sigma$ for otherwise we are done. If j is odd, then the classes of $\mathcal{P}_{i,j}^\pi$ are the components of π on classes of $\mathcal{P}_{i,j-1}^\pi$. Let $X \in \mathcal{P}_{i,j-1}^\pi$ and let A be a component of σ on X . By Lemma 14(ii), σ is connected on A . Hence, $\mathcal{P}_{i,j}^\pi \leq \mathcal{P}_{i,j}^\sigma$ and (4) follows. An analogous argument, using Lemma 14(i), can be used for even $j > 0$.

Let us consider the case $j = \infty$. We may assume that i is finite (otherwise, the claim follows directly from the induction hypothesis) and that $\pi \equiv_{(i,\infty)} \sigma$. Since $\mathcal{P}_{\infty, \infty}^\pi$ is assumed to be π -skeletal, π does not stop at (i, ∞) . Furthermore, we may assume that π does not terminate at (i, ∞) , for otherwise we immediately conclude $\pi \sqsubseteq_{(i,\infty)} \sigma$ (recalling that TERMINATE is the least element of the ordering $\leq_{\mathcal{E}}$).

Let $\mathcal{P} = \mathcal{P}_{i,\infty}^\pi$ and let γ be a quasicycle in the complement of π/\mathcal{P} in H/\mathcal{P} . Define a quasigraph γ' in the complement of σ/\mathcal{P} in \tilde{H}/\mathcal{P} as follows (see Fig. 13 for an illustration of several of the cases):

- if γ uses a hyperedge f/\mathcal{P} , where f is a hyperedge of H not incident with u , then set $\gamma'(f/\mathcal{P}) = \gamma(f/\mathcal{P})$,
- if γ uses au/\mathcal{P} and bu/\mathcal{P} , then set $\gamma'(abu/\mathcal{P}) = ab/\mathcal{P}$,
- if γ uses au/\mathcal{P} but not bu/\mathcal{P} , then set $\gamma'(abu/\mathcal{P}) = au/\mathcal{P}$ (and symmetrically with au and bu reversed),
- if γ uses ucd/\mathcal{P} (so u and d are in different classes of \mathcal{P}), then set $\gamma'(ud/\mathcal{P}) = ud/\mathcal{P}$.

A look at Figs. 11 and 13 shows that γ' is a quasicycle. Thus, σ does not terminate at (i, ∞) . We need to relate leading hyperedges of γ to those of γ' .

Any leading hyperedge of γ that is not incident with u is a leading hyperedge of γ' , and vice versa. We assert that neither au nor bu is a leading hyperedge of γ . If they were, they would be redundant (since their size is 2) and $\mathcal{P}_{\infty, \infty}^\pi$ would not be π -skeletal, contrary to the assumption. Finally, if ucd is a leading hyperedge of γ , then ud is a leading hyperedge of γ' . Note that $ucd <_E ud$.

It follows that if σ does not stop at (i, ∞) , then $d_i^\pi \leq_E d_i^\sigma$. On the other hand, if it does stop, then the same inequality holds since STOP is the greatest element of the ordering \leq_E . In both cases, we have $\pi \sqsubseteq_{(i, \infty)} \sigma$.

The last possibility left to consider is $j = 0$. We need to show that $\mathcal{P}_{i, 0}^\pi \leq \mathcal{P}_{i, 0}^\sigma$ under the assumption that $\pi \equiv_{(i-1, \infty)} \sigma$. Let $\mathcal{R} = \mathcal{P}_{i-1, \infty}^\pi$ and let $f := d_{i-1}^\pi = d_{i-1}^\sigma$. Since $\mathcal{P}_{\infty, \infty}^\pi$ is π -skeletal, $f \neq \text{STOP}$. If $f = \text{TERMINATE}$, then $\pi \equiv \sigma$. We may thus assume that f is a hyperedge, and in that case it is not incident with u (since H and \tilde{H} have no common hyperedge incident with u). Thus, $\sigma - f$ is obtained from $\pi - f$ by a switch at u . Similarly, if f is not used by π , then σ is obtained from π by a switch at u , in the hypergraph $H - f$.

If f is an X -antibrige with respect to π for some $X \in \mathcal{R}$, we may use Lemma 14 in the hypergraph $H - f$. We find that σ is anticonnected on each anticomponent of π on X in $H - f$, and hence $\pi \sqsubseteq_{(i, 0)} \sigma$. On the other hand, if f is an X -bridge with respect to π , then Lemma 14 implies that $\sigma - f$ is connected on each component of $\pi - f$ on X in H , and $\pi \sqsubseteq_{(i, 0)} \sigma$ again. This proves (4) and the lemma follows. \square

Let us now state the result we need to use in [2].

Theorem 16. *Let H be a 3-hypergraph. There exists a hypergraph \tilde{H} related to H and an acyclic quasigraph σ in \tilde{H} such that σ has no bad leaves and $V(\tilde{H})$ admits a σ -skeletal partition \mathcal{S} .*

Proof. Let \mathcal{M} be the set of all quasigraphs in 3-hypergraphs related to H . Furthermore, let $\mathcal{M}' \subseteq \mathcal{M}$ be the set of quasigraphs satisfying the following conditions:

- (1) σ is \sqsubseteq -maximal in \mathcal{M} ,
- (2) subject to (1), σ uses as few hyperedges as possible.

Let σ' be any quasigraph from \mathcal{M}' ; say, σ' is a quasigraph in a hypergraph H' related to H . Theorem 6 implies that σ' is acyclic and $\mathcal{P}_{\infty, \infty}^{\sigma'}$ is σ' -skeletal (where the partition is obtained via the plane sequence with respect to H'). In particular, it makes sense to consider bad leaves of σ' .

Let us choose σ as an element of \mathcal{M}' with as few bad leaves as possible, and define $\mathcal{S} := \mathcal{P}_{\infty, \infty}^\sigma$.

We need to prove that σ has no bad leaves. Suppose to the contrary that there is a bad leaf u for σ . By Lemma 15, $\sigma \sqsubseteq \sigma^{(u)}$. Furthermore, $\sigma^{(u)}$ uses the same number of hyperedges as σ , and has one bad leaf fewer, a contradiction with the choice of σ . \square

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