Západočeská univerzita v Plzni
Fakulta aplikovaných věd

# FUČÍKOVO SPEKTRUM DISKRÉTNÍHO DIRICHLETOVA OPERÁTORU

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Disertační práce k získání akademického titulu doktor (Ph.D.)
v oboru Aplikovaná matematika

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Katedra matematiky

Plzeň, 2021

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# THE FUČÍK SPECTRUM OF THE DISCRETE DIRICHLET OPERATOR

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Dissertation thesis submitted for the degree Doctor of Philosophy (Ph.D.)
in specialization Applied Mathematics

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Plzeň, 2021

# Declaration

I do hereby declare that the entire thesis is my original work and that I have used only the cited sources.

*Plzeň, August 27, 2021*

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Iveta Sobotková

# Abstrakt

Tato disertační práce je zaměřena na studium Fučíkova spektra pro diskrétní operátory. Vzhledem k tomu, že obecné vyšetření Fučíkova spektra diskrétních operátorů je v dnešní době stále těžce uchopitelnou výzvou, studium v této práci je zaměřené na konkrétní operátor – Dirichletův diskrétní operátor.

Tento operátor odpovídá diferenční rovnici druhého řádu s Dirichletovými okrajovými podmínkami. V disertační práci je dopodrobna vyšetřena odpovídající semilineární úloha, zaveden pojem spojitého rozšíření diskrétního řešení úlohy a hlavně je zde uveden kompletní implicitní popis Fučíkova spektra Dirichletova diskrétního operátoru. Na závěr práce jsou popsány tři typy odhadů pro Fučíkovy větve, které umožňují lokalizovat Fučíkovy větve i pro velký rozměr odpovídající matice.

Celý text disertační práce se opírá o dva autorčiny články (v příloze práce) – [25], [31]. Samotný text disertační práce je koncipován jako shrnutí klíčových výsledků odkázaných článků a obsahuje podrobná vysvětlení jednotlivých nově zavedených konceptů pro práci s Fučíkovým spektrem pro vybraný diskrétní operátor.

**Klíčová slova**: Fučíkovo spektrum, diferenční operátor, Dirichletův diskrétní operátor, Chebyshevův polynom druhého druhu, asymetrické nelinearity

# Abstract

This dissertation thesis is devoted to the study of Fučík spectrum for discrete operators. Considering the fact, that the problem of exploring Fučík spectrum for general discrete operators is still a significant challenge, in this thesis we focus on analyses in regards of a particular operator – Dirichlet discrete operator.

This operator corresponds to the second order difference equation with Dirichlet boundary conditions. In the thesis, we explore corresponding semi-linear problem, we define a continuous extension of a discrete solution and finally, we provide a complete implicit description of the Fučík spectrum of Dirichlet discrete operator. Last but not least, three bounds for Fučík curves are described. This allows for a localization of Fučík curves even for large size of a corresponding matrix.

The whole text of the thesis is based on two articles of the author [25], [31]. The main goal is to summarise key results introduced in cited articles and to explain in detail new concepts of working with Fučík spectrum for the chosen discrete operator.



**Key words**: Fučík spectrum, difference operator, Dirichlet discrete operator, Chebyshev polynomial of the second kind, asymmetric nonlinearities

# Zusammenfassung

Diese Dissertation widmet sich dem Studium des Fučík Spektrum für diskrete Operatoren. Angesichts der Tatsache, dass das Problem der Untersuchung des Fučík Spektrums für allgemeine diskrete Operatoren immer noch eine große Herausforderung darstellt, konzentrieren wir uns in dieser Arbeit auf Analysen in Bezug auf einen bestimmten Operator – den diskreten Dirichlet-Operator.

Dieser Operator entspricht der Differenzengleichung zweiter Ordnung mit Dirichlet-Randbedingungen. In der Dissertation untersuchen wir ein entsprechendes semilineares Problem, definieren eine kontinuierliche Erweiterung einer diskreten Lösung und liefern schließlich eine vollständige implizite Beschreibung des Fučík Spektrums des diskreten Dirichlet-Operatoren. Nächst werden drei Bounds von Fučík Kurven beschrieben. Diese Bounds ermöglichen eine Lokalisierung von Fučík Kurven auch bei großen Dimensionen einer entsprechenden Matrix.

Der gesamte Text der Dissertation basiert auf zwei Artikeln der Autorin: [25], [31]. Das Hauptziel besteht darin, wichtige Ergebnisse aus zitierten Artikeln zu veranschaulichen und neue Konzepte der Arbeit mit Fučík spectrum für den gewählten diskreten Operator im Detail zu erklären.

**Schlüsselwörter**: Fučík Spektrum, Differenzenoperator, diskrete Dirichlet-Operator, Tschebyschow-Polynome zweiter Art, asymmetrische Nichtlinearitäten.

# Acknowledgements

I would like to express my sincere gratitude to my supervisor, doc. Ing. Gabriela Holubová, Ph.D., and to the supervisor specialist, Ing. Petr Nečesal, Ph.D.

This work was supported by the Grant Agency of the Czech Republic, grants no. 13-00863S and 18-03253S.

# Contents

**1 Introduction** **1**
1.1 Main definitions – Problems (P1), (P2), (P3), (P4) and matrix $\mathbf{A}^{\mathsf{D}}$ . . . . . . . . . 2
1.2 Typical challenges while investigating the Fučík spectrum for matrices . . . . . . . 4
1.3 Structure of the thesis . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 10
1.4 Abstracts of published research articles of the author . . . . . . . . . . . . . . . . 11

**2 Linear problems** (P1) **and** (P2) **13**
2.1 The continuous extension and its first non-negative zero . . . . . . . . . . . . . . . 14
2.2 Chebyshev polynomials of the second kind . . . . . . . . . . . . . . . . . . . . . . . 21
2.2.1 Relationship to the linear initial value problem (P1) . . . . . . . . . . . . . 21
2.2.2 Properties of the Chebyshev polynomials of the second kind . . . . . . . . . 21
2.3 Properties of function $W_k^\lambda$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 23
2.3.1 Relationship to the linear initial value problem (P1) . . . . . . . . . . . . . 24
2.4 Linear boundary value problem (P2) . . . . . . . . . . . . . . . . . . . . . . . . . . 26

**3 Semi-linear initial value problem** (P3) **– Part I** **28**
3.1 Positive and negative semi-waves . . . . . . . . . . . . . . . . . . . . . . . . . . . . 28
3.2 Relationship between function $W_k^\lambda$ and solution of a semi-linear problem (P3) . . . 29
3.3 Sequences $(p_k)$ and $(\vartheta_k)$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 31

**4 Semi-linear initial value problem** (P3) **– Part II** **35**
4.1 Length of a semi-wave . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 36
4.2 The distance $\rho_{\alpha,\beta}$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 37
4.3 Examples . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 40

**5 Investigation of Fučík spectrum of matrix $\mathbf{A}^{\mathsf{D}}$ – Semi-linear BVP** (P4) **44**
5.1 Description of Fučík spectrum – Theory from Part I . . . . . . . . . . . . . . . . . 44
5.2 Description of Fučík spectrum – Theory from Part II . . . . . . . . . . . . . . . . . 46
5.3 Comparison of descriptions of Fučík spectrum . . . . . . . . . . . . . . . . . . . . . 47
5.4 Basic bounds of Fučík curves and their consequence . . . . . . . . . . . . . . . . . 50
5.5 Delta bounds of Fučík curves . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 54
5.6 Rho bounds of Fučík curves . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 61
5.7 Comparison between delta and rho bounds . . . . . . . . . . . . . . . . . . . . . . 62

**6 Conclusion** **67**

**Bibliography** **69**

**Appendix – Figures**

**Appendix – Published research articles of the author**

**Co-author statement**

# Chapter 1

# Introduction

Svatopluk Fučík and other mathematicians studied solvability of a problem

$$-u''(x) = f(x, u(x)),$$

on some interval with various boundary conditions. Solvability of such problem with

$$f(\cdot, s) \sim \lambda s \text{ for } s \to \pm\infty$$

is dependent on the fact whether $\lambda$ is (or is not) an eigenvalue of the corresponding operator. Main results are due to S. Fučík [11] and E.N. Dancer [5] who considered a different asymptotic behaviour of $f$, in particular

$$f(\cdot, s) \sim \mu s \text{ for } s \to +\infty, \quad f(\cdot, s) \sim \nu s \text{ for } s \to -\infty.$$

Solvability of the problem can be answered using information about all pairs $(\mu, \nu) \in \mathbb{R}^2$ such that the following problem (together with corresponding boundary conditions)

$$-u''(x) - \mu u^+(x) + \nu u^-(x) = 0$$

has a non-trivial solution. Traditionally, a set of all such pairs is called the Fučík spectrum. For more information, see [8].

Fučík spectrum for discrete operators was investigated by R. Švarc (see e.g. [38], [40]). In [40], R. Švarc considered two particular square matrices of size 4 and gave a description of their Fučík spectra. These matrices were chosen in such a way that their Fučík spectra (even for small matrices of size four) exhibit rather strange behaviour.

Authors G. Holubová and P. Nečesal [17] discussed similarities of structures in Fučík spectra for continuous and discrete operators. They also suggested an algorithm for numerical reconstruction of the Fučík spectrum for reasonably small matrices. They focused on the case of all general real square matrices of size 2 and shown all feasible structures in their Fučík spectrum. They also suggested that there are more than 300 qualitatively different patterns of the Fučík spectrum even for matrices of size 3. This illustrates that the problem of finding Fučík spectra for general matrices is a significant challenge that has not been solved yet.

Various physical phenomena are represented by continuous initial or boundary value problems. Moreover, the theory of Fučík spectrum for these problems is applied in practice for analyses of (mechanical) systems with pronounced asymmetry / asymmetric structure. One of the typical examples are suspension bridges – explored in [22, 9, 15] and the book [13] with a focus on models with asymmetric nonlinearities. Also, asymmetric nonlinearities appear in the study of competing systems of species with large interactions in biology (see [4, 6, 27]) and the Fučík spectrum of the Dirichlet Laplacian (the Laplace operator $u \mapsto -\Delta u$ with zero Dirichlet boundary conditions) is needed (see [6] for details).

Hence we contemplate that the exploration of discrete problems might be useful for practical applications. Sometimes, even though the problem is naturally discrete, researchers tend to make a simplification and look at this as a continuous problem (such examples can be found e.g. in the area of mathematical finance). On the other hand, sometimes, due to complexity of the physical phenomena, researchers tend to use a discretization of the studied continuous problem. This way, one might obtain superior analytical results or a more suitable numerical solution. Thus, we conclude that discrete problems might be relevant for both continuous and discreet natural phenomena. We note that sometimes the discrete problem can be solved in a simpler way, but quite often the discrete structure of such problems can lead to specific difficulties which pose further challenges.

We are going to make a brief comparison of the Fučík spectrum for continuous and discrete operators. We will illustrate that discrete domain brings extra challenges in finding the Fučík spectrum and we will solve several challenges for a particular problem within this thesis and in the referenced articles of the author.

Let us also mention some other articles where the structure of Fučík spectrum is studied – [1, 2, 3, 7, 10, 16, 19, 20, 21, 23, 28, 30, 34, 35, 36].

In the following paragraph, we will recall a well known result for the Fučík spectrum of the continuous second order boundary value problem.

The Fučík spectrum $\Sigma$ for the continuous second order boundary value problem with Dirichlet boundary conditions, i.e.

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

is defined as the set

$$\Sigma := \left\{(\alpha,\beta) \in \mathbb{R}^2 : \text{ the problem (1.1) has a nontrivial solution } u\right\}.$$

The description of the set $\Sigma$ is well known. In fact, as shown in [11, 12], the Fučík spectrum $\Sigma$ consists of two lines $\mathbf{C}_0^{\pm} : (\alpha - \pi^2)(\beta - \pi^2) = 0$ and countably many curves $\mathbf{C}_l^{\pm}$ given by ($j \in \mathbb{N}$)

$$\mathbf{C}_{2j-1}^{\pm} : \frac{j\pi}{\sqrt{\alpha}} + \frac{j\pi}{\sqrt{\beta}} = 1, \quad \mathbf{C}_{2j}^{+} : \frac{(j+1)\pi}{\sqrt{\alpha}} + \frac{j\pi}{\sqrt{\beta}} = 1, \quad \mathbf{C}_{2j}^{-} : \frac{j\pi}{\sqrt{\alpha}} + \frac{(j+1)\pi}{\sqrt{\beta}} = 1.$$

On the other hand, investigating the Fučík spectrum for the corresponding discrete problem is a much more elaborate process to which we will devote remaining parts of the thesis.

## 1.1 Main definitions – Problems (P1), (P2), (P3), (P4) and matrix $\mathbf{A}^{\mathrm{D}}$

In this section, we will introduce main problems of our interest and several concepts associated with the studied problems.

**Studied problems**:

i. linear initial value problem

$$\begin{cases} \Delta^2 u(k-1) + \lambda u(k) = 0, \quad k \in \mathbb{Z}, \\ u(0) = C_0,\ u(1) = C_1, \end{cases} \tag{P1}$$

ii. linear boundary value problem

$$\begin{cases} \Delta^2 u(k-1) + \lambda u(k) = 0, \quad k \in \mathbb{T}, \\ u(0) = u(n+1) = 0, \end{cases} \tag{P2}$$

iii. semi-linear initial value problem

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, & k \in \mathbb{Z}, \\ u(0) = 0, \ u(1) = C_1, \end{cases} \tag{P3}$$

iv. semi-linear boundary value problem

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, & k \in \mathbb{T}, \\ u(0) = u(n+1) = 0, \end{cases} \tag{P4}$$

where $n \in \mathbb{N}$, $n \geq 2$, $\mathbb{T} = \{1, \dots, n\}$, $\hat{\mathbb{T}} = \{0, \dots, n+1\}$, $u : \hat{\mathbb{T}} \to \mathbb{R}$, $u^+, u^-$ stand for the positive and negative parts of $u$, i.e. $u^+(k) := \max\{+u(k), 0\}$, $u^-(k) := \max\{-u(k), 0\}$ and $\alpha, \beta, \lambda \in \mathbb{R}$. In case of problem (P1), $C_0, C_1 \in \mathbb{R}$ are constants such that $C_0^2 + C_1^2 \neq 0$. In case of problem (P3), $C_1 \in \mathbb{R} \setminus \{0\}$. The second order forward difference operator is given by $\Delta^2 u(k-1) := u(k-1) - 2u(k) + u(k+1)$.

In case of problem (P3), we consider $(\alpha, \beta) \in D = ((0,4) \times (0,+\infty)) \cup ((0,+\infty) \times (0,4))$ for Part I (Chapter 3) and $(\alpha, \beta) \in \mathcal{D} = (0,4) \times (0,+\infty)$ for Part II (Chapter 4).

1. **Sign property of a vector**

   Let us define a sign property of a vector $\mathbf{u} = [u_1, u_2, \dots, u_n]^T$ of size $n$ as

   $$\operatorname{sign} \mathbf{u} = [\operatorname{sign}(u_1), \operatorname{sign}(u_2), \dots, \operatorname{sign}(u_n)]^T$$

   and simplify the notation. For $x \in \mathbb{R}$

   $$\text{instead of } \operatorname{sign}(x) = \begin{cases} 1 & \text{for } x > 0, \\ -1 & \text{for } x < 0, \\ 0 & \text{for } x = 0, \end{cases} \quad \text{we denote } \operatorname{sign}(x) = \begin{cases} + & \text{for } x > 0, \\ - & \text{for } x < 0, \\ 0 & \text{for } x = 0. \end{cases}$$

2. **Positive and negative part of a vector**

   For vector $\mathbf{u}$ of size $n$, $n \in \mathbb{N}$, $\mathbf{u} = [u(1), \dots, u(n)]^T$, we define its positive part $\mathbf{u}^+ := [u^+(1), \dots, u^+(n)]^T$, and its negative part $\mathbf{u}^- := [u^-(1), \dots, u^-(n)]^T$ (see Figure 1.1).

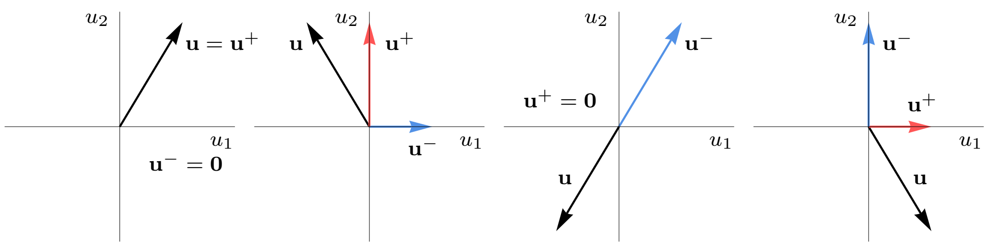



Figure 1.1: Illustration of positive $\mathbf{u}^+$ (red) and negative $\mathbf{u}^-$ (blue) part of vector $\mathbf{u} = [u_1, u_2]^T$. In this particular case, we assume $n = 2$.

3. **The Fučík spectrum of a matrix**

   The Fučík spectrum of a real square matrix $\mathbf{B}$ of size $n \times n$, $n \in \mathbb{N}$, $n \geq 2$, is the set:

   $$\Sigma(\mathbf{B}) = \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \text{ the problem } \mathbf{B}\mathbf{u} = \alpha \mathbf{u}^+ - \beta \mathbf{u}^- \text{ has a non-trivial solution } \mathbf{u} \right\}. \tag{1.2}$$

   The pair $(\alpha, \beta) \in \Sigma(\mathbf{B})$ is called the Fučík eigenpair and the non-trivial solution $\mathbf{u}$ is called the Fučík eigenvector for the matrix $\mathbf{B}$.

4. **The Dirichlet matrix**

   Matrix $\mathbf{A}^{\mathsf{D}}$ is called the Dirichlet matrix and will be used throughout the thesis:

$$\mathbf{A}^{\mathsf{D}} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}. \tag{1.3}$$

5. **Fučík curves**

   For Fučík spectrum $\Sigma(\mathbf{A}^{\mathsf{D}})$, where Dirichlet matrix $\mathbf{A}^{\mathsf{D}}$ is of size $n \times n$ (we are going to see the relationship between matrix $\mathbf{A}^{\mathsf{D}}$ and semi-linear boundary value problem (P4) further in the text), we define Fučík curves $\mathcal{C}_l^+$, $\mathcal{C}_l^-$, $l = 0, \dots, n-1$ as (the term of generalized zero is defined in Definition 5)

$$\mathcal{C}_l^+ := \big\{(\alpha, \beta) \in \mathbb{R}^2 : \quad \text{the problem (P4) has a non-trivial solution } u \\ \text{with exactly } l \text{ generalized zeros on } \mathbb{T} \text{ and } u(1) > 0\big\},$$

$$\mathcal{C}_l^- := \big\{(\alpha, \beta) \in \mathbb{R}^2 : \quad \text{the problem (P4) has a non-trivial solution } u \\ \text{with exactly } l \text{ generalized zeros on } \mathbb{T} \text{ and } u(1) < 0\big\},$$

   which we jointly denote by the following simplified notation:

$$\mathcal{C}_l^{\pm} := \mathcal{C}_l^+ \cup \mathcal{C}_l^-.$$

## 1.2 Typical challenges while investigating the Fučík spectrum for matrices

Having in mind that investigating the Fučík spetrum for general matrices is at this time unsolved as far as we know, we specify a particular matrix which comes from the discretization of the continuous problem (1.1) (which has also practical applications, see [25] and [31]). We consider the following discrete problem with Dirichlet boundary conditions (P4)

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, \quad k \in \mathbb{T}, \\ u(0) = u(n+1) = 0, \end{cases}$$

where $n \in \mathbb{N}$, $n \geq 2$ and $\alpha, \beta \in \mathbb{R}$.

Equivalently, the problem (P4) can be rephrased using a matrix notation

$$\mathbf{A}^{\mathsf{D}}\mathbf{u} = \alpha \mathbf{u}^+ - \beta \mathbf{u}^-,$$

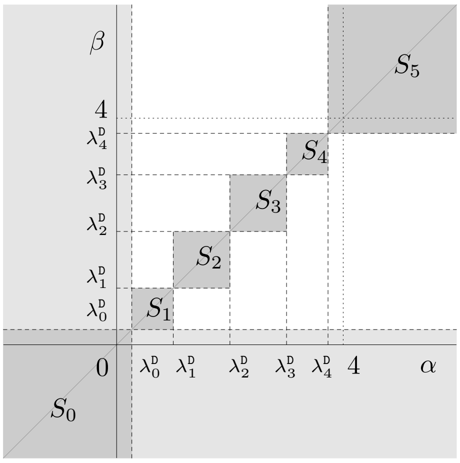
where matrix $\mathbf{A}^{\mathsf{D}}$ is the Dirichlet matrix (1.3) and $\mathbf{u} = [u(1), \dots, u(n)]^T$, $\mathbf{u}^+ = [u^+(1), \dots, u^+(n)]^T$, $\mathbf{u}^- = [u^-(1), \dots, u^-(n)]^T$.

In particular, studying the set of all pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that the problem (P4) has a non-trivial solution $u$, is equivalent to the investigation of the set $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$

$$\Sigma\left(\mathbf{A}^{\mathsf{D}}\right) = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \text{ the problem } \mathbf{A}^{\mathsf{D}}\mathbf{u} = \alpha\mathbf{u}^+ - \beta\mathbf{u}^- \text{ has a non-trivial solution } \mathbf{u}\right\},$$

and similarly to the general notation within this thesis, $\Sigma(\mathbf{A}^{\mathsf{D}})$ is called the Fučík spectrum of matrix $\mathbf{A}^{\mathsf{D}}$. To find the set $\Sigma(\mathbf{A}^{\mathsf{D}})$ will be the main purpose of our investigation.

Let us point out that Fučík spectrum is symmetric with respect to the line $\alpha = \beta$, i.e. $(\alpha, \beta) \in \Sigma(\mathbf{A}^{\mathsf{D}})$ with Fučík eigenvector $\mathbf{v}$ if and only if $(\beta, \alpha) \in \Sigma(\mathbf{A}^{\mathsf{D}})$ with Fučík eigenvector $-\mathbf{v}$ (see



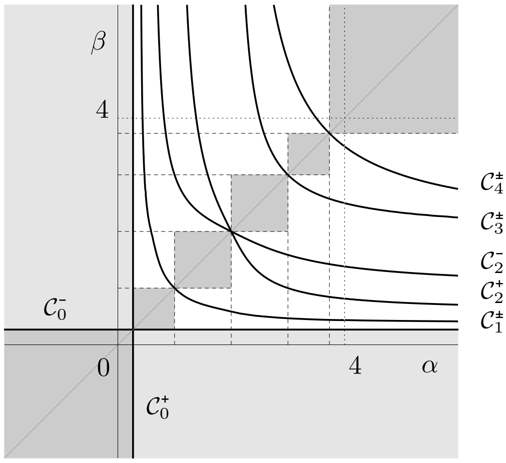




Figure 1.2: Inadmissible areas (defined in [18]) for the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$ (left, $n = 5$) and the particular Fučík curves $\mathcal{C}_k^\pm$ (black curves) of the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$ (right, $n = 5$).

Figures 1.4 and 1.5). Before diving into particular challenges, let us recall some known results about $\Sigma(\mathbf{A}^{\mathsf{D}})$ (for more details, see [25] and [31]). The eigenvalues of $\mathbf{A}^{\mathsf{D}}$ are of the form

$$\lambda_j^{\mathsf{D}} = 4\sin^2\frac{(j+1)\pi}{2(n+1)}, \qquad j = 0, \dots, n-1$$

and $\lambda_j^{\mathsf{D}} \in (0, 4)$. Note that the eigenvalues $\lambda_j^{\mathsf{D}}$ of matrix $\mathbf{A}^{\mathsf{D}}$ belong to the Fučík spectrum in the sense $(\lambda_j^{\mathsf{D}}, \lambda_j^{\mathsf{D}}) \in \Sigma(\mathbf{A}^{\mathsf{D}})$, i.e. $(\lambda_j^{\mathsf{D}}, \lambda_j^{\mathsf{D}})$ is the Fučík eigenpair for matrix $\mathbf{A}^{\mathsf{D}}$. For the Fučík spectrum of $\mathbf{A}^{\mathsf{D}}$ we have

$$\Sigma\left(\mathbf{A}^{\mathsf{D}}\right) = \bigcup_{l=0}^{n-1} \mathcal{C}_l^\pm,$$

where $\mathcal{C}_l^+$ and $\mathcal{C}_l^-$ are Fučík curves (see Section 1.1 – point 5).

In [18], authors were exploring inadmissible areas of Fučík spectrum (i.e. Fučík spectrum has empty intersection with these areas in $(\alpha, \beta)$ plane – see [18] for proper definition of an inadmissible area). Since $\lambda_0^{\mathsf{D}}$ is a principal eigenvalue of $\mathbf{A}^{\mathsf{D}}$, it implies that

$$\left\{(\alpha, \beta) \in \mathbb{R}^2 : \left(\alpha - \lambda_0^{\mathsf{D}}\right)\left(\beta - \lambda_0^{\mathsf{D}}\right) < 0\right\} \cap \Sigma\left(\mathbf{A}^{\mathsf{D}}\right) = \emptyset,$$

i.e. both shifted quadrants are inadmissible areas for the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$. For illustration, see Figure 1.2 where we can see inadmissible areas for the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$. Thus, it is enough to investigate the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$ only on the set $D = ((0,4)\times(0,+\infty))\cup((0,+\infty)\times(0,4))$.

Also, it is enough to investigate only all Fučík curves $\mathcal{C}_l^+$ $(l = 1, \dots, n-1)$, since

$$\mathcal{C}_l^- = \left\{(\alpha, \beta) \in D : (\beta, \alpha) \in \mathcal{C}_l^+\right\}.$$

Authors Ma, Xu and Gao introduced the matching-extension method for solutions of the Fučík spectrum problem for matrix $\mathbf{A}^{\mathsf{D}}$ in [26]. P. Stehlík studied the qualitative properties of the first non-trivial Fučík curve of the matrix $\mathbf{A}^{\mathsf{D}}$ in [37]. Although this topic was studied by several authors, corresponding analytic description was not introduced prior to author's articles [25] and [31] (as far as we know).

Before looking into individual results, we contemplate what possible challenges can appear while investigating the Fučík spectrum for matrix $\mathbf{A}^{\mathsf{D}}$, using illustrative examples. In Example 1 we will investigate the Fučík spectrum of matrix $\mathbf{A}^{\mathsf{D}}$ of size $n = 2$.

**Example 1.** Let $n = 2$, thus let us deal with the Dirichlet matrix in the form

$$\mathbf{A}^{\mathtt{D}} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Eigenvalues and corresponding eigenvectors are of the form

$$\lambda_0^{\mathtt{D}} = 1,\ \mathbf{v}_0 = [1,1]^T, \quad \lambda_1^{\mathtt{D}} = 3,\ \mathbf{v}_1 = [1,-1]^T.$$

All possible sign properties for Fučík eigenvectors are

$$[+,+]^T, [+,-]^T, [+,0]^T, [-,+]^T, [-,-]^T, [-,0]^T, [0,+]^T, [0,-]^T, [0,0]^T.$$

Similar to the case of eigenvalue problems, the sign properties $[+,+]^T$ and $[-,-]^T$ lead to the Fučík eigenvectors where we have opposite signs of the entries. The same works for couples $[+,-]^T$ and $[-,+]^T$, for $[+,0]^T$ and $[-,0]^T$ and for $[0,+]^T$ and $[0,-]^T$. Thus, it is enough to consider only $[+,+]^T$, $[+,-]^T$, $[+,0]^T$, $[0,+]^T$ and $[0,0]^T$.

1. Case $[0,0]^T$: Such case cannot happen since the Fučík eigenvector cannot be trivial.
2. Case $[0,+]^T$: The first entry of the Fučík eigenvector is zero, thus the solution of problem (P4) is zero in two consequential points (due to the zero boundary conditions). The difference equation in (P4) can be written as

   $$u(k+1) = 2u(k) - u(k-1) - \alpha u^+(k) + \beta u^-(k),$$

   thus if the solution $u$ is zero in two consequential points, it has to be zero everywhere. That is a contradiction with the sign property $[0,+]^T$.
3. Case $[+,0]^T$: There is the same issue as in the previous case.
4. Case $[+,+]^T$: In this case the Fučík eigenvector does not change sign thus it is equivalent to the eigenvalue problem for $\lambda_0^{\mathtt{D}}$. We have $(\alpha,\beta) \in \Sigma(\mathbf{A}^{\mathtt{D}}) : \alpha = \lambda_0^{\mathtt{D}} = 1,\ \beta \in \mathbb{R}$ with Fučík eigenvector $[1,1]^T$ and $(\alpha,\beta) \in \Sigma(\mathbf{A}^{\mathtt{D}}) : \beta = \lambda_0^{\mathtt{D}} = 1,\ \alpha \in \mathbb{R}$ with Fučík eigenvector $[-1,-1]^T$. The Fučík curves $\mathcal{C}_0^{\pm}$ are trivial ones

   $$\mathcal{C}_0^{+} = \left\{(\alpha,\beta) :\ \alpha = \lambda_0^{\mathtt{D}},\ \beta \in \mathbb{R}\right\}, \qquad \mathcal{C}_0^{-} = \left\{(\alpha,\beta) :\ \beta = \lambda_0^{\mathtt{D}},\ \alpha \in \mathbb{R}\right\}.$$

5. Case $[+,-]^T$: From the sign property of vector $\mathbf{u} = [u_1,u_2]^T$ we have $\mathbf{u}^+ = [u_1,0]^T$ and $\mathbf{u}^- = [0,-u_2]^T$, where

   $$u_1 > 0 \text{ and } u_2 < 0. \tag{1.4}$$

   We can rewrite the problem $\mathbf{A}^{\mathtt{D}}\mathbf{u} = \alpha\mathbf{u}^+ - \beta\mathbf{u}^-$ as

   $$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \alpha\begin{bmatrix} u_1 \\ 0 \end{bmatrix} - \beta\begin{bmatrix} 0 \\ -u_2 \end{bmatrix}. \tag{1.5}$$

   Matrix equation in (1.5) is equivalent to

   $$\begin{bmatrix} 2-\alpha & -1 \\ -1 & 2-\beta \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \det\begin{bmatrix} 2-\alpha & -1 \\ -1 & 2-\beta \end{bmatrix} = 0. \tag{1.6}$$

   The determinant in (1.6) is zero if

   $$(2-\alpha)(2-\beta) - 1 = 0. \tag{1.7}$$

   This leads to

   $$\beta = 2 - \frac{1}{2-\alpha},\ \mathbf{u} = \left[-\frac{1}{2-\alpha}, -1\right]^T.$$

Let us go back to the sign property in (1.4). It is satisfied when

$$-\frac{1}{2-\alpha} > 0 \;\Leftrightarrow\; \alpha > 2.$$

If we would consider sign property $\operatorname{sign} \mathbf{u} = [-,+]^T$ we would get the same result, thus the Fučík curves $\mathcal{C}_1^{\pm}$ are

$$\mathcal{C}_1^{+} = \mathcal{C}_1^{-} = \left\{(\alpha,\beta):\ \beta = 2 - \frac{1}{2-\alpha},\ \alpha > 2\right\}.$$

While going through all possible sign properties for the Fučík eigenvectors we were able to find complete description of the Fučík spectrum of matrix $\mathbf{A}^{\mathsf{D}}$ of size $n = 2$ as

$$\Sigma(\mathbf{A}^{\mathsf{D}}) = \mathcal{C}_0^{\pm} \cup \mathcal{C}_1^{\pm},$$

where $\mathcal{C}_0^{\pm}$ and $\mathcal{C}_1^{\pm}$ are given as above. See Figure 1.3 for illustration of this example.

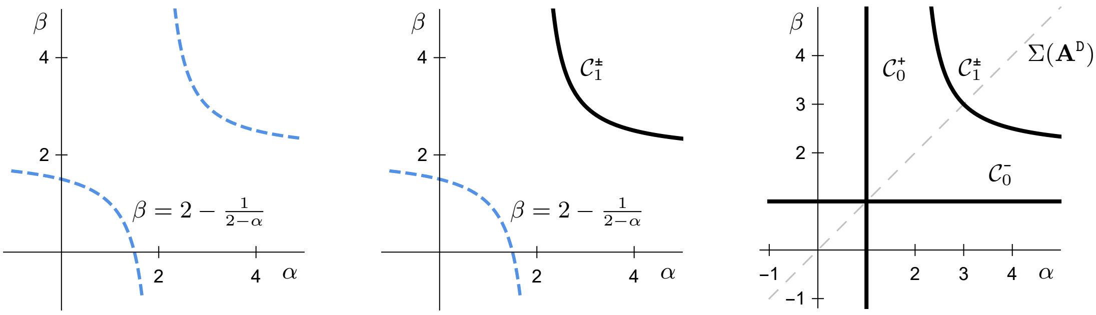



Figure 1.3: Graph of the function $\beta = 2 - \frac{1}{2-\alpha}$ (left), Fučík curves $\mathcal{C}_1^{\pm}$ (black) as part of the graph of the function $\beta = 2 - \frac{1}{2-\alpha}$ (middle) and the Fučík spectrum $\Sigma(\mathbf{A}^{\mathsf{D}})$ of matrix $\mathbf{A}^{\mathsf{D}}$ for $n = 2$.

In the following example we will consider $n = 6$, to illustrate a dimension complexity of the problem.

**Example 2.** In this example we will consider matrix $\mathbf{A}^{\mathsf{D}}$ of size $n = 6$. We will show all the possible sign properties for the Fučík eigenvectors. It is enough to investigate sign properties with positive first entries (since the Fučík spectrum is symmetric with respect to the line $\alpha = \beta$ and the Fučík eigenvectors have opposite signs). Also, for the sake of simplicity, we can investigate sign properties $\operatorname{sign}(u(k)) = 0$ and $\operatorname{sign}(u(k)) = 1$ (for some $k \in \{1, 2, \dots, n\}$) together. After this simplification, we need to investigate $2^{n-1}$ different sign properties.

All sign properties which we need to investigate are written in Table 1.1. Each column has 6 entries and represents one sign property for vector. Those sign properties which are in blue color are sign properties which at least one of the Fučík eigenvectors has (in this thesis it will be shown how to select the right ones).

To illustrate a curse of dimensionality of the studied problems, let us compare two cases of matrix $\mathbf{A}^{\mathsf{D}}$ dimension: $n_1 = 2$ and $n_2 = 6$. Within Example 1, we have shown that we need to investigate only 2 cases or more generally $2^{n_1-1}$ cases. However, in this example we are solving $2^{n_2-1} = 2^5 = 32$ different sign properties, each leads towards investigation of a different eigenvalue / eigenvector problem.

In particular, let us take one of the sign properties: $[+,+,+,-,+,-]^T$. For this sign property

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + |
| + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + |
| + | + | + | + | + | + | + | + | − | − | − | − | − | − | − | − |
| + | + | + | + | − | − | − | − | + | + | + | + | − | − | − | − |
| + | + | − | − | + | + | − | − | + | + | − | − | + | + | − | − |
| + | − | + | − | + | − | + | − | + | − | + | − | + | − | + | − |

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + |
| − | − | − | − | − | − | − | − | − | − | − | − | − | − | − | − |
| + | + | + | + | + | + | + | + | − | − | − | − | − | − | − | − |
| + | + | + | + | − | − | − | − | + | + | + | + | − | − | − | − |
| + | + | − | − | + | + | − | − | + | + | − | − | + | + | − | − |
| + | − | + | − | + | − | + | − | + | − | + | − | + | − | + | − |

Table 1.1: Considered sign properties for $n = 6$. The blue sign properties are sign properties satisfied by some Fučík eigenvectors.

we need to solve

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \alpha \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \\ u_5 \\ 0 \end{bmatrix} - \beta \begin{bmatrix} 0 \\ 0 \\ 0 \\ -u_4 \\ 0 \\ -u_6 \end{bmatrix}.$$

This leads to the determinant equation

$$\det(\mathbf{A}^{\mathtt{D}} - \mathbf{\Lambda}) = \det \left( \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \end{bmatrix} \right) = 0, \quad (1.8)$$

i.e.

$$\det \begin{bmatrix} 2-\alpha & -1 & 0 & 0 & 0 & 0 \\ -1 & 2-\alpha & -1 & 0 & 0 & 0 \\ 0 & -1 & 2-\alpha & -1 & 0 & 0 \\ 0 & 0 & -1 & 2-\beta & -1 & 0 \\ 0 & 0 & 0 & -1 & 2-\alpha & -1 \\ 0 & 0 & 0 & 0 & -1 & 2-\beta \end{bmatrix} = 0.$$

Since we are dealing with tridiagonal matrix, we can easily calculate its determinant and the determinant equation is a polynomial equation

$$\alpha^4\beta^2 - 4\alpha^4\beta + 4\alpha^4 - 8\alpha^3\beta^2 + 29\alpha^3\beta - 26\alpha^3 + 22\alpha^2\beta^2 - 70\alpha^2\beta + 53\alpha^2 - 24\alpha\beta^2$$
$$+65\alpha\beta - 38\alpha + 8\beta^2 - 18\beta + 7 = 0.$$

By comparing this with (1.7) (for $n = 2$), we can see that the dimension of the problem brings a lot of difficulties. We can find two values of $\beta$ (dependent of the value of $\alpha$ as it was done in Example 1) for which we can derive that neither one of them has a corresponding eigenvector with the sign property $[+,+,+,-,+,-]^T$. That means that there does not exist Fučík eigenvector for matrix $\mathbf{A}^{\mathtt{D}}$ of size $n = 6$ with such sign property.

Since this problem depends highly on the dimension of the matrix $\mathbf{A}^{\mathtt{D}}$ (we are dealing with $2^{n-1}$ different eigenvalue / eigenvector problems based on the number of possible sign properties), our computational possibilities might be limiting for practical applications using the illustrated approach [1].

Let us summarize some of the challenges which appear in the investigation of the Fučík spectrum $\Sigma(\mathbf{A}^{\mathsf{D}})$ of matrix $\mathbf{A}^{\mathsf{D}}$ of size $n$:

- Number of possible sign properties is $2^{n-1}$ (after the simplification which was done in Example 2).
- Only some of them are sign properties satisfied by Fučík eigenvectors of $\Sigma(\mathbf{A}^{\mathsf{D}})$.
- We note that for a general matrix, one might struggle with computation of the matrix determinant. Whereas for the Dirichlet matrix, $\det(\mathbf{A}^{\mathsf{D}} - \mathbf{\Lambda})$ (see (1.8) in Example 2) can be calculated recurrently (due to having a tridiagonal symmetric matrix).
- For each sign property we need to verify which parts (if any) of the solution (curve) of $\det(\mathbf{A}^{\mathsf{D}} - \mathbf{\Lambda}) = 0$ are actually in the Fučík spectrum $\Sigma(\mathbf{A}^{\mathsf{D}})$.

On Figure 1.4 we can see the Fučík spectrum $\Sigma(\mathbf{A}^{\mathsf{D}})$ of size $n = 9$. In this thesis, we will introduce how to deal with the curse of dimensionality and other challenges mentioned above.

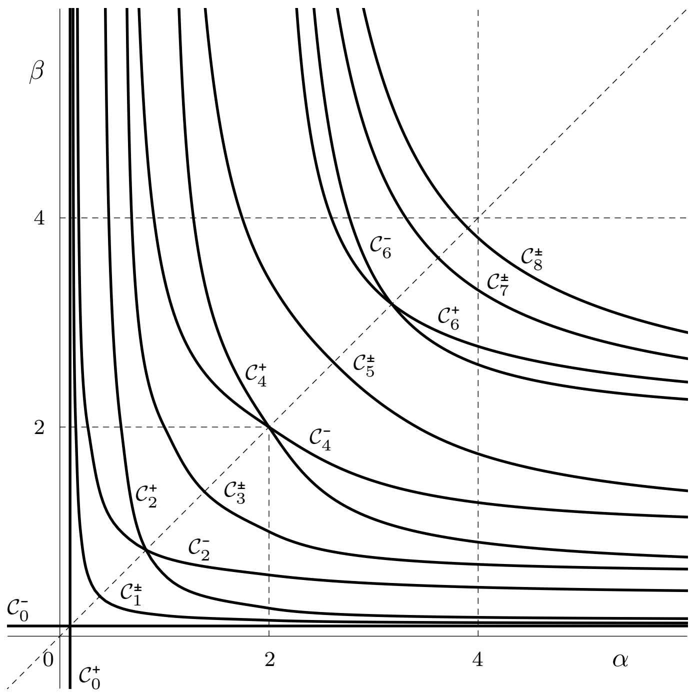



Figure 1.4: The Fučík spectrum $\Sigma(\mathbf{A}^{\mathsf{D}})$ of the Dirichlet matrix $\mathbf{A}^{\mathsf{D}}$ of size $n = 9$ and its Fučík curves $\mathcal{C}_l^\pm$, $l = 0, 1, \ldots, 8$.

[1] E.g. in a relatively reasonable time we might be able to find (numerically) Fučík spectrum up to $n = 16$.

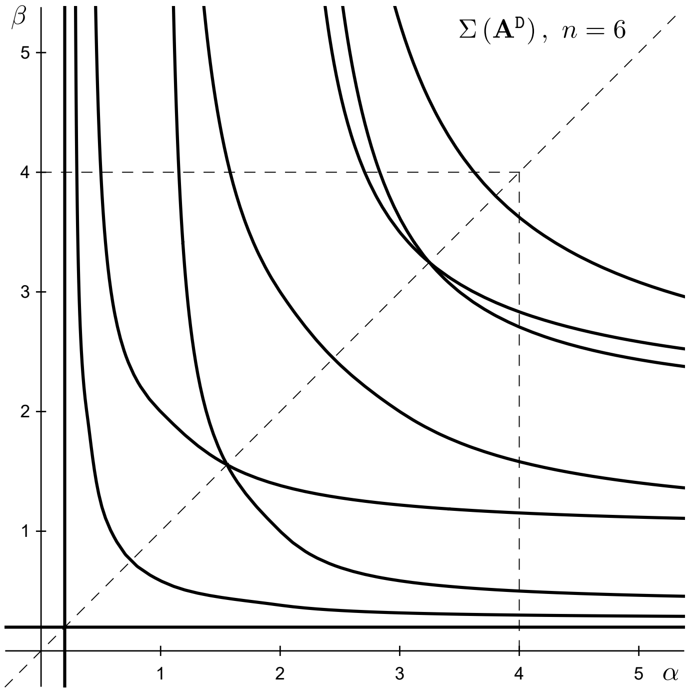



Figure 1.5: The Fučík spectrum $\Sigma(\mathbf{A}^{\mathtt{D}})$ of the Dirichlet matrix $\mathbf{A}^{\mathtt{D}}$ of size $n = 6$.

## 1.3 Structure of the thesis

First of all, we would like to note that this thesis is mainly based on research articles of the author: [25] and [31]. The aim of the thesis is not to provide in-depth technical details for all newly introduced concepts in [25] and [31], but rather to provide a comprehensive overview, illustrate the concepts on particular examples and to explain connections between individual concepts.

We note that in order to provide such comprehensive text, we also extend the results in aforementioned articles by supporting lemmas / theorems, new illustrations / examples and other new results. However, for most of the proofs of original theorems and lemmas we refer the reader to the articles which are attached to the thesis. If there is no citation (excluding citation to [25] and [31] – articles of the author) in the definitions, lemmas and theorems, then the results presented there are original and (as far as we know) not published anywhere else. We note that results in Section 5.5 are completely new and not published anywhere yet. The thesis is organized as follows.

Chapter 1 provides an introduction to the problems and showcases possible issues which may appear while investigating discrete Fučík spectrum. In the following chapters, we are going to investigate in detail four problems, introduced in Section 1.1:

i. linear initial value problem (P1);

ii. linear boundary value problem (P2);

iii. semi-linear initial value problem (P3);

iv. semi-linear boundary value problem (P4).

Chapter 2 is devoted to the study of linear problems (P1) and (P2). We are going to define one of the most important tool-kits in this thesis – the continuous extension of respective solutions. Exploring such continuous extension will allow us to explore nodal properties of the solution. A generalization of this result will be very valuable in the analysis of semi-linear problems. Let us note that even though we are spending a substantial part of this thesis (and likewise a substantial part of research articles [25] and [31]) studying simple linear problems (P1) and (P2), the results in this chapter are new and (as far as we know) not published anywhere. We need to construct a robust theory for the linear case in order to explore semi-linear case.

In Chapters 3 and 4, we are solving and investigating semi-linear initial value problem (P3). Generalizing the theory from the linear case (such as continuous extension) will allow us to "anchor" positive and negative semi-waves. This will lead to the detailed investigation of zeros of a continuous extension of the solution. Chapter 3 leverages the main results from [25] and Chapter 4 references results from [31].

Finally, Chapter 5 is devoted to the investigation of the Fučík spectrum of matrix $\mathbf{A}^{\mathsf{D}}$ (i.e. the corresponding semi-linear boundary value problem (P4)) – which is our main goal in this thesis (and in the research articles [25] and [31]). Several descriptions of the Fučík spectrum (analytical and implicit) are introduced. As far as we know, this is the first time anyone was able to find an analytical (and implicit) description of Fučík spectrum of matrix (excluding trivial cases) for any dimension $n$. In Chapter 5, we also introduce bounds of the Fučík spectrum. Such bounds can be used for efficient numerical estimations as illustrated therein.

Last but not least, we provide published articles [25] and [31]. Introduction sections in both articles describe historical references related to the Fučík spectrum and also our motivation for studying this topic in detail (including more details about practical applications).

## 1.4 Abstracts of published research articles of the author

### Research articles in impacted journals:

### Abstract of [25]:

I. Looseová (Sobotková), P. Nečesal, The Fučík spectrum of the discrete Dirichlet operator, Linear Algebra Appl. 553 (2018) 58–103

In this paper, we deal with the discrete Dirichlet operator of the second order and we investigate its Fučík spectrum, which consists of a finite number of algebraic curves. For each non-trivial Fučík curve, we are able to detect a finite number of its points, which are given explicitely. We provide the exact implicit description of all non-trivial Fučík curves in terms of Chebyshev polynomials of the second kind. Moreover, for each non-trivial Fučík curve, we give several different implicit descriptions, which differ in the level of depth of used nested functions. Our approach is based on the Möbius transformation and on the appropriate continuous extension of solutions of the discrete problem. Let us note that all presented descriptions of Fučík curves have the form of necessary and sufficient conditions. Finally, our approach can be also directly used in the case of difference operators of the second order with other local boundary conditions.

This article was published in Linear Algebra and Its Applications (Elsevier). For 2020, it has impact factor 1.401, cite score 2.1 and it belongs to Q1 in "Algebra and Number Theory" and "Discrete Mathematics and Combinatorics" fields of Mathematics.

### Abstract of [31]:

P. Nečesal, I. Sobotková, Localization of Fučík curves for the second order discrete Dirichlet operator, Bulletin des Sciences Mathématiques 171 (2021) 103014

In this paper, we deal with the second order difference equation with asymmetric nonlinearities on the integer lattice and we investigate the distribution of zeros of continuous extensions of positive semi-waves. The distance between two consecutive zeros of two different positive semi-waves depends not only on the parameters of the problem but also on the position of one of these zeros with respect to the integer lattice. We provide an explicit formula for this distance, which allows us to obtain a new simple implicit description of all non-trivial Fučík curves for the discrete Dirichlet operator. Moreover, for fixed parameters of the problem, we show that this distance is bounded and attains its global extrema that are explicitly described in terms of Chebyshev polynomials of the second kind. Finally, for each non-trivial Fučík curve, we provide suitable bounds by two curves with a simple description similar to the description of the first non-trivial Fučík curve.

This article was published in Bulletin des Sciences Mathématiques (Elsevier). For 2020, it has impact factor 1.118, cite score 1.6 and it belongs to Q1 in “Mathematics (miscellaneous)” field.

## Other activities:

### Abstract of [24] in Proceedings:

I. Looseová (Sobotková), Conjecture on Fučík curve asymptotes for a particular discrete operator, in: S. Pinelas, T. Caraballo, P. Kloeden, J. R. Graef (eds.), Differential and Difference Equations with Applications, Springer International Publishing, Cham, 2018

In this paper we study properties of the Neumann discrete problem. We investigate so called polar Pareto spectrum of a specific matrix which represents the Neumann discrete operator. There is a known relation between polar Pareto spectrum of any discrete operator and its Fučík spectrum. We also state a conjecture about asymptotes of Fučík curves with respect to the matrix and we illustrate a variety of polar Pareto eigenvectors corresponding to a fixed polar Pareto eigenvalue.

### Conferences:

1. Equadiff 13, Praha, 26.-30.8.2013, The asymptotes of Fučík curves for asymmetric difference operator
2. XXIX Seminar in Differential Equations, Monínec, 14.-18.4.2014, Properties of the Fučík spectrum for difference operator
3. Setkání studentů matematické analýzy a diferenciálních rovnic, Praha 2016, The Fučík spectrum of the Neumann discrete operator
4. XXX Seminar in Differential Equations, Ostrov u Tisé, 30. 5. – 3. 6. 2016, The Fučík spectrum of the second order discrete operators
5. International Conference on Differential & Difference Equations and Applications 2017, Amadora, Portugal, 5. 6. – 9. 6. 2017, The Fučík spectrum of the discrete Dirichlet operator

# Chapter 2

# Linear problems (P1) and (P2)

Let us consider the linear initial value problem (P1)

$$\begin{cases} \Delta^2 u(k-1) + \lambda u(k) = 0, \quad k \in \mathbb{Z}, \\ u(0) = C_0, u(1) = C_1, \end{cases}$$

where $\lambda \in \mathbb{R}$ and $C_0, C_1 \in \mathbb{R}$ are constants such that $C_0^2 + C_1^2 \neq 0$.

This problem is the easiest one to solve (considering all problems (P1), (P2), (P3) and (P4)). Yet, a complete understanding of how one can get the solution and what are the properties of such a solution, leads to valuable knowledge and tools for further study of more difficult problems such as linear boundary value problem (P2), semi-linear initial value problem (P3) and even semi-linear boundary value problem (P4).

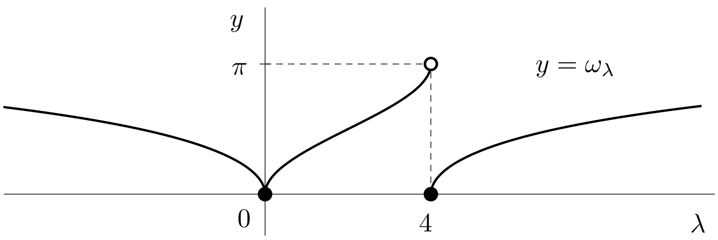



Figure 2.1: The graph of $\omega_\lambda$ as a function of $\lambda$.

The following lemma is used to find a solution of linear initial value problem (P1) which will be also utilized later on for more complex problems.

**Lemma 3.** *([25], Lemma 1, p. 66)*
*For given $\lambda \in \mathbb{R}$ and $C_0, C_1 \in \mathbb{R}$, the linear initial value problem* (P1) *has a unique solution of the form*

$$u(k) = C_0 F^\lambda(1-k) + C_1 F^\lambda(k), \quad k \in \mathbb{Z},$$

*where the function $F^\lambda : \mathbb{R} \to \mathbb{R}$ is defined as*

$$F^\lambda(t) := \begin{cases} \sinh(\omega_\lambda t)/\sinh\omega_\lambda & \textit{for } \lambda < 0, \\ t & \textit{for } \lambda = 0, \\ \sin(\omega_\lambda t)/\sin\omega_\lambda & \textit{for } \lambda \in (0,4), \\ -t\cos(\pi t) & \textit{for } \lambda = 4, \\ -\cos(\pi t)\sinh(\omega_\lambda t)/\sinh\omega_\lambda & \textit{for } \lambda > 4, \end{cases} \quad \omega_\lambda := \begin{cases} \operatorname{arcosh}\frac{2-\lambda}{2} & \textit{for } \lambda \le 0, \\ \arccos\frac{2-\lambda}{2} & \textit{for } \lambda \in (0,4), \\ \operatorname{arcosh}\frac{\lambda-2}{2} & \textit{for } \lambda \ge 4. \end{cases}$$

## 2.1 The continuous extension and its first non-negative zero

For the solution $u$ of the discrete problem (P1), let us define its continuous extension $u^{\mathsf{c}}$ on $\mathbb{R}$ as

$$u^{\mathsf{c}}(t) := C_0 F^{\lambda}(1-t) + C_1 F^{\lambda}(t), \quad t \in \mathbb{R}.$$

The continuous extension $u^{\mathsf{c}}$ builds on Lemma 3, but this time we extend the variable to the whole real axis. Continuous extension is the main tool which we will use while investigating properties of solution $u$ of initial value problem (P1). On Figures 2.2, 2.4, 2.5 and 2.6 we can see continuous extensions $u^{\mathsf{c}}$ of solutions $u$ of initial value problem (P1) for four different values of $\lambda$ (solution has a different form based on the value of $\lambda$ – see Lemma 3 in which we distinguish between $\lambda < 0$, $\lambda = 0$, $\lambda \in (0,4)$, $\lambda = 4$ and $\lambda \geq 4$). And on the Figure 2.3, we can see which role the length $\frac{\pi}{\omega_\lambda}$ has with regards to the continuous extension $u^{\mathsf{c}}$ of a solution $u$ for special case when $\lambda \in (0,4)$.

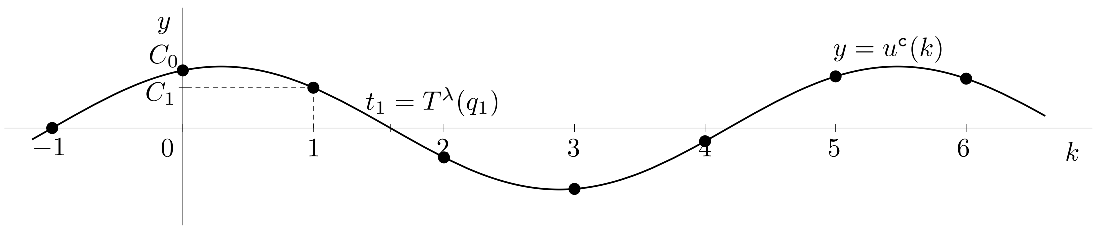



Figure 2.2: Continuous extension $u^{\mathsf{c}}$ of solution $u$ of the initial value problem (P1) for $\lambda \in (0,4)$, $\lambda = 1.3$ and the first non-negative zero $t_1$ of $u^{\mathsf{c}}$; $q_1 = \frac{C_1}{C_0}$ (defined in Definition 4).

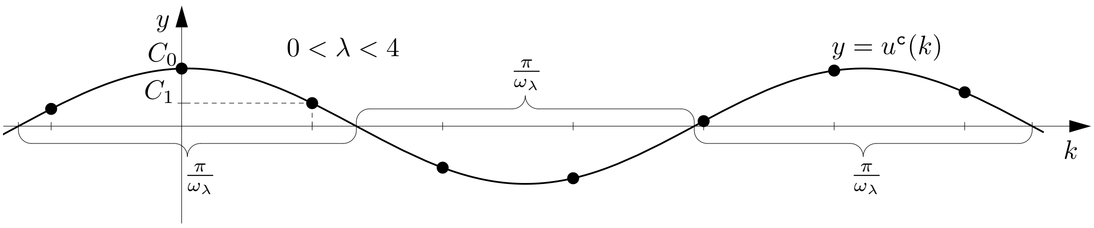



Figure 2.3: Continuous extension $u^{\mathsf{c}}$ of solution $u$ of the initial value problem (P1) for $\lambda \in (0,4)$, $\lambda = 1.3$ and meaning of value $\frac{\pi}{\omega_\lambda}$ as the length of continuous extension's "half-wave".

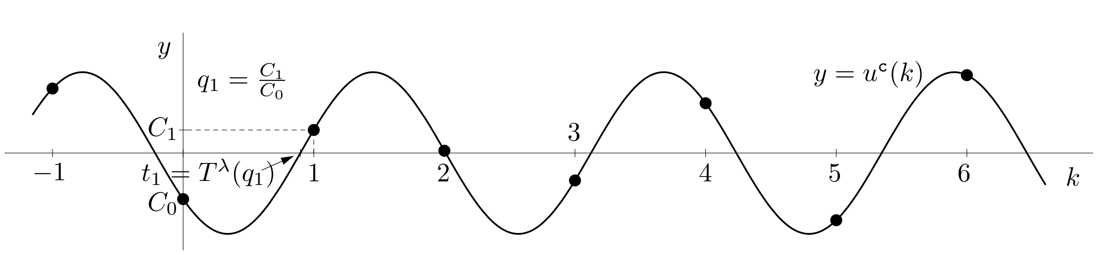



Figure 2.4: Continuous extension $u^{\mathsf{c}}$ of solution $u$ of the initial value problem (P1) for $\lambda \in (0,4)$, $\lambda = 3.9$ and the first non-negative zero $t_1$ of $u^{\mathsf{c}}$.

One of the properties which we are interested in is first non-negative zero of the continuous extension $u^{\mathsf{c}}$ of solution $u$ of the initial value problem (P1). First of all, we define the bi-infinite sequence $(q_k)_{k\in\mathbb{Z}}$ of ratios of values of $u$ in two consecutive integers:

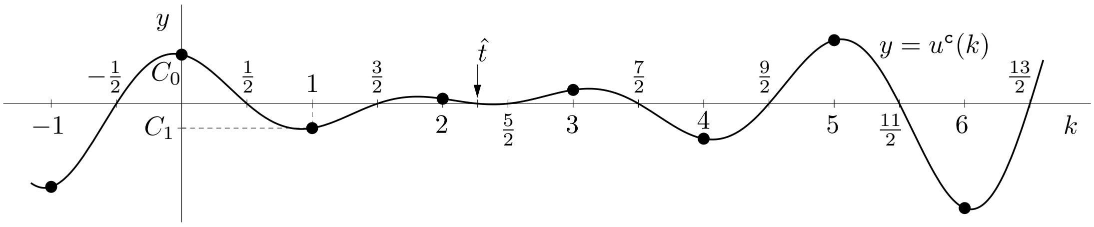



Figure 2.5: Continuous extension $u^c$ of solution $u$ of the initial value problem (P1) for $\lambda > 4$, $\lambda = 4.2$ and the determining zero point $\hat{t}$ of $u^c$ (the concept of determining zero point $\hat{t}$ is explained in Remark 6).

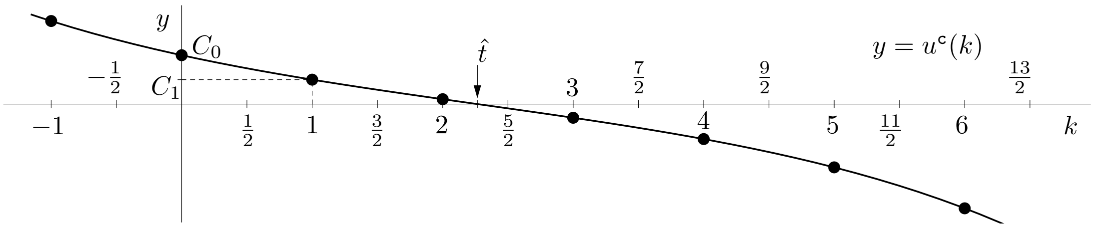



Figure 2.6: Continuous extension $u^c$ of solution $u$ of the initial value problem (P1) for $\lambda < 0$, $\lambda = -0.2$ and the determining zero point $\hat{t}$ of $u^c$ (the concept of determining zero point $\hat{t}$ is explained in Remark 6).

**Definition 4.** ([25], p. 67)
Let us define the sequence $(q_k)_{k\in\mathbb{Z}}$ as a mapping from $\mathbb{Z}$ to $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$ (the one-point compactification of $\mathbb{R}$) as

$$q_k := \frac{u(k)}{u(k-1)}, \qquad k \in \mathbb{Z}.$$

The sequence $(q_k)$ is defined correctly since value of $u$ in two consecutive integers cannot be zero. If $u(0) = C_0 = 0$, then $q_1 = \frac{C_1}{C_0} = \frac{C_1}{0} = \infty$ independent of the sign of $C_1$. Sequence $(q_k)$ will be very important in the investigation of initial value problems and the properties of the solution. In the following text, we are not going to focus on the values of $u$ themselves but on these ratios $(q_k)$. Using such approach will allow us to study the problem in detail, find zeros of solution $u$ and describe any term of such sequence $(q_k)$ (all will be explained later in the text).

Let us also define a generalized zero of the solution of the discrete problem (P1) (for the original definition of a generalized zero see [14]).

**Definition 5.** Solution $u$ of the discrete problem (P1) has a generalized zero at $k \in \mathbb{Z}$ if

$$u(k) = 0 \quad \text{or} \quad u(k)u(k-1) < 0.$$

From the definition of $(q_k)$ we have that $u$ has a generalized zero at $k \in \mathbb{Z}$ if and only if $q_k \leq 0$ and $q_k \neq \infty$.

**Remark 6.** We can distinguish between three different cases dependent on the value of $\lambda$ and find the number of generalized zeros – see Lemma 7. For $\lambda \leq 0$, if there exists a generalized zero (conditions when such generalized zero exists are in the lemma), we denote the corresponding zero point of the continuous extension as the determining zero $\hat{t}$. Similarly, in the case of $\lambda \geq 4$, there also might exist a determining zero $\hat{t}$ (see lemma for details). In general, we can say that the determining zero point $\hat{t}$ is a zero point of $u^c$ (and it does not have to always exist). For $\lambda \leq 0$, the solution has at most one zero point (compare to other cases), thus the continuous extension changes sign at most once (at the determining zero) – see Figure 2.6. For $\lambda \geq 4$, the continuous extension has infinitely many zero points, but one of them stands out – the determining zero point (if it exists). Such solution changes sign at every integer; except for two integers which are defined as $k_1 = \lfloor \hat{t} \rfloor$ and $k_2 = \lceil \hat{t} \rceil$, thus we have $\operatorname{sign}(u(k_1)) = \operatorname{sign}(u(k_2))$ (for illustration see Figure 2.5). For $\lambda \in (0, 4)$, the determining zero always exists and it is the same as the first non-negative zero of a continuous extension $u^{\mathrm{c}}$ (we will explain this in the text following Definition 8).

**Lemma 7.** *The number of generalized zeros of solution $u$ of linear initial value problem* (P1) *is:*

1. *For $\lambda \leq 0$, the solution $u$ of* (P1) *has no generalized zero if $q_1 = \frac{C_1}{C_0} \in [\mathrm{e}^{-\omega_\lambda}, \mathrm{e}^{\omega_\lambda}]$ and has exactly one generalized zero for $q_1 \notin [\mathrm{e}^{-\omega_\lambda}, \mathrm{e}^{\omega_\lambda}]$.*
2. *For $\lambda \geq 4$, the solution $u$ of* (P1) *has infinitely many generalized zeros. For continuous extension $u^{\mathrm{c}}$, we distinguish between zero points $t^k = \frac{1}{2} + k,\ k \in \mathbb{Z}$ and between determining zero point $\hat{t}$ which exists if $q_1 = \frac{C_1}{C_0} \notin [-\mathrm{e}^{\omega_\lambda}, -\mathrm{e}^{-\omega_\lambda}]$.*
3. *For $\lambda \in (0, 4)$, the solution $u$ of* (P1) *has infinitely many generalized zeros. In this case, $0 < \omega_\lambda < \pi$ and the continuous extension $u^{\mathrm{c}}$ is $\frac{2\pi}{\omega_\lambda}$-periodic function.*

*Proof.* (a) For $\lambda = 0$, we have $\omega_\lambda = 0$ and $u^{\mathrm{c}}(t) = C_0 + t(C_1 - C_0)$. Thus $u^{\mathrm{c}}$ is a linear function and has no generalized zero if $C_1 = C_0$ (i.e. $q_1 = 1$), and one generalized zero if $q_1 \neq 1$.

(b) For $\lambda < 0$ the situation is as following. The continuous extension $u^{\mathrm{c}}$ can be written as

$$\begin{aligned} u^{\mathrm{c}}(t) \quad &= \quad \left(\tfrac{C_0}{2} - \tfrac{\cosh\omega_\lambda C_0 - C_1}{2\sinh\omega_\lambda}\right)\big(\cosh(\omega_\lambda t) + \sinh(\omega_\lambda t)\big) \\ &\quad + \left(\tfrac{C_0}{2} + \tfrac{\cosh\omega_\lambda C_0 - C_1}{2\sinh\omega_\lambda}\right)\big(\cosh(\omega_\lambda t) - \sinh(\omega_\lambda t)\big) \\ &= \quad C_0\cosh(\omega_\lambda t) - \tfrac{C_0\cosh\omega_\lambda - C_1}{\sinh\omega_\lambda}\sinh(\omega_\lambda t). \end{aligned}$$

Such function has no zero points if and only if

$$|C_0| \geq \left|\frac{C_0\cosh\omega_\lambda - C_1}{\sinh\omega_\lambda}\right|. \tag{2.1}$$

For $C_0 > 0$ the inequality in (2.1) is satisfied if

$$C_1 \leq C_0(\cosh\omega_\lambda + \sinh\omega_\lambda) = C_0\,\mathrm{e}^{\omega}_{\lambda} \quad \wedge \quad C_0(\cosh\omega_\lambda - \sinh\omega_\lambda) = C_0\,\mathrm{e}^{-\omega_\lambda} \leq C_1,$$

i.e. $\mathrm{e}^{-\omega_\lambda} \leq q_1 \leq \mathrm{e}^{\omega_\lambda}$. For $C_0 < 0$ we will get the same condition.

(c) For $\lambda = 4$, we have $\omega_\lambda = 0$ and

$$u^{\mathrm{c}}(t) = -C_0\cos\big(\pi(1-t)\big)(1-t) - C_1\cos(\pi t)t = \cos(\pi t)\big(C_0 + t(-C_1 - C_0)\big).$$

Such function has zero points if $\cos(\pi t) = 0$ or if $C_0 + t(-C_1 - C_0) = 0$. First condition gives us infinitely many zero points $t^k = \frac{1}{2} + k,\ k \in \mathbb{Z}$. Second condition leads to an existence of determining zero point $\hat{t}$. Determining zero point $\hat{t}$ does not exist if $C_0 = -C_1$.

(d) For $\lambda > 4$, the continuous extension $u^{\mathrm{c}}$ can be written in the form

$$u^{\mathrm{c}}(t) = \cos(\pi t)\left(C_0\cosh(\omega_\lambda t) + \frac{-C_0\cosh\omega_\lambda - C_1}{\sinh\omega_\lambda}\sinh(\omega_\lambda t)\right).$$

Such function has infinitely many zero points if $\cos(\pi t) = 0$. Determining zero point does not exist if

$$|C_0| \geq \left|\frac{-C_0\cosh\omega_\lambda - C_1}{\sinh\omega_\lambda}\right|.$$

Such inequality is satisfied if $-\mathrm{e}^{\omega}_{\lambda} \leq q_1 \leq -\mathrm{e}^{-\omega_\lambda}$. See Figure 2.5 for zero points $t^k = \frac{1}{2} + k,\ k \in \mathbb{Z}$ and for determining zero point $\hat{t}$.

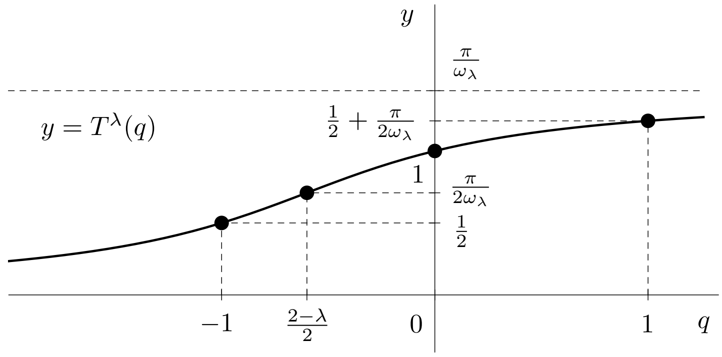



Figure 2.7: The graph of $T^\lambda$ as a function of $q$, case $\lambda \in (0,4)$.

(e) For $\lambda \in (0,4)$, the continuous extension $u^{\mathrm{c}}$ can be written as

$$u^{\mathrm{c}}(t) = C_0 \cos(\omega_\lambda t) - \frac{C_0 \cos \omega_\lambda - C_1}{\sin \omega_\lambda} \sin(\omega_\lambda t),$$

which is a sum of two $\frac{2\pi}{\omega_\lambda}$-periodic functions and has infinitely many zeros.

We note that (a) – (b) prove assertion 1. in the lemma, (c) – (d) are connected to assertion 3. and finally (e) is a proof of assertion 2. of the lemma.

□

Furthermore, we define function $T^\lambda$ which gives us an answer to our question about the first non-negative zero of continuous extension $u^{\mathrm{c}}$ of solution $u$ to (P1). See Figures 2.7, 2.8 and 2.9 for graphs of $T^\lambda$. In [25], function $T^\lambda$ is defined only for $\lambda \in (0,4)$ (see [25], Definition 2, p. 68) but we can define function $T^\lambda$ for $\lambda \in \mathbb{R}$ and use such definition for finding the first non-negative zero of continuous extension $u^{\mathrm{c}}$ (and the determining zero $\hat{t}$).

**Definition 8.** For $\lambda \in \mathbb{R}$, let us define the function $T^\lambda :\ \mathbb{R}^* \to \mathbb{R}$, $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$, as

$$\begin{aligned}
\mathrm{Dom}(T^\lambda) \quad &:= \quad \begin{cases} \mathbb{R}^* \setminus [\mathrm{e}^{-\omega_\lambda}, \mathrm{e}^{\omega_\lambda}] & \text{for } \lambda \leq 0, \\ \mathbb{R}^* & \text{for } \lambda \in (0,4), \\ \mathbb{R}^* \setminus [-\mathrm{e}^{\omega_\lambda}, -\mathrm{e}^{-\omega_\lambda}] & \text{for } \lambda \geq 4, \end{cases} \\
T^\lambda(\infty) \quad &:= \quad 0, \\
T^\lambda(q) \quad &:= \quad \begin{cases} \frac{1}{\omega_\lambda} \operatorname{arcoth}\left(\frac{\cosh \omega_\lambda - q}{\sinh \omega_\lambda}\right) & \text{for } \lambda < 0, \\ \frac{1}{1-q} & \text{for } \lambda = 0, \\ \frac{1}{\omega_\lambda} \operatorname{arccot}\left(\frac{\cos \omega_\lambda - q}{\sin \omega_\lambda}\right) & \text{for } \lambda \in (0,4), \\ \frac{1}{1+q} & \text{for } \lambda = 4, \\ \frac{1}{\omega_\lambda} \operatorname{arcoth}\left(\frac{\cosh \omega_\lambda + q}{\sinh \omega_\lambda}\right) & \text{for } \lambda > 4. \end{cases}
\end{aligned}$$

In this thesis, we are going to use that inverse cotangent (arccotangent) has the usual principal values, thus it is defined for all real numbers and its range is interval $(0,\pi)$.

Let us explain the role the function $T^\lambda$ has for the continuous extension $u^{\mathrm{c}}$ of solution $u$ of the initial value problem (P1). We denote $t_1$ the first non-negative zero of $u^{\mathrm{c}}$ (if it exits). We will show the relationship between $t_1$ and $\hat{t}$ (the determining zero) for different values of $\lambda$. Let us note that $u^{\mathrm{c}}(\hat{t}) = 0$ and $\hat{t} = T^\lambda(q_1) = T^\lambda(\frac{C_1}{C_0})$ (if it exists) (details in the text below).

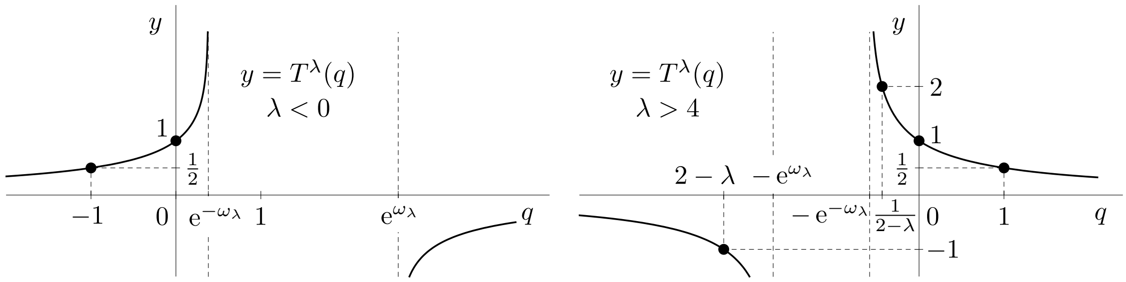



Figure 2.8: The graph of $T^\lambda$ as a function of $q$, case $\lambda < 0$ ($\lambda = -1$, left) and $\lambda > 4$ ($\lambda = 4.3$, right).

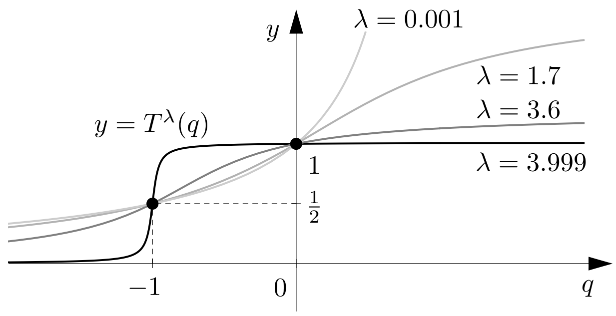



Figure 2.9: Graphs of $T^\lambda$ for $\lambda \in \{0.001, 1.7, 3.6, 3.999\}$.

Firstly, let us summarise. Let $u$ be a solution of (P1) and $u^{\mathrm{c}}$ its continuous extension. Then we have:

1. For $\lambda \le 0$ the continuous extension $u^{\mathrm{c}}$ has exactly one zero $T^\lambda(q_1)$ if $q_1 \in \mathrm{Dom}(T^\lambda)$ and no zero if $q_1 \notin \mathrm{Dom}(T^\lambda)$.

2. For $\lambda \ge 4$ the continuous extension $u^{\mathrm{c}}$ has infinitely many zeros $k+\frac{1}{2}$, $k \in \mathbb{Z}$. The determining zero $T^\lambda(q_1)$ exists if and only if $q_1 \in \mathrm{Dom}(T^\lambda)$.

3. For $\lambda \in (0,4)$ the continuous extension $u^{\mathrm{c}}$ has infinitely many zeros and the first non-negative zero is $T^\lambda(q_1)$.

Let $\lambda = 0$. In this case we will distinguish between $C_0 = 0$ and $C_0 \neq 0$.

(a) If $C_0 = 0$, then $u^{\mathrm{c}}(t) = C_1 t$ and $t_1 = 0$, $q_1 = \infty$. And so, $\hat{t} = T^\lambda(q_1) = T^\lambda(\infty) = 0 = t_1$.

(b) If $C_0 \neq 0$, we have $C_0(1-\hat{t}) + C_1\hat{t} = 0$. From Lemma 7 we have that $\hat{t}$ exists exactly one for $q_1 \in \mathbb{R}^* \setminus \{1\}$ and does not exist otherwise. Thus for $C_1 \neq C_0$ we have

$$\hat{t} = \frac{1}{1-\frac{C_1}{C_0}} = \frac{1}{1-q_1} = T^\lambda(q_1).$$

Because of the fact, that $\hat{t}$ can be negative, we have $t_1 \neq \hat{t}$ in general and the first non-negative zero point $t_1$ does not have to exist.

Let $\lambda \in (0,4)$. In this case, $0 < \omega_\lambda < \pi$, the continuous extension $u^{\mathrm{c}}$ is $\frac{2\pi}{\omega_\lambda}$-periodic function and all zeros of $u^{\mathrm{c}}$ are $t_k = t_1 + (k-1)\frac{\pi}{\omega_\lambda}$, $k \in \mathbb{Z}$.

(a) If $C_0 = 0$, then $u^{\mathrm{c}}(t) = C_1 \frac{\sin(\omega_\lambda t)}{\sin \omega_\lambda}$ and $q_1 = \infty$, $t_1 = 0$. And we have $T^\lambda(\infty) = 0$.

(b) If $C_0 \neq 0$, then for $t_1$, we have that

$$\sin(\omega_\lambda(1-t_1)) + q_1 \sin(\omega_\lambda t_1) = 0, \quad 0 < t_1 < \frac{\pi}{\omega_\lambda},$$

which gives us

$$\begin{aligned} q_1 &= \tfrac{-\sin\omega_\lambda \cos(\omega_\lambda t_1) + \cos\omega_\lambda \sin(\omega_\lambda t_1)}{\sin(\omega_\lambda t_1)} = \cos\omega_\lambda - \sin\omega_\lambda \cdot \cot(\omega_\lambda t_1), \\ t_1 &= \tfrac{1}{\omega_\lambda} \operatorname{arccot}\left(\tfrac{\cos\omega_\lambda - q_1}{\sin\omega_\lambda}\right). \end{aligned}$$

Therefore, $t_1 = \hat{t} = T^\lambda(q_1) = T^\lambda\left(\frac{C_1}{C_0}\right)$. See Figures 2.2 and 2.4, where we can see the first non-negative zero of $u^{\mathrm{c}}$ for two different values of $\lambda \in (0,4)$.

Let $\lambda < 0$. In this case, the situation is similar to $\lambda = 0$. Zero point $\hat{t}$ of $u^{\mathrm{c}}$ exists (exactly one and can be negative) if $q_1 = \frac{C_1}{C_0} \notin [\mathrm{e}^{-\omega_\lambda}, \mathrm{e}^{\omega_\lambda}]$ and does not exist otherwise. Therefore, existence of $t_1$ is not guaranteed. The derivation of zero point $\hat{t}$ is similar to the case $\lambda \in (0,4)$. See Figure 2.6, where zero $\hat{t}$ exists.

Let $\lambda = 4$. From Lemma 7 we have that the solution $u$ of (P1) has infinitely many generalized zeros. For continuous extension $u^{\mathrm{c}}$, we distinguish between zero points $t^k = \frac{1}{2} + k,\ k \in \mathbb{Z}$ and between determining zero point $\hat{t}$ which exists if $q_1 = \frac{C_1}{C_0} \in \mathbb{R}^* \setminus \{-1\}$. This determining zero point $\hat{t}$ is given as $\hat{t} = T^\lambda(q_1)$, which we can verify by calculation:

(a) For $C_0 = 0$, we have $-C_1 \cos(\pi t_1) t_1 = 0$, therefore $t_1 = 0$ and $q_1 = \infty$.

(b) For $C_0 \neq 0$, we have

$$\hat{t} = \frac{\cos(\pi(1-\hat{t}))}{\cos(\pi(1-\hat{t})) - q_1 \cos(\pi\hat{t})} = \frac{-\cos(\pi\hat{t})}{-\cos(\pi\hat{t}) - q_1\cos(\pi\hat{t})} = \frac{1}{1+q_1} = T^\lambda(q_1).$$

Let $\lambda > 4$. From Lemma 7, we have that the solution $u$ of (P1) has infinitely many generalized zeros. In this case, for continuous extension $u^{\mathrm{c}}$, we distinguish between zero points $t^k = \frac{1}{2} + k,\ k \in \mathbb{Z}$ and between determining zero point $\hat{t}$ which exists if $q_1 = \frac{C_1}{C_0} \notin [-\mathrm{e}^{\omega_\lambda}, -\mathrm{e}^{-\omega_\lambda}]$ and can be negative. As in the case $\lambda = 4$, this determining zero point $\hat{t}$ is given as $\hat{t} = T^\lambda(q_1)$, which we can verify by calculation similar to derivation in the case $\lambda \in (0,4)$. See Figure 2.5, where point $\hat{t}$ exists and see the difference between zero points $t^k = \frac{1}{2} + k,\ k \in \mathbb{Z}$ and the determining zero point $\hat{t}$.

We are also going to define function $Q^\lambda$, which is an inverse function to $T^\lambda$ (in [31], p. 17, we defined function $Q^\lambda$ for $\lambda \in (0,4)$).

**Definition 9.** For $\lambda \in \mathbb{R}$, let us define the function $Q^\lambda:\ \mathbb{R} \to \mathbb{R}^*$, $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$, as

$$\begin{aligned} \mathrm{Dom}(Q^\lambda) &:= \begin{cases} \left[0, \dfrac{\pi}{\omega_\lambda}\right) & \text{for } \lambda \in (0,4), \\ \mathbb{R} & \text{for } \lambda \in \mathbb{R} \setminus (0,4), \end{cases} \\ Q^\lambda(0) &:= \infty, \\ Q^\lambda(t) &:= \begin{cases} -\dfrac{\sinh(\omega_\lambda(1-t))}{\sinh(\omega_\lambda t)} & \text{for } \lambda < 0, \\ -\dfrac{1-t}{t} & \text{for } \lambda = 0, \\ -\dfrac{\sin(\omega_\lambda(1-t))}{\sin(\omega_\lambda t)} & \text{for } \lambda \in (0,4), \\ \dfrac{1-t}{t} & \text{for } \lambda = 4, \\ \dfrac{\sinh(\omega_\lambda(1-t))}{\sinh(\omega_\lambda t)} & \text{for } \lambda > 4. \end{cases} \end{aligned}$$

As it was illustrated above, it makes sense to talk about first non-negative zero point $t_1$ of $u^c$ mainly in the case $\lambda \in (0,4)$. In the following text, we will limit ourselves only to the case $\lambda \in (0,4)$.

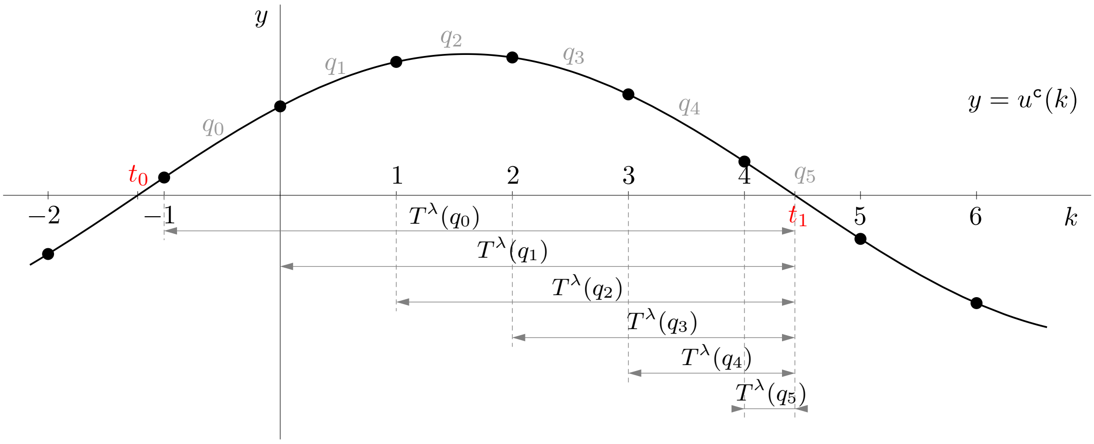



Figure 2.10: The bi-infinite sequence $(q_k)_{k\in\mathbb{Z}}$ of ratios of values of $u$ as the solution of the initial value problem (P1) and its relation to the first non-negative zero point $t_1$ using function $T^\lambda$ (case $\lambda \in (0,4)$).

Let us note that for $\lambda \in (0,4)$, $T^\lambda$ is a strictly increasing function on $\mathbb{R}$ and maps $\mathbb{R}^*$ onto $\left[0, \frac{\pi}{\omega_\lambda}\right)$. We can calculate few values (see Figure 2.7) of function $T^\lambda$:

$$\begin{aligned}
T^\lambda(0) &= \tfrac{1}{\omega_\lambda} \operatorname{arccot}\left(\tfrac{\cos\omega_\lambda}{\sin\omega_\lambda}\right) = \tfrac{1}{\omega_\lambda}\operatorname{arccot}(\cot\omega_\lambda) = 1,\\
T^\lambda(-1) &= \tfrac{1}{\omega_\lambda} \operatorname{arccot}\left(\tfrac{\cos\omega_\lambda+1}{\sin\omega_\lambda}\right) = \tfrac{1}{\omega_\lambda}\operatorname{arccot}\left(\tfrac{2\cos^2\frac{\omega_\lambda}{2}}{\sin\left(\frac{\omega_\lambda}{2}+\frac{\omega_\lambda}{2}\right)}\right) = \tfrac{1}{\omega_\lambda}\operatorname{arccot}\left(\tfrac{2\cos^2\frac{\omega_\lambda}{2}}{2\sin\frac{\omega_\lambda}{2}\cos\frac{\omega_\lambda}{2}}\right)\\
&= \tfrac{1}{\omega_\lambda}\tfrac{\omega_\lambda}{2} = \tfrac{1}{2},\\
T^\lambda(1) &= \tfrac{1}{\omega_\lambda} \operatorname{arccot}\left(\tfrac{\cos\omega_\lambda-1}{\sin\omega_\lambda}\right) = \tfrac{1}{\omega_\lambda}\operatorname{arccot}\left(\tfrac{-2\sin^2\frac{\omega_\lambda}{2}}{2\sin\frac{\omega_\lambda}{2}\cos\frac{\omega_\lambda}{2}}\right) = \tfrac{1}{\omega_\lambda}\operatorname{arccot}\left(-\tan\tfrac{\omega_\lambda}{2}\right)\\
&= \tfrac{1}{\omega_\lambda}\operatorname{arccot}\left(\cot\left(\tfrac{\omega_\lambda}{2}+\tfrac{\pi}{2}\right)\right) = \tfrac{1}{2}+\tfrac{\pi}{2\omega_\lambda},\\
T^\lambda\left(\tfrac{2-\lambda}{2}\right) &= \tfrac{1}{\omega_\lambda} \operatorname{arccot}\left(\tfrac{\cos\omega_\lambda-\frac{2-\lambda}{2}}{\sin\omega_\lambda}\right) = \tfrac{1}{\omega_\lambda}\operatorname{arccot}\left(\tfrac{\cos\omega_\lambda-\cos\omega_\lambda}{\sin\omega_\lambda}\right) = \tfrac{\pi}{2\omega_\lambda}.
\end{aligned} \tag{2.2}$$

If we take into account that the difference equation in (P1) is autonomous, we realize that the first non-negative zero $t_1$ can be calculated as (for $\lambda \in (0,4)$)

$$t_1 = j + T^\lambda(q_{1+j}), \qquad j = \lceil t_0 \rceil, \dots, 0, \dots, \lfloor t_1 \rfloor,$$

where $t_0$ is the previous zero of continuous extension $u^c$ of solution $u$. For illustration, see Figure 2.10, where is $\lceil t_0 \rceil = -1$, $\lfloor t_1 \rfloor = 4$. For such example, there are 6 possible ways how to get $t_1$ using sequence $(q_k)_{k\in\mathbb{Z}}$. We have

$$\begin{gathered}
t_1 = -1 + T^\lambda(q_0),\ t_1 = T^\lambda(q_1),\ t_1 = 1 + T^\lambda(q_2),\\
t_1 = 2 + T^\lambda(q_3),\ t_1 = 3 + T^\lambda(q_4),\ t_1 = 4 + T^\lambda(q_5).
\end{gathered}$$

Finally, in the following lemma, we will introduce a useful formula for $T^\lambda$ (which will be used in Chapter 5 in the part where we introduce bounds of Fučík curves which are referred to as the “delta bounds” in this thesis).

**Lemma 10.** *([25], Lemma 3, p. 69)*
*Let $\lambda \in (0,4)$. For $q = \infty$ and $q \leq 0$, we have*

$$T^\lambda(q) + T^\lambda\left(\tfrac{1}{q}\right) = 1.$$

## 2.2 Chebyshev polynomials of the second kind

In this section, we will give more detailed information about Chebyshev polynomials of the second kind (for more details see [29]), which we will use in Section 2.3.

**Definition 11.** Chebyshev polynomials $U_k$ of the second kind of degree $k \in \mathbb{Z}$ at the point $x \in \mathbb{R}$ are defined by the recurrence formula

$$U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x)$$

with initial conditions $U_0(x) = 1$, $U_1(x) = 2x$.

### 2.2.1 Relationship to the linear initial value problem (P1)

For all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, let us denote

$$V_k^\lambda := U_k\left(\tfrac{2-\lambda}{2}\right). \tag{2.3}$$

For all $\lambda \in \mathbb{R}$, polynomials $V_k^\lambda$ satisfy the three terms recurrence formula

$$V_{k-1}^\lambda - (2-\lambda)V_k^\lambda + V_{k+1}^\lambda = 0, \quad k \in \mathbb{Z}, \tag{2.4}$$

with initial conditions $V_0^\lambda = 1$, $V_1^\lambda = 2 - \lambda$. Initial value problem (P1) has solution in the recurrence form

$$u(k-1) - (2-\lambda)u(k) + u(k+1) = 0$$

with initial conditions $u(0) = C_0$ and $u(1) = C_1$. Therefore, $V_k^\lambda$ is the solution of the initial value problem (P1) with $C_0 = V_0^\lambda = 1$ and $C_1 = V_1^\lambda = 2 - \lambda$.

Moreover, for all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$ we have

$$F^\lambda(k) = V_{k-1}^\lambda,$$

where function $F$ is used in Lemma 3. Such property allows us to get the solution $u$ of (P1) as

$$u(k) = -C_0V_{k-2}^\lambda + C_1V_{k-1}^\lambda.$$

For illustration, see Figure 2.11, where first few Chebyshev polynomials $V_k^\lambda$ (for $k = 0, \ldots, 4$) are shown. We chose $C_0$ and $C_1$ in such a way, that it corresponds with solution $u$ from Figure 2.2. Then, these values can be displayed as points of Chebyshev polynomials $V_k^\lambda$ for fixed value $\lambda = 1.3$.

### 2.2.2 Properties of the Chebyshev polynomials of the second kind

Let us list several useful properties of Chebyshev polynomials of the second kind $U_k$ and polynomials $V_k^\lambda$.

1. Zeros of $U_k$ are
$$x = \cos\frac{m\pi}{k+1}, \qquad m = 1, \ldots, k,$$
and thus zeros of $V_k^\lambda$ are
$$\lambda = 2 - 2\cos\frac{m\pi}{k+1} = 4\sin^2\frac{m\pi}{2(k+1)}, \qquad m = 1, \ldots, k.$$

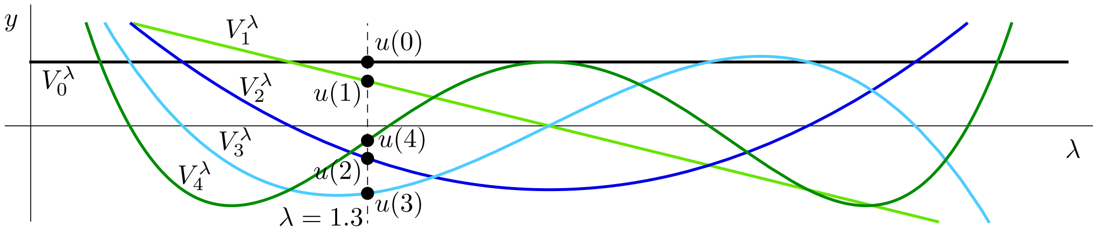



Figure 2.11: Chebyshev polynomials $V_k^\lambda$ for $k = 0, 1, \ldots, 4$, $\lambda = 1.3$. Marked points on the dashed line $\lambda = 1.3$ are values of solution $u$ of initial value problem (P1) with $C_0 = 1$, $C_1 = 2 - \lambda = 0.7$ displayed on the Figure 2.2. Chebyshev polynomial $V_0^\lambda$ is black, $V_1^\lambda$ is light green, $V_2^\lambda$ is dark blue, $V_3^\lambda$ is light blue and $V_4^\lambda$ is dark green.

2. We have that $V_{-k}^\lambda = -V_{k-2}^\lambda$ for all $k \in \mathbb{Z}$. For $\lambda \in (0, 4)$ and all $k \in \mathbb{Z}$, we have

$$V_{-k}^\lambda = U_{-k}\left(\tfrac{2-\lambda}{2}\right) = \frac{\sin((-k+1)\omega_\lambda)}{\sin \omega_\lambda} = \frac{-\sin((k-1)\omega_\lambda)}{\sin \omega_\lambda} = \frac{-\sin(((k-2)+1)\omega_\lambda)}{\sin \omega_\lambda} = -V_{k-2}^\lambda.$$

For $|\lambda| > 4$, the derivation is very similar.

3. We have

$$\ldots, V_{-3}^\lambda = -2 + \lambda, \quad V_{-2}^\lambda = -1, \quad V_{-1}^\lambda = 0, \quad V_0^\lambda = 1, \quad V_1^\lambda = 2 - \lambda,$$
$$V_2^\lambda = -\lambda^3 + 6\lambda^2 - 10\lambda + 4, \quad V_3^\lambda = \lambda^4 - 8\lambda^3 + 21\lambda^2 - 20\lambda + 5, \ldots$$

4. We have $V_k^\lambda = \det(\mathbf{B}^\lambda)$, where $\mathbf{B}^\lambda$ is a square matrix of size $k$ defined as

$$\mathbf{B}^\lambda = \begin{bmatrix} 2-\lambda & 1 & & & \\ 1 & 2-\lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2-\lambda & 1 \\ & & & 1 & 2-\lambda \end{bmatrix}.$$

For Chebyshev polynomials of the second kind $U_k$ exits an inequality (also known as Turán inequality) – see [39] :

$$U_k^2(x) - U_{k-1}(x)U_{k+1}(x) > 0 \quad \text{for } -1 < x < 1.$$

For Chebyshev polynomials $V_k^\lambda$, the Turán inequality has a special form of identity (in [25], we have proved this lemma using properties of functions $V_k^\lambda$).

**Lemma 12.** *([25], Lemma 4, p. 69)*
*For all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, we have the following identity*

$$\left(V_k^\lambda\right)^2 - V_{k+1}^\lambda V_{k-1}^\lambda = 1.$$

Let us note a few other interesting properties of Chebyshev polynomials (for reference, see [33]).

Chebyshev polynomials of the first kind $T_k$ are given by

$$T_k(\cos(\theta)) = \cos(k\theta).$$

Both Chebyshev polynomials form a sequence of orthogonal polynomials. Polynomials $T_k$ are orthogonal with respect to the weight function $\frac{1}{\sqrt{1-x^2}}$ on the interval $[-1, 1]$ and polynomials $U_k$ are orthogonal with respect to the weight function $\sqrt{1-x^2}$ on the interval $[-1, 1]$.

The Chebyshev polynomials have a lot of applications, but one stands out – numerical analysis. Functions can be expanded to a series of Chebyshev polynomials. The effectiveness of such expansion (partial sums of a Chebyshev expansion) can be found studied e.g. here [41].

Quoting [32] for a historical remark on orthogonal polynomials:

> "Chebyshev was probably the first mathematician to recognise the general concept of orthogonal polynomials. A few particular orthogonal polynomials were known before his work. Legendre and Laplace had encountered the Legendre polynomials in their work on celestial mechanics in the late eighteenth century. Laplace had found and studied the Hermite polynomials in the course of his discoveries in probability theory during the early nineteenth century. Other isolated instances of orthogonal polynomials occurring in the work of various mathematicians are mentioned later. It was Chebyshev who saw the possibility of a general theory and its applications. His work arose out of the theory of least squares approximation and probability; he applied his results to interpolation, approximate quadrature and other areas. He discovered the discrete analogue of the Jacobi polynomials but their importance was not recognized until this century. They were rediscovered by Hahn and named after him upon their rediscovery. Geronimus has pointed out that in his first paper on orthogonal polynomials, Chebyshev already had the Christoffel-Darboux formula."

## 2.3 Properties of function $W_k^\lambda$

The reason we have devoted previous pages to the introduction of Chebyshev polynomials of the second kind follows in this section. It is convenient to use them for the definition of function $W_k^\lambda$. This function determines the value of $k$-th element $q_k$ defined in Definition 4 by the value of $q_0$ (this property comes from Lemma 15).

**Definition 13.** ([25], Definition 5, p. 70)
For all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, let us define the function $W_k^\lambda :\ \mathbb{R}^* \to \mathbb{R}^*$ in the following way

$$W_k^\lambda(q) := \begin{cases} \dfrac{q \cdot V_k^\lambda - V_{k-1}^\lambda}{q \cdot V_{k-1}^\lambda - V_{k-2}^\lambda} & \text{for } q \in \mathbb{R}, \\[2ex] \dfrac{V_k^\lambda}{V_{k-1}^\lambda} & \text{for } q = \infty. \end{cases}$$

Let us recall that a Möbius transformation is given by $(a, b, c, d \in \mathbb{C},\ ad - bc \neq 0)$

$$f :\ \mathbb{C}^* \to \mathbb{C}^* :\ z \mapsto \frac{a \cdot z + b}{c \cdot z + d}.$$

The function $W_k^\lambda$ is the restriction of a Möbius transformation on $\mathbb{R}^*$. Indeed, we have that (using Lemma 12)

$$(V_{k-1}^\lambda)^2 - V_k^\lambda V_{k-2}^\lambda = 1 \neq 0, \quad k \in \mathbb{Z}.$$

For $V_{k-1}^\lambda = 0$, we have that $V_{k-2}^\lambda \neq 0$ and $V_k^\lambda \neq 0$, thus

$$W_k^\lambda(q) = \begin{cases} -\dfrac{V_k^\lambda}{V_{k-2}^\lambda} q & \text{for } q \in \mathbb{R}, \\[2ex] \infty & \text{for } q = \infty. \end{cases}$$

The following equality for function $W_k^\lambda$ allows us to generate functions $W_k^\lambda$ (for $k \in \mathbb{Z}$) easily – using recurrence.

**Lemma 14.** *For all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, we have*

$$W_{k+1}^\lambda(q) = 2 - \lambda - \frac{1}{W_k^\lambda(q)}, \quad q \in \mathbb{R}^*.$$

*Proof.* Proving this is a part of the proof of ([25], Lemma 6, p. 70). □

Using Lemma 14, for $k \in \mathbb{N}$, the value of function $W_k^\lambda$ at $q$ can be written in a form which reminds a finite continued fraction (yet, it is not a finite continued fraction, since terms $2-\lambda$ and $\lambda-2$ are not generally positive integers). For $q \neq \infty$, we have

$$\begin{aligned} W_1^\lambda(q) &= 2-\lambda+\tfrac{1}{-q} \\ W_2^\lambda(q) &= 2-\lambda+\tfrac{1}{\lambda-2+\frac{1}{q}} \\ W_3^\lambda(q) &= 2-\lambda+\tfrac{1}{\lambda-2+\frac{1}{2-\lambda+\frac{1}{-q}}} \\ &\vdots \end{aligned}$$

And finally, each ratio of bi-infinite sequence $(q_k)_{k\in\mathbb{Z}}$ can be calculated from ratio $q_0$ using the function $W_k^\lambda$. Thus, Lemma 15 allows us to calculate any ratio $q_k$ if we know $q_0$.

**Lemma 15.** *([25], Lemma 6, p. 70)*
*For all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, we have*

$$q_k = W_k^\lambda(q_0). \tag{2.5}$$

### 2.3.1 Relationship to the linear initial value problem (P1)

Let $u$ be the solution of the initial value problem (P1) and $(q_k)$ the bi-infinite sequence of ratios of value $u$ in two consecutive integers as it was defined in Definition 4.

**Remark 16.** ([25], Remark 10, p. 73)
Let us assume that we have some element of bi-infinite sequence $(q_k)$ (for example $q_1 = {}^{C_1}/_{C_0}$ given by the initial conditions). If we want to get any other element of such sequence or the first non-negative zero $t_1$ of $u^c$, we can use the following formulas.

1. For $\lambda \in \mathbb{R}$ and $i, j, k \in \mathbb{Z}$ such that $i + j = k$, we have that

$$q_k = W_j^\lambda(q_i). \tag{2.6}$$

This can be used for calculation of any term of sequence $(q_k)_{k\in\mathbb{Z}}$ from the initial condition. Let our initial condition be $C_0 = 0$, $C_1 \in \mathbb{R}\setminus\{0\}$. Then we have $q_1 = \frac{u(1)}{u(0)} = \frac{C_1}{0} = \infty$. And for any $k \in \mathbb{Z}$, we have $q_k = W_{k-1}^\lambda(q_1) = W_{k-1}^\lambda(\infty)$.

2. For $\lambda \in (0,4)$, we have for the first non-negative zero $t_1$ of $u^c$ that

$$t_1 = j + T^\lambda(W_j^\lambda(q_1)), \qquad j = \lceil t_0 \rceil, \dots, 0, \dots, \lfloor t_1 \rfloor. \tag{2.7}$$

In the last part of this section, let us consider the following linear problem which we explored in detail in ([25], p. 74) (detailed inspection of this problem will help us to understand better the semi-linear initial value problem, which we will explore in Chapter 3):

$$\begin{cases} \Delta^2 u(k-1) + a_k \cdot u(k) = 0, & k \in \mathbb{N}, \\ u(0) = 0, \quad u(1) = C_1, \end{cases} \tag{2.8}$$

where $C_1 \in \mathbb{R}$, $C_1 \neq 0$ and the sequence $(a_k)_{k\in\mathbb{N}}$ is given by

$$(a_k)_{k\in\mathbb{N}} = (\underbrace{\lambda_1, \dots, \lambda_1}_{k_1-\text{times}}, \underbrace{\lambda_2, \dots, \lambda_2}_{k_2-\text{times}}, \dots, \underbrace{\lambda_m, \dots, \lambda_m}_{k_m-\text{times}}, \dots),$$

where $(k_j)_{j\in\mathbb{N}}$ is a sequence of natural numbers, $\lambda_j \in \mathbb{R}$, $j \in \mathbb{N}$.

Let us look more closely on the linear initial value problem (2.8). Let us denote $u_k := u(k),\ k \in \mathbb{N} \cup \{0\}$. Such problem can be written in the form of infinite matrix equation

$$\begin{bmatrix} 2-a_1 & -1 & & & \\ -1 & 2-a_2 & -1 & & \\ & -1 & 2-a_3 & -1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix},$$

which has a solution

$$u_0 = 0,\ u_1 = C_1,\ u_2 = (2-a_1)C_1,\ u_3 = -C_1 + (2-a_2)u_2,\ u_4 = -u_2 + (2-a_3)u_3, \dots$$

written in the recurrence form

$$\begin{aligned} &u_0 = 0, \\ &u_1 = C_1, \\ &u_k = -u_{k-2} + (2-a_{k-1})u_{k-1},\ k \in \mathbb{N} \setminus \{1\}. \end{aligned}$$

We have that

$$\begin{aligned} &q_1 &&= \tfrac{C_1}{0} = \infty, && \\ &q_{k+1} &&= W_k^{\lambda_1}(q_1) && \text{for } 1 \le k \le k_1, \\ &q_{k+k_1+1} &&= W_k^{\lambda_2}(q_{k_1+1}) && \text{for } 1 \le k \le k_2, \\ & &&\vdots && \\ &q_{k+k_{m-1}+\dots+k_1+1} &&= W_k^{\lambda_m}(q_{k_{m-1}+\dots+k_1+1}) && \text{for } 1 \le k \le k_m, \quad m \ge 3, \\ & &&\vdots && \end{aligned}$$

Thus, for all $m \in \mathbb{N}$, we obtain $q_{k_m+\dots+k_2+k_1+1} = \left(W_{k_m}^{\lambda_m} \circ \dots \circ W_{k_2}^{\lambda_2} \circ W_{k_1}^{\lambda_1}\right)(q_1)$.

**Example 17.** Let us consider linear initial value problem (2.8) with $(a_k)_{k\in\mathbb{Z}} = (-1,-1,2,2,2,3,\dots)$, $C_1 = 1$. Then $\lambda_1 = -1,\ k_1 = 2,\ \lambda_2 = 2,\ k_2 = 3,\ \lambda_3 = 3,\ k_3 \ge 1, \dots$. Solution $u$ has values $u(0) = 0,\ u(1) = 1,\ u(2) = 3,\ u(3) = 8,\ u(4) = -3,\ u(5) = -8,\ u(6) = 3,\ u(7) = 5, \dots$. We have

$$\begin{aligned} q_1 &= \tfrac{1}{0} &&= \infty &&= \tfrac{u(1)}{u(0)}, && \\ q_2 &= W_1^{-1}(\infty) &&= 3 &&= \tfrac{u(2)}{u(1)}, && k_1 = 2, \\ q_3 &= W_2^{-1}(\infty) &&= \tfrac{8}{3} &&= \tfrac{u(3)}{u(2)}, && k_1 = 2, \\ q_4 &= W_1^{2}(q_3) &&= -\tfrac{3}{8} &&= \tfrac{u(4)}{u(3)}, && k_2 = 3, \\ q_5 &= W_2^{2}(q_3) &&= \tfrac{8}{3} &&= \tfrac{u(5)}{u(4)}, && k_2 = 3, \\ q_6 &= W_3^{2}(q_3) &&= -\tfrac{8}{3} &&= \tfrac{u(6)}{u(5)}, && k_2 = 3, \\ q_7 &= W_1^{3}(q_6) &&= \tfrac{5}{3} &&= \tfrac{u(7)}{u(6)}, && k_3 \ge 1, \\ &\vdots && && && \end{aligned}$$

Or, we can write $q_7 = \left(W_1^3 \circ W_3^2 \circ W_2^{-1}\right)(\infty)$.

## 2.4 Linear boundary value problem (P2)

In this section, we will briefly inspect linear boundary value problem with Dirichlet boundary conditions (P2)

$$\begin{cases} \Delta^2 u(k-1) + \lambda u(k) = 0, & k \in \mathbb{T}, \\ u(0) = u(n+1) = 0, \end{cases}$$

where $n \in \mathbb{N}$, $n \geq 2$, $\mathbb{T} = \{1, \dots, n\}$, $\hat{\mathbb{T}} = \{0, \dots, n+1\}$, $u : \hat{\mathbb{T}} \to \mathbb{R}$ and $\lambda \in \mathbb{R}$.

Problem (P2) can be rephrased using a matrix notation as

$$\mathbf{A}^{\mathtt{D}} \mathbf{u} = \lambda \mathbf{u}$$

where matrix $\mathbf{A}^{\mathtt{D}}$ is the Dirichlet matrix (1.3)

$$\mathbf{A}^{\mathtt{D}} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

and $\mathbf{u} = [u(1), \dots, u(n)]^T$. Thus, to find all values $\lambda \in \mathbb{R}$ for which the linear boundary value problem (P2) has a non-trivial solution is the same as to find all eigenvalues and corresponding eigenvectors of matrix $\mathbf{A}^{\mathtt{D}}$.

Problem (P2) is closely related to Chebyshev polynomials of the second kind (see Section 2.2). We can write

$$\det(\mathbf{A}^{\mathtt{D}} - \lambda \mathbf{I}) = U_n \left( \frac{2-\lambda}{2} \right) = V_n^{\lambda}.$$

Thus, eigenvalues of $\mathbf{A}^{\mathtt{D}}$ are zero points of polynomial $V_n^{\lambda}$ (see Section 2.2.2) and are of the form

$$\lambda_j^{\mathtt{D}} = 4 \sin^2 \frac{(j+1)\pi}{2(n+1)}, \qquad j = 0, \dots, n-1,$$

$\lambda_j^{\mathtt{D}} \in (0, 4)$ and $\lambda_0^{\mathtt{D}} < \lambda_1^{\mathtt{D}} < \dots < \lambda_{n-1}^{\mathtt{D}}$. And eigenvectors are of the form

$$u_j^{\mathtt{D}}(k) = \sqrt{\frac{2}{n+1}} \sin \left( \frac{k(j+1)\pi}{n+1} \right), \quad k \in \{1, 2, \dots, n\},$$

where $j$ represents the eigenvalue of the corresponding eigenvector and $k \in \{1, 2, \dots, n\}$ the entry of the vector. See Figure 2.12 where some (not all of them) non-trivial solutions of the problem (P2) are shown for $n = 13$.

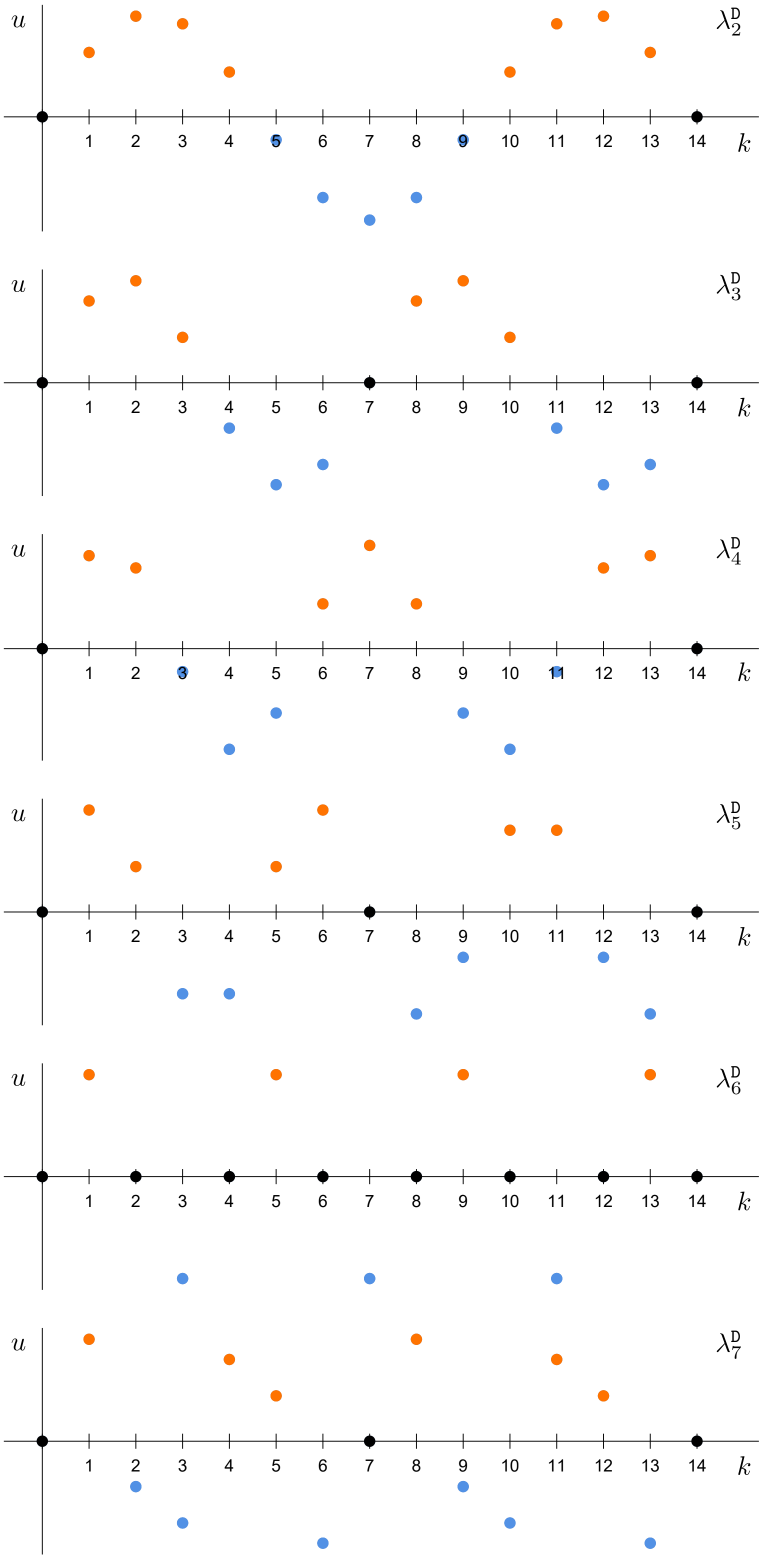



Figure 2.12: Some of the non-trivial solutions of the problem (P2) for $n = 13$ and (from top to bottom) $\lambda_2^{\mathtt{D}} \doteq 0.436337$, $\lambda_3^{\mathtt{D}} \doteq 0.75302$, $\lambda_4^{\mathtt{D}} \doteq 1.13223$, $\lambda_5^{\mathtt{D}} \doteq 1.55496$, $\lambda_6^{\mathtt{D}} = 2$, $\lambda_7^{\mathtt{D}} \doteq 2.44504$. Orange dots represent positive points, blue dots represent negative points and black dots represent zero points.

# Chapter 3

# Semi-linear initial value problem (P3) – Part I

In this chapter, we deal with the semi-linear initial value problem (P3)

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, & k \in \mathbb{Z}, \\ u(0) = 0, u(1) = C_1, \end{cases}$$

where $u^{\pm}(k) = \max\{\pm u(k), 0\}$, $C_1 \in \mathbb{R}$, $C_1 \neq 0$ and $(\alpha, \beta) \in D$,

$$D := ((0,4) \times (0,+\infty)) \cup ((0,+\infty) \times (0,4)).$$

**Remark 18.** ([25], p. 76)
Let $u$ be a solution of semi-linear initial value problem (P3). Then $u$ is also the solution of initial problem (2.8) if we take $(a_k)_{k\in\mathbb{N}}$ in the following form

$$a_k = \begin{cases} \alpha & \text{for } u(k) \geq 0, \\ \beta & \text{for } u(k) < 0. \end{cases}$$

## 3.1 Positive and negative semi-waves

In this section, we are going to describe a continuous extension of a solution $u$ of semi-linear initial value problem (P3) by defining a continuous extension $u^{\mathsf{c}}_{i,j}$ of $u$ on the interval $[i-1, j+1]$, where $i \in \mathbb{Z}$ is a generalized zero (similarly as we have defined a generalized zero in Definition 5 for problem (P1), we can define it for different problems such as (P3)) and $j \in \mathbb{Z} : j > i$ is such that for all $k = i, \ldots, j$, $u(k)$ is non-negative (or non-positive) and

$$u(j)u(j+1) < 0 \quad \text{or} \quad u(j) = 0.$$

See Figure 3.1 and Figure 3.2 (left). This means that $i$ and $(j+1)$ are two consecutive generalized zeros of $u$ if $u(j) \neq 0$. In the case of $u(j) = 0$, $i$ and $j$ are two consecutive generalized zeros of $u$.

On such interval, we construct a continuous extension in the same way as we have done it for the linear case (see Section 2.1). We define the continuous extension $u^{\mathsf{c}}_{i,j}$ of $u$ (see Figure 3.2 (right)) on the interval $[i-1, j+1]$ as (see ([25], p. 77))

$$u^{\mathsf{c}}_{i,j}(t) := \begin{cases} u(i-1)F^{\alpha}(1-(t-i+1)) + u(i)F^{\alpha}(t-i+1) & \text{for } u(i-1) < 0, \\ u(i-1)F^{\beta}(1-(t-i+1)) + u(i)F^{\beta}(t-i+1) & \text{for } u(i-1) > 0, \end{cases}$$

where functions $F^{\alpha}$ and $F^{\beta}$ are given by $F^{\lambda}$ (see Lemma 3) for $\lambda = \alpha$ and $\lambda = \beta$, respectively.

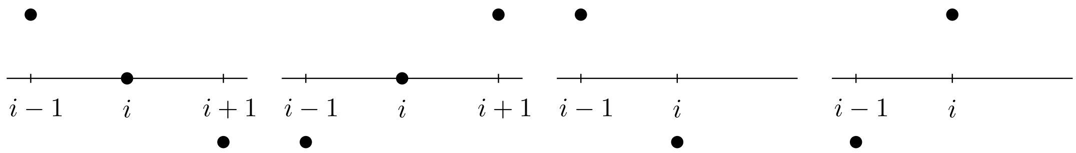



Figure 3.1: Generalized zeros at point $i$ in four possible cases: 1) $u(i) = 0, u(i-1) > 0$, 2) $u(i) = 0, u(i-1) < 0$, 3) $u(i-1)u(i) < 0, u(i-1) > 0$, 4) $u(i-1)u(i) < 0, u(i-1) < 0$.

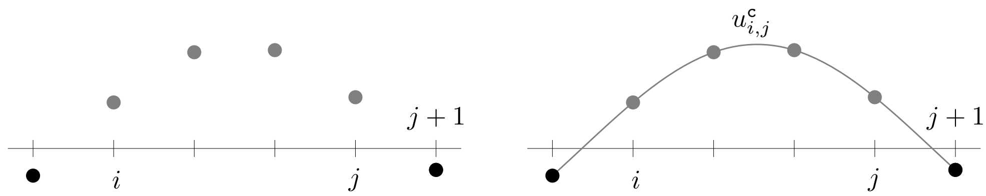



Figure 3.2: Consecutive generalized zeros $i$, $j+1$ and the continuous extension $u^{\mathsf{c}}_{i,j}$ of $u$.

Positive semi-wave is a continuous extension $u^{\mathsf{c}}_{i,j}$ of $u$ such that $u(k)$ is non-negative for all $k = i, \ldots, j$. Negative semi-wave is continuous extension $u^{\mathsf{c}}_{i,j}$ of $u$ such that $u(k)$ is non-positive for all $k = i, \ldots, j$. See Figure 3.4 where positive semi-waves are in orange color and negative semi-waves are in blue color. We say that positive and negative semi-waves are "anchored" together.

## 3.2 Relationship between function $W_k^\lambda$ and solution of a semi-linear problem (P3)

Now that we have defined what is the continuous extension of a solution of the problem (P3), we can use the theory from the linear case here.

Assuming that we would know where all generalized zeros (or anchorings of all positive / negative semi-waves) are (for solution $u$), we would quickly find all ratios $(q_k)_{k\in\mathbb{Z}}$ using function $W_k^\lambda$, where $\lambda = \alpha$ for positive semi-wave and $\lambda = \beta$ for negative semi-wave. Let us demonstrate such construction on the following example.

**Example 19.** Let us take the semi-linear initial value problem (P3) with $\alpha = 0.8$, $\beta = 3.94$, $C_1 = 1$. Let us assume that we know that solution $u$ has the following sign properties:

$$u(0) = 0,\ u(1) = 1 > 0,\ u(2) > 0,\ u(3) > 0,\ u(4) < 0,\ u(5) > 0,\ u(6) > 0,\ u(7) > 0,\ u(8) < 0,$$
$$u(9) < 0,\ u(10) > 0,\ u(11) > 0,\ u(12) > 0,\ u(13) < 0,\ \ldots$$

Since all values $u(k)$, $k \in \mathbb{Z}$, can be calculated directly from the difference equation, we can show the solution $u$ on Figure 3.3. Our goal is to calculate values of sequence $(q_k)_{k\in\mathbb{Z}}$ (let us recall definition of sequence $(q_k)$: $q_k = \frac{u(k)}{u(k-1)}$) using functions $W_k^\alpha$, $W_k^\beta$.

Firstly, let us construct continuous extension of the solution $u$. Since we know the sign properties of solution $u$, we know that we have positive semi-waves $u^{\mathsf{c}}_{0,3}$, $u^{\mathsf{c}}_{5,7}$, $u^{\mathsf{c}}_{10,12}$, …And negative semi-waves $u^{\mathsf{c}}_{4,4}$, $u^{\mathsf{c}}_{8,9}$, …See Figure 3.4 for such construction.

Finally, let us demonstrate that we have several ways how to calculate values of sequence $(q_k)_{k\in\mathbb{Z}}$. Since $q_1$ is defined by the initial conditions, we have

$$q_1 = \tfrac{1}{0} = \infty.$$

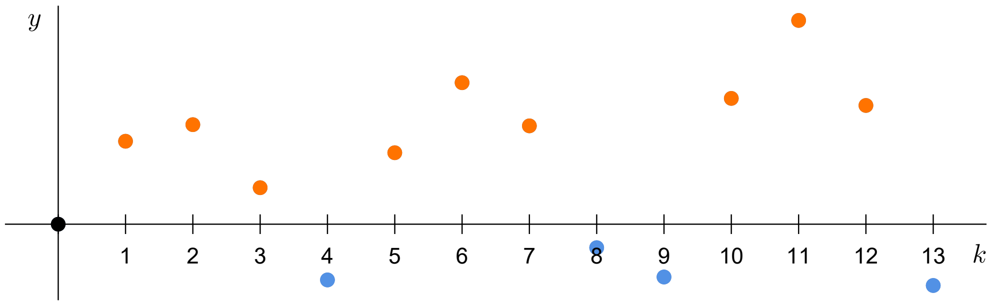



Figure 3.3: Solution $u$ of semi-linear initial value problem (P3) for $\alpha = 0.8$, $\beta = 3.94$, $C_1 = 1$.

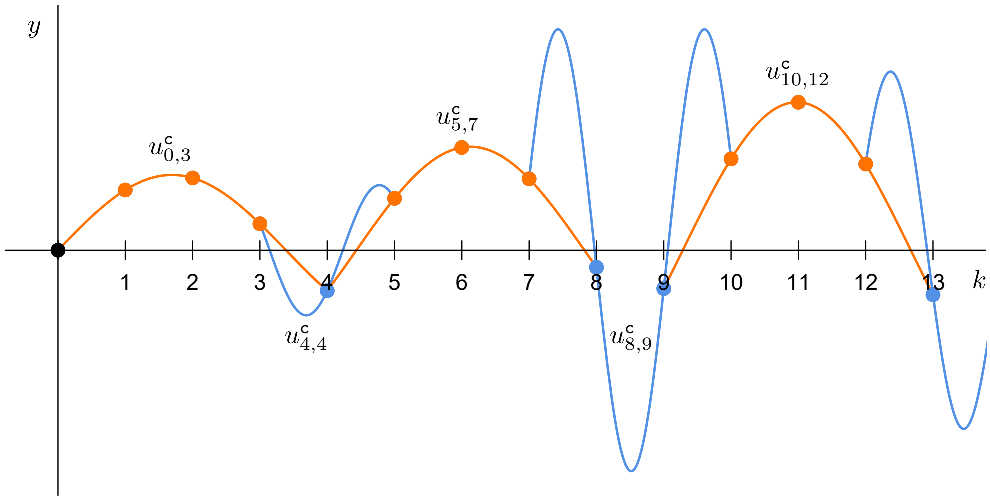



Figure 3.4: Continuous extension of solution $u$ from Figure 3.3.

We are going to use $q_1$ in the calculations of all other terms of sequence $(q_k)$. Values $u(1)$ and $u(2)$ which define $q_2$ are part of the positive semi-wave $u^{\mathrm{c}}_{0,3}$, thus for the calculation of $q_2$ we use $W_k^\alpha$ (thus $\lambda = \alpha$). Starting at $q_1$, we need to make "one jump" in the sequence $q_k$ (on the positive semi-wave) in order to get $q_2$, thus $k = 1$ and we have

$$q_2 = \tfrac{u(2)}{u(1)} = \tfrac{u(2)}{1} = u(2) = W_1^\alpha(q_1) = W_1^\alpha(\infty) = 1.2.$$

Since $q_3$ and $q_4$ are also "part" of positive semi-wave $u^{\mathrm{c}}_{0,3}$, we can calculate them in a similar way. Value $q_3$ is "one jump" away from $q_2$ (thus $k = 1$) on the positive semi-wave and "two jumps" away from $q_1$ (thus $k = 2$) on the positive semi-wave. And in the same way, value $q_4$ is "three jumps" away from the $q_1$ (on the positive semi-wave), thus $k = 3$.

$$\begin{aligned} q_3 &= \tfrac{u(3)}{u(2)} = W_1^\alpha(q_2) = W_2^\alpha(q_1) = W_2^\alpha(\infty) \doteq 0.367 \\ q_4 &= \tfrac{u(4)}{u(3)} = W_1^\alpha(q_3) = W_2^\alpha(q_2) = W_3^\alpha(q_1) = W_3^\alpha(\infty) \doteq -1.527 \end{aligned}$$

Value $q_5$ is calculated a little differently, because it is part of the negative semi-wave $u^{\mathrm{c}}_{4,4}$ (value $u(5)$ is not part of the positive semi-wave $u^{\mathrm{c}}_{0,3}$). In order to get value $q_5$, we need to make "one jump" on the negative-semi wave from $q_4$, thus $q_5 = W_1^\beta(q_4)$. If we want to start with the value $q_3$, we need to make firstly "one jump" on the positive semi-wave to the value $q_4$ and then "one jump" on the negative semi-wave, thus $q_5 = W_1^\beta(W_1^\alpha(q_3))$. If we want to start with the value

$q_1 = \infty$, we can easily calculate the number of particular "jumps":

$$\begin{aligned} q_5 &= \tfrac{u(5)}{u(4)} = W_1^\beta(q_4) = W_1^\beta(W_1^\alpha(q_3)) = W_1^\beta(W_2^\alpha(q_2)) = W_1^\beta(W_3^\alpha(q_1)) \\ &= W_1^\beta(W_3^\alpha(\infty)) \doteq -1.282 \end{aligned}$$

Values $q_6, q_7, q_8$ are part of positive semi-wave $u^{\mathfrak{c}}_{5,7}$, thus we calculate them as:

$$\begin{aligned} q_6 &= W_1^\alpha(W_1^\beta(W_3^\alpha(\infty))) \doteq 1.98 \\ q_7 &= W_2^\alpha(W_1^\beta(W_3^\alpha(\infty))) \doteq 0.695 \\ q_8 &= W_3^\alpha(W_1^\beta(W_3^\alpha(\infty))) \doteq -0.239 \end{aligned}$$

Values $q_9, q_{10}$ are parts of negative semi-wave $u^{\mathfrak{c}}_{8,9}$:

$$\begin{aligned} q_9 &= W_1^\beta(W_3^\alpha(W_1^\beta(W_3^\alpha(\infty)))) \doteq 2.247 \\ q_{10} &= W_2^\beta(W_3^\alpha(W_1^\beta(W_3^\alpha(\infty)))) \doteq -2.382 \end{aligned}$$

And finally, we have:

$$\begin{aligned} q_{11} &= W_1^\alpha(W_2^\beta(W_3^\alpha(W_1^\beta(W_3^\alpha(\infty))))) \doteq 1.62 \\ q_{12} &= W_2^\alpha(W_2^\beta(W_3^\alpha(W_1^\beta(W_3^\alpha(\infty))))) \doteq 0.583 \\ q_{13} &= W_3^\alpha(W_2^\beta(W_3^\alpha(W_1^\beta(W_3^\alpha(\infty))))) \doteq -0.516 \end{aligned}$$

## 3.3 Sequences $(p_k)$ and $(\vartheta_k)$

In Example 19, we have shown how to use functions $W_k^\alpha$ and $W_k^\beta$ in calculations of sequence $(q_k)_{k\in\mathbb{Z}}$ of the semi-linear problem (P3). The assumption we have used there was that we knew sign properties of the solution $u$. Without it, we wouldn't know when to "switch" between positive and negative semi-waves. Thus our main goal in this section is to find out how to calculate these sign properties.

In the following definition, we define (recurrently given) sequences $(p_k)_{k\in\mathbb{Z}}$, $(P_k)_{k\in\mathbb{Z}}$, $(\vartheta_k)_{k\in\mathbb{Z}}$, $(\mathcal{W}_k^+)_{k\in\mathbb{Z}}$ and $(\mathcal{W}_k^-)_{k\in\mathbb{Z}}$. In the text following this definition, we will explain for the simplest case $0 < \alpha, \beta < 4$ what these sequences represent.

**Definition 20.** ([25], Definition 17, p. 82)
For all $j \in \mathbb{Z}$, let us denote

$$\phi_j := \begin{cases} \alpha & \text{for } j \text{ odd,} \\ \beta & \text{for } j \text{ even.} \end{cases}$$

On the domain $D = ((0,4)\times(0,+\infty)) \cup ((0,+\infty)\times(0,4))$, let us define sequences of functions $(p_i)$ and $(\vartheta_i)$, which are given recurrently for $i \in \mathbb{N}$ in the following way

$$\vartheta_0(\alpha,\beta) := \infty,$$

$$p_i(\alpha,\beta) := \begin{cases} \left\lfloor T^{\phi_i}(\vartheta_{i-1}(\alpha,\beta)) + \frac{\pi}{\omega_{\phi_i}} \right\rfloor & \text{for } \phi_i < 4, \\ \left\lfloor T^{\phi_{i+1}}(\vartheta_{i-1}(\alpha,\beta)) + T^{\phi_{i+1}}(2-\phi_i) + 1 \right\rfloor & \text{for } \phi_i \geq 4, \end{cases}$$

$$\vartheta_i(\alpha,\beta) := W^{\phi_i}_{p_i(\alpha,\beta)}(\vartheta_{i-1}(\alpha,\beta)).$$

Moreover, for all $k \in \mathbb{N}$, let us define function $P_k : D \to \mathbb{N}$ and composite functions $\mathcal{W}_k^\pm : \mathbb{R}^* \to \mathbb{R}^*$ as

$$P_k(\alpha,\beta) := \sum_{i=1}^k p_i(\alpha,\beta), \qquad \begin{aligned} \mathcal{W}_k^+ &:= W^{\phi_k}_{p_k(\alpha,\beta)} \circ \cdots \circ W^{\phi_2}_{p_2(\alpha,\beta)} \circ W^{\phi_1}_{p_1(\alpha,\beta)}, \\ \mathcal{W}_k^- &:= W^{\phi_{k+1}}_{p_k(\beta,\alpha)} \circ \cdots \circ W^{\phi_3}_{p_2(\beta,\alpha)} \circ W^{\phi_2}_{p_1(\beta,\alpha)}. \end{aligned}$$

We are going to illustrate what sequences in Definiton 20 mean for special case $0 < \alpha, \beta < 4$ and $C_1 > 0$. Let $u$ be a solution of the semi-linear initial value problem (P3). Since we are looking for description of the sign properties of the solution $u$, we are interested in all positive generalized zeros of $u$.

For our restriction $(0 < \alpha, \beta < 4)$, first few terms in the sequences $(p_j)_{j\in\mathbb{Z}}$ and $(\vartheta_j)_{j\in\mathbb{Z}}$ defined in Definition 20 are:

$$\begin{array}{rclrcl}
p_1(\alpha,\beta) & := & \left\lfloor \frac{\pi}{\omega_\alpha} \right\rfloor, & \vartheta_1(\alpha,\beta) & := & W^{\alpha}_{p_1(\alpha,\beta)}(\infty), \\
p_2(\alpha,\beta) & := & \left\lfloor T^{\beta}(\vartheta_1(\alpha,\beta)) + \frac{\pi}{\omega_\beta} \right\rfloor, & \vartheta_2(\alpha,\beta) & := & W^{\beta}_{p_2(\alpha,\beta)}(\vartheta_1(\alpha,\beta)), \\
p_3(\alpha,\beta) & := & \left\lfloor T^{\alpha}(\vartheta_2(\alpha,\beta)) + \frac{\pi}{\omega_\alpha} \right\rfloor, & \vartheta_3(\alpha,\beta) & := & W^{\alpha}_{p_3(\alpha,\beta)}(\vartheta_2(\alpha,\beta)), \\
p_4(\alpha,\beta) & := & \left\lfloor T^{\beta}(\vartheta_3(\alpha,\beta)) + \frac{\pi}{\omega_\beta} \right\rfloor, & \vartheta_4(\alpha,\beta) & := & W^{\beta}_{p_4(\alpha,\beta)}(\vartheta_3(\alpha,\beta)), \\
& \vdots & & & \vdots &
\end{array}$$

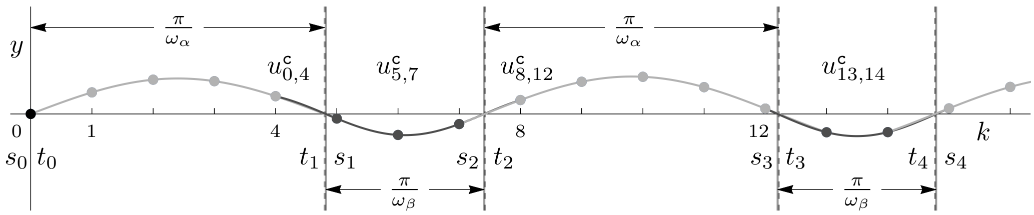



Figure 3.5: Positive and negative semi-waves of a solution of the semi-linear initial value problem (P3) for $0 < \alpha, \beta < 4$ and $C_1 > 0$ $(0 = s_0 = t_0 < t_1 < s_1 < s_2 < t_2 < s_3 < t_3 < t_4 < s_4)$.

In this part of the text, for simplification, we are going to write $p_1$ instead of $p_1(\alpha, \beta)$ and similarly for other terms of all sequences defined in Definition 20. For easier understanding of the following text, see Figure 3.5.

(a) First positive semi-wave: The first positive semi-wave of $u$ (we have $C_1 > 0$) is $u^{c}_{0,p_1}$, thus $p_1$ represents the length which we need to add to $t = 0$ in order to find interval where positive semi-wave is anchored with negative semi-wave.

Positive semi-wave $u^{c}_{0,p_1}$ is defined on $[-1, p_1 + 1]$ and has two zeros $t_0 = 0$ and $t_1 = \frac{\pi}{\omega_\alpha}$. For zero $t_1$ we have (remember, that function $T^{\alpha}(q_{p_1+1})$ returns position of zero of positive semi-wave calculated from $p_1$, since $q_{p_1+1} = \vartheta_1$ is the ratio $\frac{u(p_1+1)}{u(p_1)}$ – see Example 19 to compare how we calculated ratios $(q_k)$)

$$t_1 = p_1 + T^{\alpha}(q_{p_1+1}) = p_1 + T^{\alpha}(\vartheta_1).$$

The first positive generalized zero of $u$ is $z_1 = p_1 + 1$ if $\vartheta_1 < 0$ or $z_1 = p_1 = t_1$ if $\vartheta_1 = \infty$.

(b) First negative semi-wave: The next semi-wave of $u$ is negative. It has two zeros $s_1$ and $s_2$ and is defined on $[\lceil s_1 \rceil - 1, \lfloor s_2 \rfloor + 1]$. Its first zero $s_1$ can be calculated as

$$s_1 = \lfloor t_1 \rfloor + T^{\beta}\left(q_{\lfloor t_1 \rfloor + 1}\right) = p_1 + T^{\beta}\left(\vartheta_1\right).$$

And its second zero $s_2$ is

$$s_2 = s_1 + \frac{\pi}{\omega_\beta},$$

since we are just adding length of negative wave $\frac{\pi}{\omega_\beta}$ to the first zero $s_1$. For $s_2$ we have

$$\lfloor s_2 \rfloor = p_1 + p_2$$

which implies

$$q_{p_2+p_1+1} = W^{\beta}_{p_2}(p_1+1) = W^{\beta}_{p_2}(\vartheta_1) = W^{\beta}_{p_2}(W^{\alpha}_{p_1}(\infty)) = \vartheta_2$$

and

$$s_2 = p_1 + p_2 + T^{\beta}\left(\vartheta_2\right).$$

The second positive generalized zero of $u$ is $z_2 = p_1 + p_2 + 1$ if $\vartheta_2 < 0$ or $z_2 = p_1 + p_2 = s_2$ if $\vartheta_2 = \infty$.

(c) Second positive semi-wave: The next semi-wave of $u$ is the positive semi-wave $u^{\mathsf{c}}_{\lceil t_2 \rceil, \lfloor t_3 \rfloor}$, which has two zeros $t_2$ and $t_3$ and is defined on $[\lceil t_2 \rceil - 1, \lfloor t_3 \rfloor + 1]$. We have that $t_3 - t_2 = \frac{\pi}{\omega_\alpha}$ and

$$\begin{aligned} t_2 &= \lfloor s_2 \rfloor + T^{\alpha}\left(q_{\lfloor s_2 \rfloor + 1}\right) = p_1 + p_2 + T^{\alpha}\left(\vartheta_2\right), \\ \vartheta_3 &= q_{p_3+p_2+p_1+1} = W^{\alpha}_{p_3}(W^{\beta}_{p_2}(W^{\alpha}_{p_1}(\infty))), \\ t_3 &= \lfloor t_3 \rfloor + T^{\alpha}\left(q_{\lfloor t_3 \rfloor + 1}\right) = p_1 + p_2 + p_3 + T^{\alpha}(\vartheta_3). \end{aligned}$$

The third positive generalized zero of $u$ is $z_3 = p_1 + p_2 + p_3 + 1$ if $\vartheta_3 < 0$ or $z_3 = p_1 + p_2 + p_3 = t_3$ if $\vartheta_3 = \infty$.

**Example 21.** Let us go back to our problem from Example 19, thus let us take the semi-linear initial value problem (P3) with $\alpha = 0.8$, $\beta \doteq 3.94$, $C_1 = 1$. In the Example 19, sign properties were known, but this time, we are going to calculate them.

Sequences $(p_k)_{k\in\mathbb{Z}}$ and $(\vartheta_k)_{k\in\mathbb{Z}}$ from Definition 20 are (the reader is advised to compare these values with Example 19)

$$\begin{aligned} p_1 &= 3, & \vartheta_1 &\doteq -1.527, \\ p_2 &= 1, & \vartheta_2 &\doteq -1.282, \\ p_3 &= 3, & \vartheta_3 &\doteq -0.239, \\ p_4 &= 2, & \vartheta_4 &\doteq -2.382, \\ p_5 &= 3, & \vartheta_5 &\doteq -0.516, \\ &\vdots & &\vdots \end{aligned}$$

From sequence $(p_k)$ we have all positive and negative semi-waves. From $p_1 = 3$, we have that the first positive semi-wave is

$$u^{\mathsf{c}}_{0,p_1} = u^{\mathsf{c}}_{0,3}.$$

First negative semi-wave is

$$u^{\mathsf{c}}_{p_1+1,p_1+p_2} = u^{\mathsf{c}}_{4,4}.$$

Second positive semi-wave is

$$u^{\mathsf{c}}_{p_1+p_2+1,p_1+p_2+p_3} = u^{\mathsf{c}}_{5,7}.$$

Second negative semi-wave is

$$u^{\mathsf{c}}_{p_1+p_2+p_3+1,p_1+p_2+p_3+p_4} = u^{\mathsf{c}}_{8,9}.$$

Third positive semi-wave is

$$u^{\mathsf{c}}_{p_1+p_2+p_3+p_4+1,p_1+p_2+p_3+p_4+p_5} = u^{\mathsf{c}}_{10,12}.$$

The following lemma uses sequences from Definition 20 and shows how to calculate all generalized zeros of solution $u$ of a semi-linear initial value problem (P3) in a general case.

**Lemma 22.** *([25], Lemma 19, p. 82)*
*Let $(\alpha, \beta) \in D$ and let $u$ be a solution of the initial value problem* (P3) *with $C_1 > 0$. All generalized zeros of $u$ form a sequence $(z_m)_{m\in\mathbb{Z}}$, where*

$$z_{-i} = -P_i(\beta, \alpha), \qquad z_i = \begin{cases} P_i(\alpha,\beta) + 1 & \text{if } \vartheta_i(\alpha,\beta) \neq \infty, \\ P_i(\alpha,\beta) & \text{if } \vartheta_i(\alpha,\beta) = \infty, \end{cases} \qquad i \in \mathbb{N}.$$

*Moreover, the solution $u$ consists of infinitely many positive and negative semi-waves.*

*For $0 < \alpha < 4$ and $\beta > 0$, all zero points of all positive semi-waves form a sequence $(t_m)_{m\in\mathbb{Z}}$, where*

$$t_{-i} = -P_i(\beta,\alpha) - T^{\alpha}(\vartheta_i(\beta,\alpha)), \qquad t_0 = 0, \qquad t_i = P_i(\alpha,\beta) + T^{\alpha}(\vartheta_i(\alpha,\beta)), \qquad i \in \mathbb{N}.$$

*The $m$-th semi-wave, $m \in \mathbb{Z} \setminus \{0\}$, is positive one if and only if $m > 0$ is odd or $m < 0$ is even and it has exactly two zero points $t_{m-1}$ and $t_m$ for $0 < \alpha < 4$ and $\beta > 0$.*

*For $\alpha > 0$ and $0 < \beta < 4$, all zero points of all negative semi-waves form a sequence $(s_m)_{m\in\mathbb{Z}}$, where*

$$s_{-i} = -P_i(\beta,\alpha) - T^{\beta}(\vartheta_i(\beta,\alpha)), \qquad s_0 = 0, \qquad s_i = P_i(\alpha,\beta) + T^{\beta}(\vartheta_i(\alpha,\beta)), \qquad i \in \mathbb{N}.$$

*The $m$-th semi-wave, $m \in \mathbb{Z} \setminus \{0\}$, is negative one if and only if $m > 0$ is even or $m < 0$ is odd and it has exactly two zero points $s_{m-1}$ and $s_m$ for $\alpha > 0$ and $0 < \beta < 4$.*

The main goal of this thesis (and articles [25], [31]) is to study Fučík spectrum of matrix $\mathbf{A}^{\mathtt{D}}$ – thus values $(\alpha, \beta) \in \mathbb{R}^2$ such that the semi-linear boundary value problem (P4) has a non-trivial solution. Understanding how all generalized zeros of such solution can be retrieved (see Lemma 22), is crucial to an implicit description of Fučík curves $\mathcal{C}_k^{\pm}$, $k = 1, \ldots, n-1$. In Section 5.1, we will use sequences from Definition 20 in Theorem 32.

# Chapter 4

# Semi-linear initial value problem (P3) – Part II

In this chapter, we are going to investigate problem (P3) from a different angle. A continuous extension of solution $u$ of (P3) will be constructed in a manner considering positive semi-waves only. We will calculate the distance between every two consecutive zeros of two different (consecutive) positive semi-waves. This will allow us not only to study nodal properties of solution $u$ of (P3) in more detail, it will also allow us to find different implicit description of all Fučík curves $\mathcal{C}_k^{\pm}$, $k = 1, \ldots, n-1$ (which can be found in Chapter 5).

Let us recall problem (P3):

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, \quad k \in \mathbb{Z}, \\ u(0) = 0, u(1) = C_1, \end{cases}$$

where $u^{\pm}(k) = \max\{\pm u(k), 0\}$, $C_1 \in \mathbb{R}$, $C_1 \neq 0$.

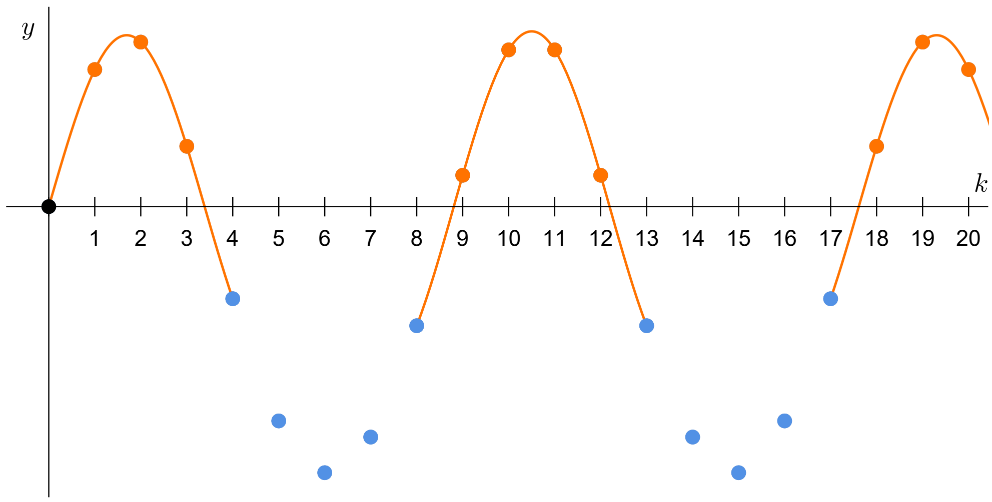



Figure 4.1: Continuous extension of only positive semi-waves for solution $u$ of problem (P3) for $\alpha = 0.8$, $\beta = 0.33$ and $C_1 = 1 > 0$.

Continuous extension – positive semi-waves only – can be seen on the Figure 4.1. If we would have $0 < \alpha < 4$ only, then the length of all positive semi-waves is the same and is equal to $\frac{\pi}{\omega_\alpha}$. This way, localization of intervals where positive semi-waves are anchored can be rephrased to – "what is the distance between every two consecutive zeros of two different consecutive positive semi-waves."

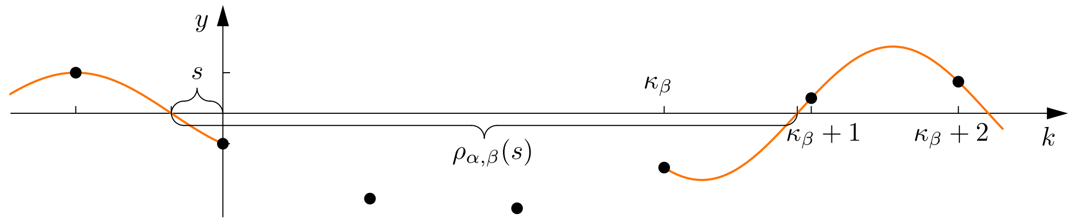



Figure 4.2: The distance $\rho_{\alpha,\beta}$ – the distance between two consecutive zeros (last and first) of two different positive semi-waves (orange color). Continuous extension for $\beta < 4$ ($\alpha = 3.5$, $\beta = 0.53$).

We denote such distance as $\rho_{\alpha,\beta}$ (we will define such function later in the text in Definition 28) – see Figure 4.2 for better understanding of the distance $\rho_{\alpha,\beta}$. Let us define half-strip $\mathcal{D}$ as

$$\mathcal{D} := (0,4) \times (0,+\infty).$$

In the following text, without any loss of generality, we are going to assume that $(\alpha,\beta) \in \mathcal{D}$ (it is enough to investigate $(\alpha,\beta) \in \mathcal{D}$ due to the symmetry of the Fučík spectrum). We note that it is easier to deal with zeros of positive semi-waves when $\alpha \in (0,4)$.

## 4.1 Length of a semi-wave

Let us define map $\kappa_\beta : (0,+\infty) \to \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, as (see [31], p. 9)

$$\kappa_\beta := \begin{cases} \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor - 1 & \text{for } 0 < \beta < 4, \\ 0 & \text{for } \beta \geq 4. \end{cases}$$

Such map divides half-strip $D$ into “rectangles” by $\kappa_\beta = k$, $k \in \mathbb{N}_0$ (see Figure 4.3), thus we have

$$\mathcal{D} = \big((0,4) \times (\xi_2,+\infty]\big) \cup \big((0,4) \times (\xi_3,\xi_2]\big) \cup \cdots \cup \big((0,4) \times (\xi_{k+2},\xi_{k+1}]\big) \cup \ldots,$$

where $\xi_k$ is given by the formula:

$$\xi_k := 4\sin^2 \frac{\pi}{2k}, \quad k \in \mathbb{N} \setminus \{1\}.$$

In the Table 4.1, we can see approximate numerical values of $\xi_k$.

| $\xi_2$ | $\xi_3$ | $\xi_4$ | $\xi_5$ | $\xi_6$ | $\xi_7$ | $\xi_8$ | $\xi_9$ | $\xi_{10}$ |
|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 0.586 | 0.382 | 0.268 | 0.198 | 0.152 | 0.121 | 0.098 |

Table 4.1: Approximate numerical values of $\xi_k$, $k = 2, 3, \ldots, 10$.

Values $\xi_k$ play important role in Chebyshev polynomials $V_k^\lambda$ of the second kind (for definition of $V_k^\lambda$, see (2.3) in Section 2.2). By comparison of definition of $\xi_k$ and zeros of Chebyshev polynomials $V_k^\lambda$ (refer to Section 2.2.2), we can see that the first positive zero of $V_k^\lambda$ is $\xi_{k+1}$ and similarly, the first positive zero of $V_{k+1}^\lambda$ is $\xi_{k+2}$. This is illustrated on Figure 4.4.

Function $\kappa_\lambda$ allows us to determine the length of a semi-wave (as continuous extension) – see the following lemma which describes semi-wave $u_{i,j}^{\mathfrak{c}}$. The semi-wave is defined on an interval $[i-1, j+1]$. Knowing $i$ and using the ratio $q_i = \frac{u(i)}{u(i-1)}$, we can (using the value $\kappa_\lambda$) determine $j$ (for illustration, see Figure 4.5).

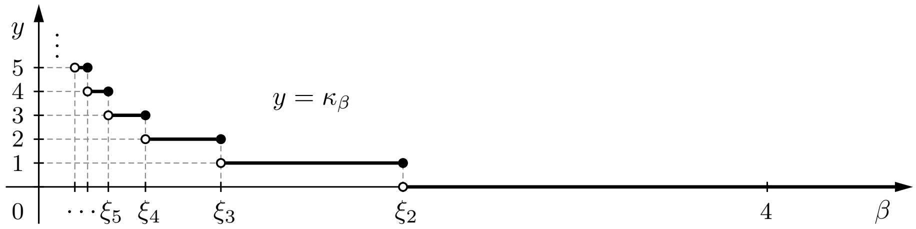



Figure 4.3: The graph of the piecewise constant function $\beta \mapsto \kappa_\beta$.

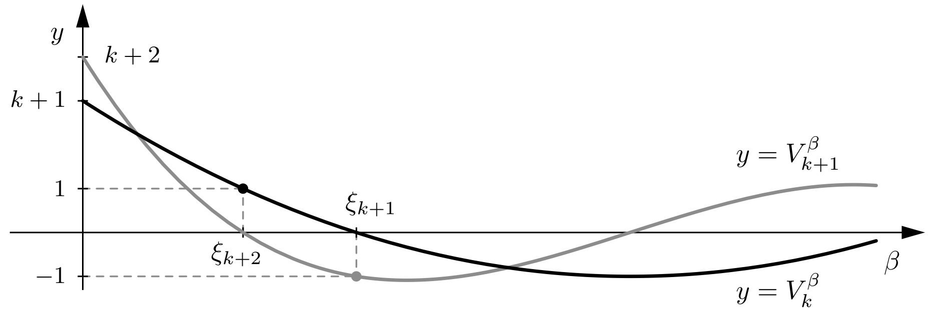



Figure 4.4: Graphs of functions $\beta \mapsto V_k^{\beta}$ (black curve) and $\beta \mapsto V_{k+1}^{\beta}$ (grey curve).

**Lemma 23.** *([31], Lemma 10, p. 24)*
*Let $(\alpha, \beta) \in D$ and $u$ be the solution of the initial value problem* (P3)*. Moreover, let $i, j \in \mathbb{Z}$ be such that $i \le j$ and*

$$u(i-1) < 0, \quad u(k) \ge 0 \quad \text{for } k = i, \dots, j, \quad u(j+1) < 0, \tag{4.1}$$

*or*

$$u(i-1) > 0, \quad u(k) \le 0 \quad \text{for } k = i, \dots, j, \quad u(j+1) > 0. \tag{4.2}$$

*Then we have*

$$j = \begin{cases} i + \kappa_\lambda & \text{for} \quad W^{\lambda}_{\kappa_\lambda+1}(q_i) < 0, \\ i + \kappa_\lambda + 1 & \text{for} \quad W^{\lambda}_{\kappa_\lambda+1}(q_i) \ge 0, \end{cases}$$

*where we denoted $q_i := \frac{u(i)}{u(i-1)} \le 0$ and $\lambda = \alpha$ if* (4.1) *holds or $\lambda = \beta$ if* (4.2) *holds. Moreover, we have $u(k) \ne 0$ for $k \in \mathbb{Z}$ such that $i < k < j$, and $u(j) = 0$ if and only if $W^{\lambda}_{\kappa_\lambda+1}(q_i) = 0$.*

## 4.2 The distance $\rho_{\alpha,\beta}$

Function $\kappa_\beta$ plays important role when finding the distance $\rho_{\alpha,\beta}$. We will use it as an order $k$ ($\kappa_\beta$ is a piece-wise linear function) for Chebyshev polynomials $V_k^{\lambda}$. In the following definition, we will define three functions $\eta_{\alpha,\beta}$, $\tau_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$. These functions (for fixed $\alpha, \beta$) represent important values for distance $\rho_{\alpha,\beta}$ – see Theorem 46.

**Definition 24.** ([31], Definition 1, p. 10)
For $0 < \alpha < 4$ and $\beta > 0$, let us define

$$\eta_{\alpha,\beta} := T^{\alpha}\left(\frac{V^{\beta}_{\kappa_\beta+1} - 1}{V^{\beta}_{\kappa_\beta}}\right), \qquad \tau_{\alpha,\beta} := T^{\alpha}\left(\frac{V^{\beta}_{\kappa_\beta+1}}{V^{\beta}_{\kappa_\beta}}\right), \qquad \mu_{\alpha,\beta} := T^{\alpha}\left(\frac{V^{\beta}_{\kappa_\beta+1}}{V^{\beta}_{\kappa_\beta} + 1}\right).$$

Using $\tau_{\alpha,\beta}$, we can formulate an implicit description of the first non-trivial Fučík curve $\mathcal{C}_1^{\pm}$ – see ([31], p. 6).

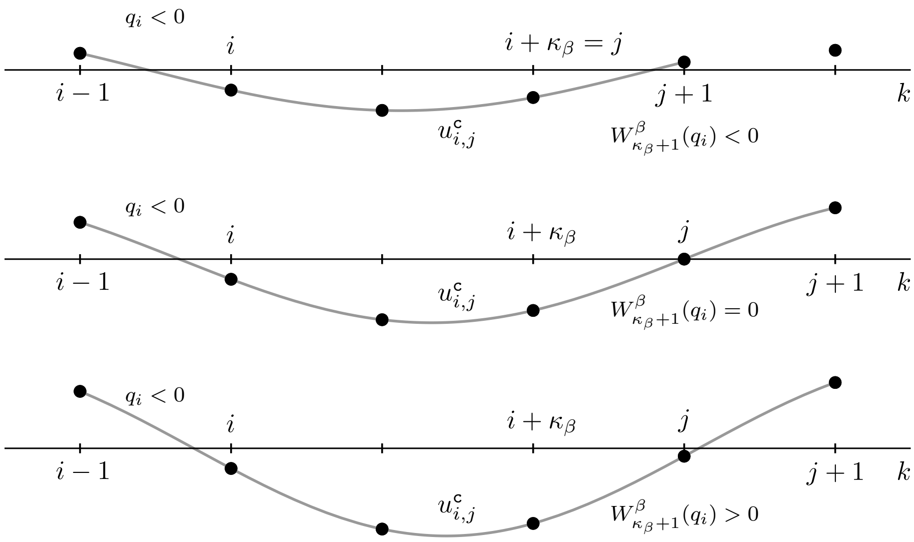



Figure 4.5: The length of the interval $[i-1, j+1]$ for a negative semi-wave $u^{\mathsf{c}}_{i,j}$ of the solution $u$ of (P3) for fixed $(\alpha, \beta) \in D$ according to the sign of $W^\beta_{\kappa_\beta+1}(q_i)$: $j = i + \kappa_\beta + 1$ and $u(j) < 0$ (bottom), $j = i + \kappa_\beta + 1$ and $u(j) = 0$ (middle) and $j = i + \kappa_\beta$ and $u(j) < 0$ (top).

Let us have two consecutive continuous positive semi-waves $u^{\mathsf{c}}_1$ and $u^{\mathsf{c}}_2$ of $u$ with zeros: the second zero of $u^{\mathsf{c}}_1$ is $t_1 \in (i-1, i]$ and the first zero of $u^{\mathsf{c}}_2$ is $t_2 \in [j, j+1)$. In the following lemma, we show how to reconstruct the zero $t_2$ according to values of $t_1$, $\alpha$ and $\beta$. For this reconstruction, we use $\tau_{\alpha,\beta}$ to distinguish between two disjoint cases

$$j = i + \kappa_\beta \qquad \text{and} \qquad j = i + \kappa_\beta + 1.$$

**Lemma 25.** *([31], Lemma 14, p. 30)*
*Let $u$ be the solution of the initial value problem* (P3) *for $0 < \alpha < 4$ and $\beta > 0$ and let $u^{\mathsf{c}}_1$ and $u^{\mathsf{c}}_2$ be two consecutive continuous positive semi-waves of $u$. Moreover, let $t_1$ be the second zero of $u^{\mathsf{c}}_1$ and let $t_2$ be the first zero of $u^{\mathsf{c}}_2$. If we denote*

$$s = \lceil t_1 \rceil - t_1,$$

*then we have*

$$t_2 = \begin{cases} t_1 + s + \kappa_\beta + T^\alpha\left(W^\beta_{\kappa_\beta+1}\left(Q^\alpha(1-s)\right)\right) & \text{for } s > \tau_{\alpha,\beta}, \\ t_1 + s + \kappa_\beta + 1 + T^\alpha\left(W^\beta_{\kappa_\beta+2}\left(Q^\alpha(1-s)\right)\right) & \text{for } s \le \tau_{\alpha,\beta}. \end{cases} \tag{4.3}$$

**Remark 26.** In Lemma 25, we have denoted

$$s = \lceil t_1 \rceil - t_1.$$

Such length represents the distance between $t_1$ and the nearest larger integer. Thus, we can reformulate this as the following. Let us have a positive semi-wave which has its second zero in the interval $[i-1, i]$. Then, we denote $s$ the length between that zero and the nearest larger integer. Such length can be retrieved from the ratio $q_i$ as $q_i = Q^\alpha(1-s)$, i.e. $s = 1 - T^\alpha(q_i)$. See Figure 4.10 for visual idea of how to retrieve $s$.

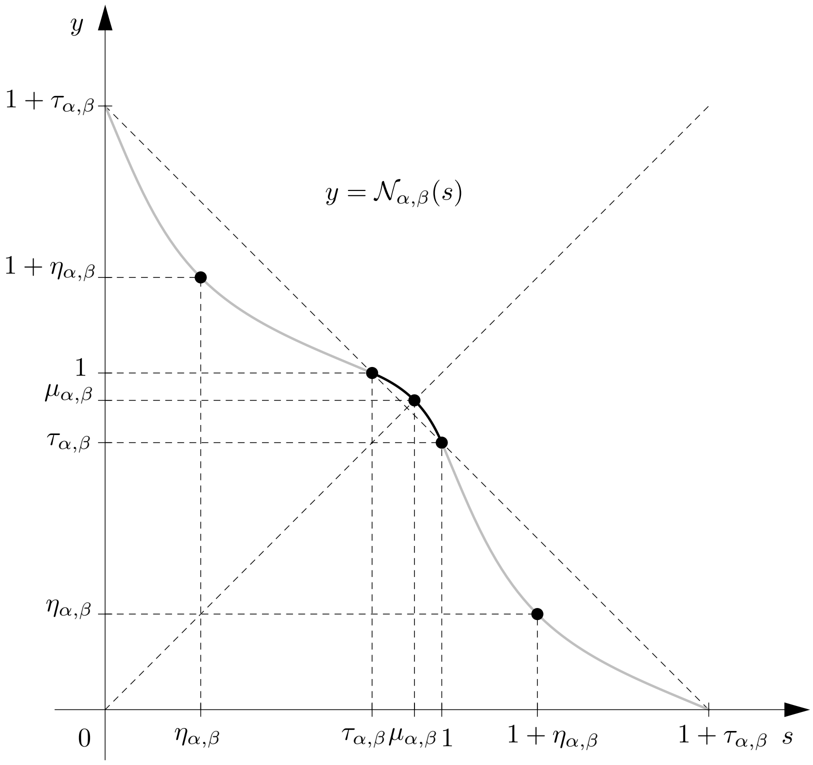



Figure 4.6: Function $\mathcal{N}_{\alpha,\beta}$ for $\alpha > \beta$ ($\alpha = 3.2$, $\beta = 1.2$).

**Definition 27.** ([31], Definition 17, p. 32)
For $0 < \alpha < 4$ and $\beta > 0$, let us define

$$\mathrm{Dom}(\mathcal{N}_{\alpha,\beta}) := [0, 1+\tau_{\alpha,\beta}], \qquad \mathcal{N}_{\alpha,\beta}(s) := \begin{cases} \bar{\bar{M}}_{\alpha,\beta}(s) + 1 & \text{for } s \in [0, \tau_{\alpha,\beta}], \\ \bar{M}_{\alpha,\beta}(s) & \text{for } s \in (\tau_{\alpha,\beta}, 1), \\ \bar{\bar{M}}_{\alpha,\beta}(s-1) & \text{for } s \in [1, 1+\tau_{\alpha,\beta}], \end{cases}$$

where

$$\begin{aligned} \bar{M}_{\alpha,\beta}(s) &:= T^{\alpha}\left(W^{\beta}_{\kappa_\beta+1}\left(Q^{\alpha}(1-s)\right)\right), \quad s \in [\tau_{\alpha,\beta}, 1], \\ \bar{\bar{M}}_{\alpha,\beta}(s) &:= T^{\alpha}\left(W^{\beta}_{\kappa_\beta+2}\left(Q^{\alpha}(1-s)\right)\right), \quad s \in [0, \tau_{\alpha,\beta}]. \end{aligned}$$

On Figure 4.6 we can see function $\mathcal{N}_{\alpha,\beta}$ in the case of $\alpha > \beta$, on Figure 6.1 we can see function $\mathcal{N}_{\alpha,\beta}$ in the case of $\alpha < \beta$ and finally on Figure 6.2 we can see three different shapes of graph of function $\mathcal{N}_{\alpha,\beta}$ when the values $\alpha$ are fixed and values of $\beta$ are changed (the reader is asked to notice convexity versus concavity in all of these cases).

In [31], we have investigated function $\mathcal{N}_{\alpha,\beta}$ in a lot of detail. In ([31], Lemma 20, p. 34) we have proved that function $\mathcal{N}_{\alpha,\beta}$ is a continuous involution, i.e.

$$\forall s \in [0, 1+\tau_{\alpha,\beta}] : \ \mathcal{N}_{\alpha,\beta}(\mathcal{N}_{\alpha,\beta}(s)) = s.$$

Moreover, we have $\mathcal{N}_{\alpha,\beta}(0) = 1 + \tau_{\alpha,\beta}$ and $\mathcal{N}_{\alpha,\beta}(\tau_{\alpha,\beta}) = 1$ (proved in the same lemma). Next, we have proved (in [31], Lemma 21, p. 35) that points $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ are fixed points of $\bar{\bar{M}}_{\alpha,\beta}$ and $\bar{M}_{\alpha,\beta}$, respectively. And that

$$\mathcal{N}_{\alpha,\beta}(\eta_{\alpha,\beta}) = \eta_{\alpha,\beta} + 1, \qquad \mathcal{N}_{\alpha,\beta}(\mu_{\alpha,\beta}) = \mu_{\alpha,\beta}.$$

But mainly, all of this leads towards definition of function $\rho_{\alpha,\beta}$ – the distance between two consecutive zeros (second and first) of two different consecutive positive semi-waves.

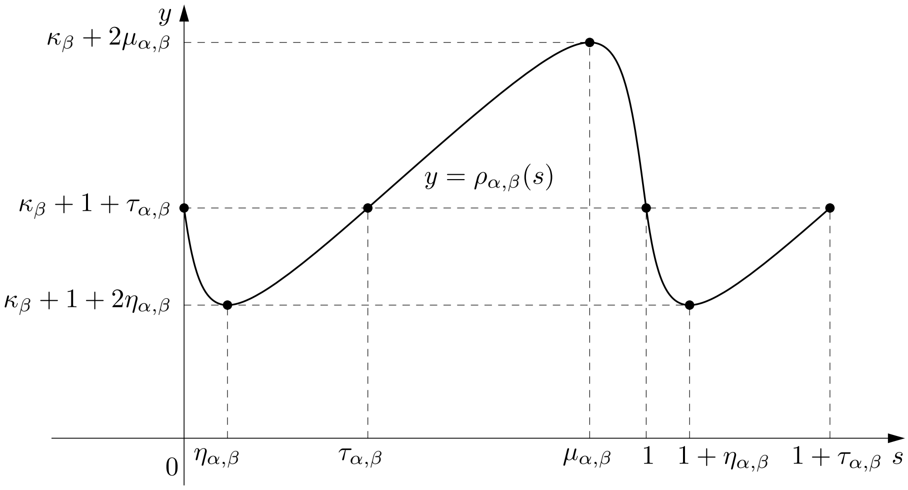



Figure 4.7: The graph of the function $\rho_{\alpha,\beta}$ for $\alpha > \beta$ ($\alpha = 3.9$, $\beta = 3.1$).

**Definition 28.** ([31], Definition 19, p. 33)
Let $0 < \alpha < 4$ and $\beta > 0$. Let us define

$$\rho_{\alpha,\beta}(s) := s + \kappa_\beta + \mathcal{N}_{\alpha,\beta}(s), \qquad 0 \le s \le 1 + \tau_{\alpha,\beta}.$$

See Figures 4.7 and 6.6 to see examples of graphs of function $\rho_{\alpha,\beta}$ for different $\alpha$ and $\beta$. The property of function $\rho_{\alpha,\beta}$ (what it measures) can be rephrased as the following. For zeros $t_1$ and $t_2$ in Lemma 25, we have that

$$t_2 = t_1 + \rho_{\alpha,\beta}(\lceil t_1 \rceil - t_1) \tag{4.4}$$

(we have denoted $s = \lceil t_1 \rceil - t_1$). Since this equality holds for all possible two consecutive positive semi-waves ($0 < \alpha < 4$), we can show how to find a sequence of these zeroes (of positive semi-waves) in Example 29.

## 4.3 Examples

Let us denote $(t_k^+)_{k\in\mathbb{N}}$ a sequence of positive zeros of all positive semi-waves (as continuous extension) for a solution $u$ of (P3) with the property, that $u(1) > 0$ (thus the first semi-wave is positive). Similarly, let us denote $(t_k^-)_{k\in\mathbb{N}}$ a sequence of positive zeros of all positive semi-waves (as continuous extension) for a solution $u$ of (P3) with the property, that $u(1) < 0$ (thus the first semi-wave is negative).

**Example 29.** Let us take the semi-linear initial value problem (P3) with $\alpha = 2.462$, $\beta \doteq 1.37$, $C_1 = 1 > 0$. Its solution $u$ is displayed on the Figure 4.8. We are going to show how to calculate positive zeros of positive semi-waves using distances $\rho_{\alpha,\beta}$.

Firstly, zero $t_1^+$ is calculated simply as

$$t_1^+ = \frac{\pi}{\omega_\alpha} \doteq 1.742.$$

The length $s$ can be calculated either as $s = \lceil t_1^+ \rceil - t_1^+ \doteq 2 - 1.742 = 0.258$ or as (see Remark 26) $1 - T^\alpha(q_2) = 1 - T^\alpha(-0.462) \doteq 0.258$. For $s$ we have $s \in [0, \tau_{\alpha,\beta}]$ ($\tau_{\alpha,\beta} \doteq 0.521$). Thus the second positive zero $t_2^+$ (which can be obtained using function $\rho_{\alpha,\beta}$) is

$$t_2^+ \;=\; t_1^+ + \rho_{\alpha,\beta}(\lceil t_1^+ \rceil - t_1^+) \doteq 4.189.$$

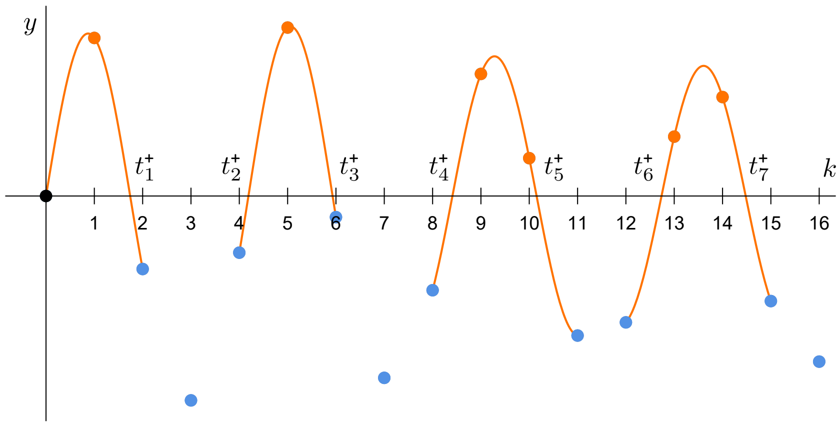



Figure 4.8: Solution $u$ from Example 29, continuous extension of positive semi-waves and zeros $(t_k^+)_{k\in\mathbb{N}}$.

Another zero $t_3^+$ is

$$t_3^+ = t_2^+ + \frac{\pi}{\omega_\alpha} \doteq 5.931$$

Next, we have $\lceil t_3^+ \rceil - t_3^+ \doteq 0.069$ and zero $t_4^+$ is

$$t_4^+ = t_3^+ + \rho_{\alpha,\beta}(\lceil t_3^+ \rceil - t_3^+) \doteq 8.41.$$

Zero $t_5^+$ is

$$t_5^+ = t_4^+ + \frac{\pi}{\omega_\alpha} \doteq 10.152.$$

For $t_6^+$ we have $\lceil t_5^+ \rceil - t_5^+ \doteq 0.848$ (notice that $(\lceil t_5^+ \rceil - t_5^+) \in [\tau_{\alpha,\beta}, 1]$ and there is a different number of negative values between $t_5^+$ and $t_6^+$ with comparison to previous cases – there will be one less negative point than in the previous case)

$$t_6^+ = t_5^+ + \rho_{\alpha,\beta}(\lceil t_5^+ \rceil - t_5^+) \doteq 12.737.$$

And finally, zero $t_7^+$ is

$$t_7^+ = t_6^+ + \frac{\pi}{\omega_\alpha} \doteq 14.48.$$

**Example 30.** Let us take the semi-linear initial value problem (P3) with $\alpha = 1.2$, $\beta \doteq 6.959$, $C_1 = -1 < 0$. Its solution $u$ is displayed on Figure 4.9. As in the previous example, we are going to show how to calculate positive zeros of positive semi-waves using distances $\rho_{\alpha,\beta}$.

The first zero $t_1^-$ is the first zero of the first positive semi-wave. It can be retrieved using function $\rho_{\alpha,\beta}$. The length $s$ is in this case equal to $s = 0$ (if we would extended the solution to negative values of $k$, the previous positive semi-wave would end at the point $k = 0$). Thus, we have

$$t_1^- = \rho_{\alpha,\beta}(0) \doteq 1.146.$$

For the next value of a sequence $(t_k^-)$ we have

$$t_2^- = t_1^- + \frac{\pi}{\omega_\alpha} \doteq 3.856.$$

For $t_3^-$, the value $s$ is equal to $s = \lceil t_2^- \rceil - t_2^- \doteq 0.144$. Since $\tau_{\alpha,\beta} \doteq 0.1461$, we have $(\lceil t_2^- \rceil - t_2^-) \in [0, \tau_{\alpha,\beta}]$ (which means that there will be more negative points between two positive semi-waves than in the other case) and

$$t_3^- = t_2^- + \rho_{\alpha,\beta}(\lceil t_2^- \rceil - t_2^-) \doteq 5.045.$$

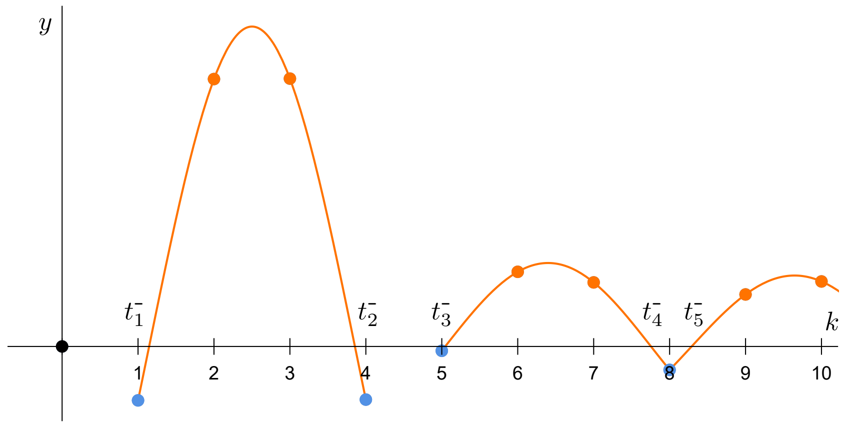



Figure 4.9: Solution $u$ from Example 30, continuous extension of positive semi-waves and zeros $(t_k^-)_{k\in\mathbb{N}}$.

Value $t_4^-$ is

$$t_4^- = t_3^- + \frac{\pi}{\omega_\alpha} \doteq 7.755.$$

And finally, for $t_5^-$ we have $\lceil t_4^- \rceil - t_4^- \doteq 0.245$, thus $(\lceil t_4^- \rceil - t_4^-) \in [\tau_{\alpha,\beta}, 1]$ (which means that there will be one less negative point between two positive semi-waves than in the previous case) and

$$t_5^- = t_4^- + \rho_{\alpha,\beta}(\lceil t_4^- \rceil - t_4^-) \doteq 8.29.$$

**Example 31.** This example shows how differently the solution can behave when changing a few parameters.

1. Let us assume $0 < \beta < 4$ and anchoring of two consecutive positive semi-waves. The length $\rho_{\alpha,\beta}$ depends not only on $s$ and $\mathcal{N}_{\alpha,\beta}$, but also on $\kappa_\beta$, since $\kappa_\beta$ is not equal to zero in general (for $\beta \leq 2$). On Figures 4.10 and 4.11 we illustrated examples of two such anchorings. In these two cases, we have selected solution in a way that the value $s$ is the same in both cases. They differ by value of $\beta$ ($\beta = 0.53$ vs. $\beta = 0.39$). This itself is enough for the length between positive semi-waves to differ by one – notice that the second positive semi-wave starts in the interval $[\kappa_\beta, \kappa_\beta + 1]$ for Figure 4.10 and starts in the interval $[\kappa_\beta + 1, \kappa_\beta + 2]$ for Figure 4.11.

2. On the other hand, in Figures 4.12 and 4.13 we have fixed $\alpha$ and $\beta$ on the same values ($\alpha = 2.1$, $\beta = 4.1$) but we have changed the value $s$ so that for these two solutions the second positive semi-wave starts in the different interval. In Figure 4.13, we can see the determining zero $\hat{t}$ (see Section 2.1 for more details about determining zero of a continuous extension) of negative semi-wave which is in the interval $[0, 1]$.

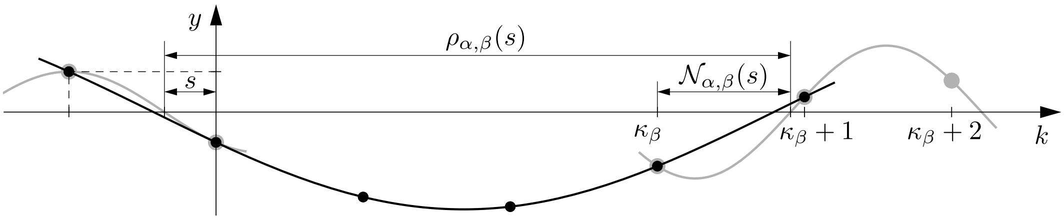



Figure 4.10: Continuous extension for $\beta < 4$ ($\alpha = 3.5$, $\beta = 0.53$) – the distance $\rho_{\alpha,\beta}$ which depends on $s$, $\kappa_\beta$ and $\mathcal{N}_{\alpha,\beta}$. The second positive semi-wave starts in the interval $[\kappa_\beta, \kappa_\beta + 1]$.

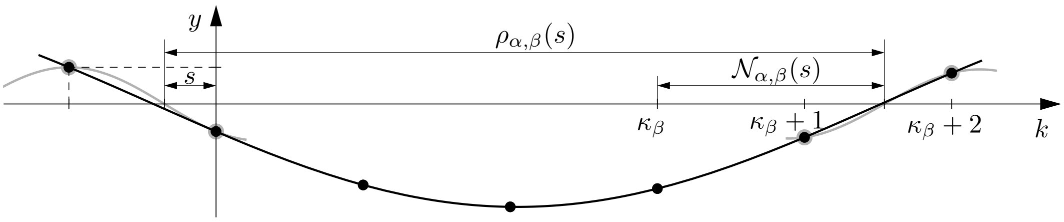



Figure 4.11: Continuous extension for $\beta < 4$ ($\alpha = 3.5$, $\beta = 0.39$) – the distance $\rho_{\alpha,\beta}$ which depends on $s$, $\kappa_\beta$ and $\mathcal{N}_{\alpha,\beta}$. The second positive semi-wave starts in the interval $[\kappa_\beta + 1, \kappa_\beta + 2]$.

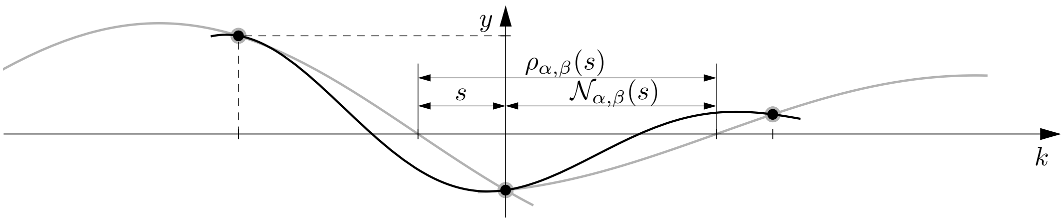



Figure 4.12: Continuous extension for $\beta > 4$ ($\alpha = 2.1$, $\beta = 4.1$) – the distance $\rho_{\alpha,\beta}$ which depends on $s$ and $\mathcal{N}_{\alpha,\beta}$ ($\kappa_\beta = 0$ for $\beta > 2$). There is just one negative value between two consecutive positive semi-waves.

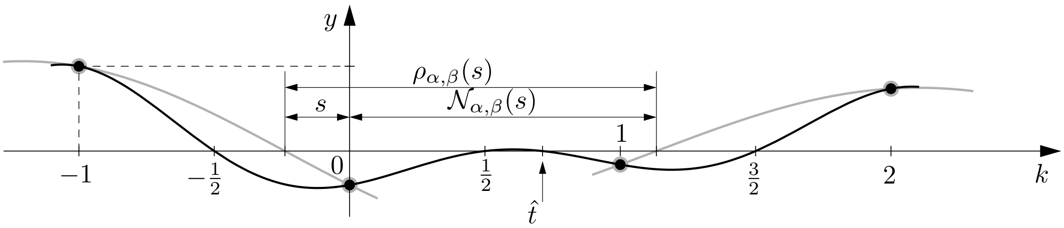



Figure 4.13: Continuous extension for $\beta > 4$ ($\alpha = 2.1$, $\beta = 4.1$) – the distance $\rho_{\alpha,\beta}$ which depends on $s$ and $\mathcal{N}_{\alpha,\beta}$ ($\kappa_\beta = 0$ for $\beta > 2$). There are two negative values between two consecutive positive semi-waves.

# Chapter 5

# Investigation of Fučík spectrum of matrix $\mathbf{A}^{\mathsf{D}}$ – Semi-linear BVP (P4)

This chapter is devoted to the investigation of semi-linear boundary value problem (P4)

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, & k \in \mathbb{T}, \\ u(0) = u(n+1) = 0, \end{cases}$$

where $n \in \mathbb{N}$, $n \geq 2$, $u^{\pm}(k) = \max\{\pm u(k), 0\}$ and $\alpha, \beta \in \mathbb{R}$.

Equivalently, the problem (P4) can be rephrased using a matrix notation

$$\mathbf{A}^{\mathsf{D}}\mathbf{u} = \alpha \mathbf{u}^+ - \beta \mathbf{u}^-,$$

where matrix $\mathbf{A}^{\mathsf{D}}$ is the Dirichlet matrix defined in (1.3). Thus, our main goal is to investigate Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$. Some known results were already discussed in Section 1.2.

In this chapter, we will focus on the description of Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$ using our two approaches described in Chapters 3 (Part I) and 4 (Part II).

## 5.1 Description of Fučík spectrum – Theory from Part I

In Chapter 3, we have studied semi-linear initial value problem (P3) in detail. Obtained sequences of functions $(p_k)_{k\in\mathbb{Z}}$, $(P_k)_{k\in\mathbb{Z}}$, $(\vartheta_k)_{k\in\mathbb{Z}}$, $(\mathcal{W}_k^+)_{k\in\mathbb{Z}}$ and $(\mathcal{W}_k^-)_{k\in\mathbb{Z}}$ from Definition 20 allow us to give the description of Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$ – see Theorem 32 (remember that it is enough to investigate Fučík curves $\mathcal{C}_k^+$, $k = 1, \ldots, n-1$).

**Theorem 32.** *([25], Theorem 22, p. 87)*
*For $k = 1, \ldots, n-1$, we have that*

$$\begin{aligned} \mathcal{C}_k^+ = \; & \{(\alpha,\beta) \in (0,4) \times (0,+\infty) : \; P_{k+1}(\alpha,\beta) + T^{\alpha}(\vartheta_{k+1}(\alpha,\beta)) = n+1\} \; \cup \\ & \left\{(\alpha,\beta) \in (0,+\infty) \times (0,4) : \; P_{k+1}(\alpha,\beta) + T^{\beta}(\vartheta_{k+1}(\alpha,\beta)) = n+1\right\}. \end{aligned}$$

*Moreover, if we denote*

$$\Omega_k^+ := \{(\alpha,\beta) \in D : \; P_{k+1}(\alpha,\beta) = n+1\}, \qquad k = 1, \ldots, n-1,$$

*then we have that*

$$\mathcal{C}_k^+ = \left\{(\alpha,\beta) \in \Omega_k^+ : \; \mathcal{W}_{k+1}^+(\infty) = \infty\right\}.$$

An example of sets $\Omega_k^+$ for $n = 4$ can be seen on Figure 5.1. Also, see Figure 5.3 for the complete Fučík spectrum of matrix $\mathbf{A}^{\mathsf{D}}$ for $n = 4$ and $n = 7$ (including Fučík curves $\mathcal{C}_k^-$).

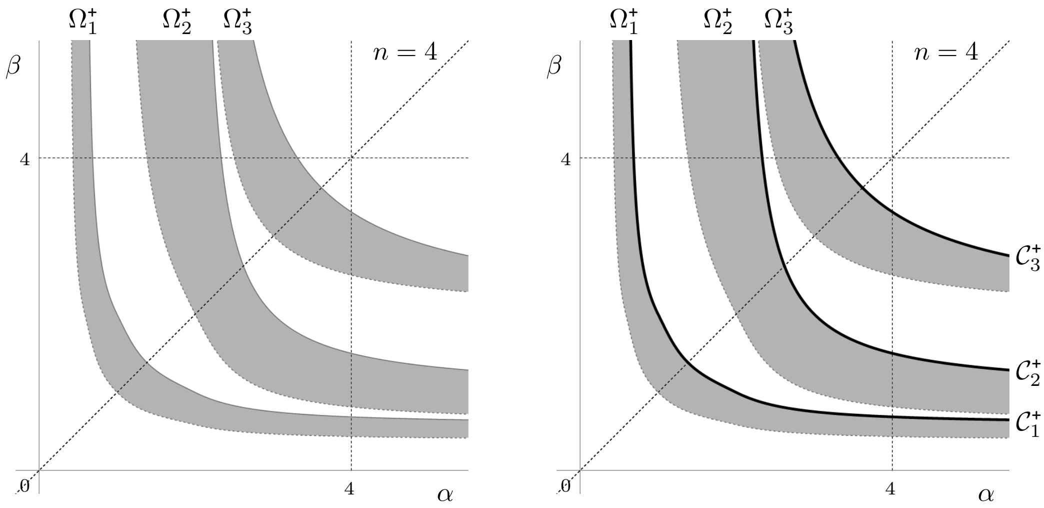



Figure 5.1: The sets $\Omega_k^+$ as grey regions for $n = 4$ (left) and the Fučík curves $\mathcal{C}_k^+$ as black curves (right) due to Theorem 32.

**Remark 33.** Theorem 32 basically says that $(\alpha, \beta)$ belongs to the Fučík curve $\mathcal{C}_k^+$ if the corresponding solution $u$ satisfies the following. It needs to start with a positive semi-wave (i.e. $u(1) > 0$), it needs to have $k$ generalized zeros on $\mathbb{T}$ (that is why we are using function $P_{k+1}$) and the ratio $q_{n+1}$ needs to be zero.

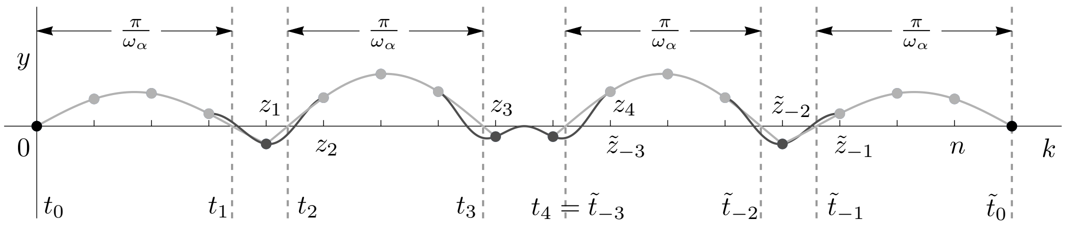



Figure 5.2: The non-trivial solution $u$ of (P4) for $(\alpha, \beta) \in \mathcal{C}_6^+$ (parameters from Theorem 34 are: $n = 16$, $i = 4$, $j = 3$, $k = 6$) with six generalized zeros of $u$ on $\mathbb{T}$ ($z_1 < z_2 < z_3 < z_4 = \tilde{z}_{-3} < \tilde{z}_{-2} < \tilde{z}_{-1}$) and six zeros of positive semi-waves strictly between 0 and $n+1$ ($t_1 < t_2 < t_3 < t_4 = \tilde{t}_{-3} < \tilde{t}_{-2} < \tilde{t}_{-1}$).

We can also provide a different description when we will "glue" together solutions from both end points – from $k = 0$ to the right and from $k = n+1$ to the left. Thus, we consider solutions of two initial value problems starting at $k = 0$ and at $k = n+1$ and we require that their selected zero points of positive (or negative) semi-waves coincide (see Figure 5.2 and note that $t_4 = \tilde{t}_{-3}$).

**Theorem 34.** *([25], Theorem 26, p. 91)*
*Let $k, n \in \mathbb{N}$ be such that $k \leq n-1$, $n \geq 2$. Moreover, let $i, j \in \mathbb{N}$ be such that $i + j = k + 1$.*

1. *If $k$ is odd then*

$$\begin{aligned} \mathcal{C}_k^+ = \; & \{(\alpha,\beta) \in (0,4) \times (0,+\infty) : \; P_i(\alpha,\beta) + P_j(\beta,\alpha) + T^\alpha(\vartheta_i(\alpha,\beta)) + T^\alpha(\vartheta_j(\beta,\alpha)) = n+1\} \; \cup \\ & \left\{(\alpha,\beta) \in (0,+\infty) \times (0,4) : \; P_i(\alpha,\beta) + P_j(\beta,\alpha) + T^\beta(\vartheta_i(\alpha,\beta)) + T^\beta(\vartheta_j(\beta,\alpha)) = n+1\right\}. \end{aligned}$$

2. *If $k$ is even then*

$$\begin{aligned} \mathcal{C}_k^+ = \; & \{(\alpha,\beta) \in (0,4) \times (0,+\infty) : \; P_i(\alpha,\beta) + P_j(\alpha,\beta) + T^\alpha(\vartheta_i(\alpha,\beta)) + T^\alpha(\vartheta_j(\alpha,\beta)) = n+1\} \; \cup \\ & \left\{(\alpha,\beta) \in (0,+\infty) \times (0,4) : \; P_i(\alpha,\beta) + P_j(\alpha,\beta) + T^\beta(\vartheta_i(\alpha,\beta)) + T^\beta(\vartheta_j(\alpha,\beta)) = n+1\right\}. \end{aligned}$$

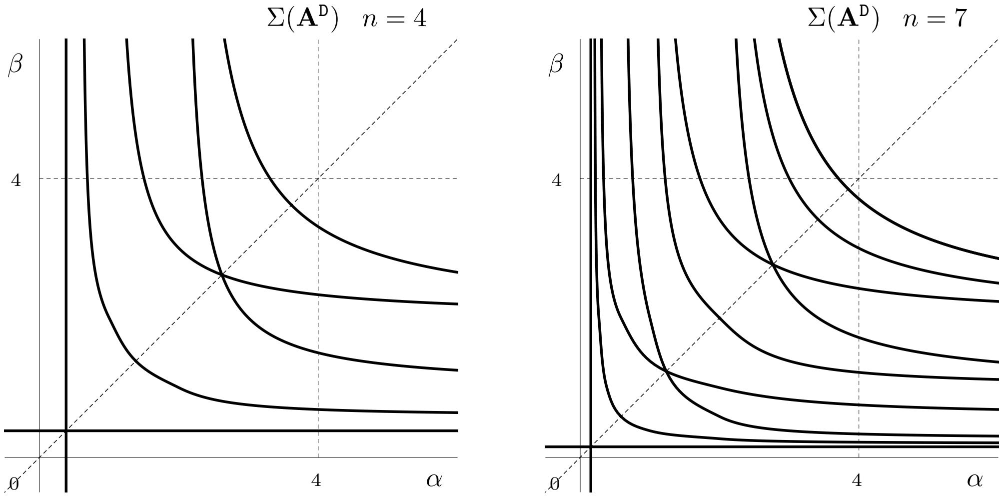



Figure 5.3: Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathrm{D}}\right)$ for $n=4$ (left) and $n=7$ (right).

See Figure 6.3 for example of solutions $u$ for $(\alpha,\beta)\in\mathcal{C}_5^{\pm}$ for $\Sigma(\mathbf{A}^{\mathrm{D}})$ of $n=9$. In this figure, each of the smaller figures has a bit different set of $(\alpha,\beta)\in\mathcal{C}_5^{\pm}$ – we are "moving" alongside the Fučík curve $\mathcal{C}_5^{\pm}$ and we can see how the corresponding solutions change (they even change the sign property of the solutions).

## 5.2 Description of Fučík spectrum – Theory from Part II

Chapter 4 was devoted to the investigation of zeros of all positive semi-waves. We have been dealing with $\rho_{\alpha,\beta}$ which is a function that measures the length between two consecutive zeros of every two different consecutive positive semi-waves.

**Theorem 35.** *([31], Theorem 5, p. 14)*
*In the domain $\mathcal{D}=(0,4)\times(0,+\infty)$, we have the following description of Fučík curves $\mathcal{C}_l^{\pm}$, $l=1,\dots,n-1$,*

$$\begin{aligned}
\mathcal{C}_{2j-1}^{\pm}\cap\mathcal{D} &= \left\{(\alpha,\beta)\in\mathcal{D}:\ t_j^{+}(\alpha,\beta)+t_j^{-}(\alpha,\beta)=n+1\right\},\\
\mathcal{C}_{2j}^{+}\cap\mathcal{D} &= \left\{(\alpha,\beta)\in\mathcal{D}:\ t_{j+1}^{+}(\alpha,\beta)+t_j^{+}(\alpha,\beta)=n+1\right\},\\
\mathcal{C}_{2j}^{-}\cap\mathcal{D} &= \left\{(\alpha,\beta)\in\mathcal{D}:\ t_{j+1}^{-}(\alpha,\beta)+t_j^{-}(\alpha,\beta)=n+1\right\},
\end{aligned}$$

*where*

$$t_1^{+}:=\frac{\pi}{\omega_\alpha},\qquad t_j^{+}:=\begin{cases} t_{j-1}^{+}+\rho_{\alpha,\beta}\left(\lceil t_{j-1}^{+}\rceil-t_{j-1}^{+}\right) & \text{for } j \text{ even},\\ t_{j-1}^{+}+\frac{\pi}{\omega_\alpha} & \text{for } j \text{ odd},\end{cases}$$

$$t_1^{-}:=\rho_{\alpha,\beta}(0),\qquad t_j^{-}:=\begin{cases} t_{j-1}^{-}+\frac{\pi}{\omega_\alpha} & \text{for } j \text{ even},\\ t_{j-1}^{-}+\rho_{\alpha,\beta}\left(\lceil t_{j-1}^{-}\rceil-t_{j-1}^{-}\right) & \text{for } j \text{ odd}.\end{cases}$$

**Remark 36.** In Examples 4.8 and 4.9 we have explored in detail semi-linear initial value problem (P3) for two cases – one for the solution that starts with a positive semi-wave and has first positive zero denoted as $t_1^{+}$ (Example 4.8) and the second for the solution that starts with a negative semi-wave and has first positive zero denoted as $t_1^{-}$ (Example 4.9). Notice, that in Theorem 35, we are defining sequences $(t_k^{+})$ and $(t_k^{-})$ exactly in the same way as we did in those two examples. The way that Fučík spectrum is retrieved is based on an idea of "anchoring the solution from both ends" – thus for the Fučík curves with solutions with $2j$ generalized zeros on $\mathbb{T}$, we need only (at maximum) $(j+1)$th terms of sequences $(t_k^{-})$ and $(t_k^{+})$.

## 5.3 Comparison of descriptions of Fučík spectrum

Natural question at this point is why to have more descriptions of Fučík spectrum $\Sigma(\mathbf{A}^{\mathsf{D}})$ using different functions (function sequences $(p_k), (\vartheta_k)$ versus distance $\rho_{\alpha,\beta}$) and which description is better. In all approaches, the description is implicit and recurrent but the second approach (using function $\rho_{\alpha,\beta}$) is more suitable for numerical calculations because it is easier to implement.

**Example 37.** In this example, we are going to showcase both different approaches of description of Fučík spectrum $\Sigma(\mathbf{A}^{\mathsf{D}})$. In order to use the same level of nested functions, we are going to compare Theorems 34 and 35. In both cases, we are going to take the Fučík curve $\mathcal{C}_3^+$. Solution for one $(\alpha,\beta) \in \mathcal{C}_3^+$ is shown in Figure 5.4. For simplicity, we are going to assume $0 < \alpha, \beta < 4$ in both cases. We will take the description of the Fučík curve $\mathcal{C}_3^+$ and specify it in extended (detailed) form for the purpose of showcasing how to work with defined functions and how complicated the description is – thus numerical implementation is needed.

1. Using Theorem 34 we have that $(\alpha,\beta) \in \mathcal{C}_3^+$ has to satisfy
$$P_2(\alpha,\beta) + P_2(\beta,\alpha) + T^\alpha(\vartheta_2(\alpha,\beta)) + T^\alpha(\vartheta_2(\beta,\alpha)) = n+1. \tag{5.1}$$
Sequences $(P_k)$ and $(\vartheta_k)$ are recurrent. Equality in (5.1) is
$$p_1(\alpha,\beta) + p_2(\alpha,\beta) + p_1(\beta,\alpha) + p_2(\beta,\alpha) + T^\alpha(\vartheta_2(\alpha,\beta)) + T^\alpha(\vartheta_2(\beta,\alpha)) = n+1.$$
We know $p_1(\alpha,\beta) = \left\lfloor \frac{\pi}{\omega_\alpha} \right\rfloor$ and $p_1(\beta,\alpha) = \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor$. We have
$$\begin{aligned} \vartheta_1(\alpha,\beta) &= W^\alpha_{p_1(\alpha,\beta)}(\infty), \\ \vartheta_1(\beta,\alpha) &= W^\beta_{p_1(\beta,\alpha)}(\infty). \end{aligned}$$
Next:
$$\begin{aligned} p_2(\alpha,\beta) &= \left\lfloor T^\beta\big(\vartheta_1(\alpha,\beta)\big) + \tfrac{\pi}{\omega_\beta} \right\rfloor &&= \left\lfloor T^\beta\big(W^\alpha_{p_1(\alpha,\beta)}(\infty)\big) + \tfrac{\pi}{\omega_\beta} \right\rfloor, \\ p_2(\beta,\alpha) &= \left\lfloor T^\alpha\big(\vartheta_1(\beta,\alpha)\big) + \tfrac{\pi}{\omega_\alpha} \right\rfloor &&= \left\lfloor T^\alpha\big(W^\beta_{p_1(\beta,\alpha)}(\infty)\big) + \tfrac{\pi}{\omega_\alpha} \right\rfloor. \end{aligned}$$
And finally:
$$\begin{aligned} \vartheta_2(\alpha,\beta) &= W^\beta_{p_2(\alpha,\beta)}(\vartheta_1(\alpha,\beta)) \\ &= W^\beta_{\left\lfloor T^\beta\left(W^\alpha_{p_1(\alpha,\beta)}(\infty)\right) + \frac{\pi}{\omega_\beta}\right\rfloor}\left(W^\alpha_{p_1(\alpha,\beta)}(\infty)\right), \\ \vartheta_2(\beta,\alpha) &= W^\alpha_{p_2(\beta,\alpha)}(\vartheta_1(\beta,\alpha)) \\ &= W^\alpha_{\left\lfloor T^\alpha\left(W^\beta_{p_1(\beta,\alpha)}(\infty)\right) + \frac{\pi}{\omega_\alpha}\right\rfloor}\left(W^\beta_{p_1(\beta,\alpha)}(\infty)\right). \end{aligned}$$
That means that we are looking for $(\alpha,\beta) \in ((0,4)\times(0,4))$ such that the following equation is satisfied:
$$\begin{aligned} &\left\lfloor \tfrac{\pi}{\omega_\alpha} \right\rfloor + \left\lfloor T^\beta\big(W^\alpha_{\lfloor \frac{\pi}{\omega_\alpha} \rfloor}(\infty)\big) + \tfrac{\pi}{\omega_\beta} \right\rfloor + \left\lfloor \tfrac{\pi}{\omega_\beta} \right\rfloor + \left\lfloor T^\alpha\big(W^\beta_{\lfloor \frac{\pi}{\omega_\beta} \rfloor}(\infty)\big) + \tfrac{\pi}{\omega_\alpha} \right\rfloor \\ &+T^\alpha\left( W^\beta_{\left\lfloor T^\beta\left(W^\alpha_{\lfloor \frac{\pi}{\omega_\alpha} \rfloor}(\infty)\right) + \frac{\pi}{\omega_\beta}\right\rfloor}\left(W^\alpha_{\lfloor \frac{\pi}{\omega_\alpha} \rfloor}(\infty)\right)\right) \\ &+T^\beta\left( W^\alpha_{\left\lfloor T^\alpha\left(W^\beta_{\lfloor \frac{\pi}{\omega_\beta} \rfloor}(\infty)\right) + \frac{\pi}{\omega_\alpha}\right\rfloor}\left(W^\beta_{\lfloor \frac{\pi}{\omega_\beta} \rfloor}(\infty)\right)\right) \\ &= n+1. \end{aligned}$$

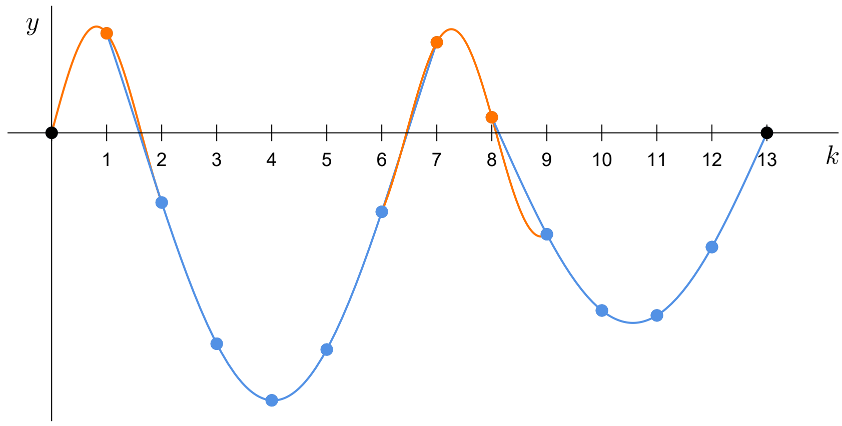



Figure 5.4: Solution $u$ for $\alpha = 2.7$, $\beta \doteq 0.4$, $(\alpha, \beta) \in \mathcal{C}_3^+$ of $\Sigma(\mathbf{A}^{\mathsf{D}})$ for $n = 12$.

2. On the other hand, using Theorem 35 we are solving

$$t_2^+(\alpha, \beta) + t_2^-(\alpha, \beta) = n + 1. \tag{5.2}$$

Since $t_1^-(\alpha, \beta) = \rho_{\alpha,\beta}(0)$ we have

$$t_2^-(\alpha, \beta) = t_1^-(\alpha, \beta) + \frac{\pi}{\omega_\alpha} = \rho_{\alpha,\beta}(0) + \frac{\pi}{\omega_\alpha}.$$

Term $\rho_{\alpha,\beta}(0)$ can be written as

$$\rho_{\alpha,\beta}(0) = \kappa_\beta + 1 + \tau_{\alpha,\beta} = \kappa_\beta + 1 + T^\alpha\left(\frac{V^\beta_{\kappa_\beta+1}}{V^\beta_{\kappa_\beta}}\right) = \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor + T^\alpha\left(\frac{V^\beta_{\left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor}}{V^\beta_{\left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor - 1}}\right).$$

Similarly, since $t_1^+(\alpha, \beta) = \dfrac{\pi}{\omega_\alpha}$, term $t_2^+(\alpha, \beta)$ is

$$t_2^+(\alpha, \beta) = t_1^+(\alpha, \beta) + \rho_{\alpha,\beta}\left(\lceil t_1^+(\alpha, \beta) \rceil - t_1^+(\alpha, \beta)\right) = \frac{\pi}{\omega_\alpha} + \rho_{\alpha,\beta}\left(\left\lceil \frac{\pi}{\omega_\alpha} \right\rceil - \frac{\pi}{\omega_\alpha}\right).$$

For term $\rho_{\alpha,\beta}\left(\left\lceil \frac{\pi}{\omega_\alpha} \right\rceil - \frac{\pi}{\omega_\alpha}\right)$ we have

$$\begin{aligned} \rho_{\alpha,\beta}\left(\left\lceil \frac{\pi}{\omega_\alpha} \right\rceil - \frac{\pi}{\omega_\alpha}\right) &= \left\lceil \frac{\pi}{\omega_\alpha} \right\rceil - \frac{\pi}{\omega_\alpha} + \kappa_\beta + \mathcal{N}_{\alpha,\beta}\left(\left\lceil \frac{\pi}{\omega_\alpha} \right\rceil - \frac{\pi}{\omega_\alpha}\right) \\ &= \left\lceil \frac{\pi}{\omega_\alpha} \right\rceil - \frac{\pi}{\omega_\alpha} + \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor - 1 + \mathcal{N}_{\alpha,\beta}\left(\left\lceil \frac{\pi}{\omega_\alpha} \right\rceil - \frac{\pi}{\omega_\alpha}\right). \end{aligned}$$

Function $\mathcal{N}_{\alpha,\beta}$ is a piecewise function which we can write as the following. Since $\left(\left\lceil \frac{\pi}{\omega_\alpha} \right\rceil - \frac{\pi}{\omega_\alpha}\right) \in [0, 1)$, we will assume only two distinct cases (first and second term in the definition of $\mathcal{N}_{\alpha,\beta}$):

$$\mathcal{N}_{\alpha,\beta}\left(\left\lceil\frac{\pi}{\omega_\alpha}\right\rceil - \frac{\pi}{\omega_\alpha}\right) = \begin{cases} T^\alpha\left(W^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor+1}\left(Q^\alpha\left(1-\left(\left\lceil\frac{\pi}{\omega_\alpha}\right\rceil - \frac{\pi}{\omega_\alpha}\right)\right)\right)\right)+1 \\ \quad \text{for } \left(\left\lceil\frac{\pi}{\omega_\alpha}\right\rceil - \frac{\pi}{\omega_\alpha}\right) \in \left[0, T^\alpha\left(\frac{V^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor}}{V^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor-1}}\right)\right], \\ T^\alpha\left(W^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor}\left(Q^\alpha\left(1-\left(\left\lceil\frac{\pi}{\omega_\alpha}\right\rceil - \frac{\pi}{\omega_\alpha}\right)\right)\right)\right) \\ \quad \text{for } \left(\left\lceil\frac{\pi}{\omega_\alpha}\right\rceil - \frac{\pi}{\omega_\alpha}\right) \in \left(T^\alpha\left(\frac{V^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor}}{V^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor-1}}\right), 1\right). \end{cases}$$

That means that we are looking for $(\alpha, \beta) \in ((0,4)\times(0,4))$ such that the following is satisfied (we are going to write equation in (5.2) also piecewise):

For $\left(\left\lceil\frac{\pi}{\omega_\alpha}\right\rceil - \frac{\pi}{\omega_\alpha}\right) \in \left[0, T^\alpha\left(\frac{V^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor}}{V^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor-1}}\right)\right]$:

$$\left\lceil\frac{\pi}{\omega_\alpha}\right\rceil + 2\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor + T^\alpha\left(W^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor+1}\left(Q^\alpha\left(1-\left(\left\lceil\frac{\pi}{\omega_\alpha}\right\rceil - \frac{\pi}{\omega_\alpha}\right)\right)\right)\right) + T^\alpha\left(\frac{V^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor}}{V^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor-1}}\right) = n+1,$$

for $\left(\left\lceil\frac{\pi}{\omega_\alpha}\right\rceil - \frac{\pi}{\omega_\alpha}\right) \in \left(T^\alpha\left(\frac{V^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor}}{V^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor-1}}\right), 1\right)$:

$$\left\lceil\frac{\pi}{\omega_\alpha}\right\rceil + 2\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor - 1 + T^\alpha\left(W^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor}\left(Q^\alpha\left(1-\left(\left\lceil\frac{\pi}{\omega_\alpha}\right\rceil - \frac{\pi}{\omega_\alpha}\right)\right)\right)\right) + T^\alpha\left(\frac{V^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor}}{V^\beta_{\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor-1}}\right) = n+1.$$

Let us note, that calculating Fučík curves is actually a little easier using function $\rho_{\alpha,\beta}$ (using numerical implementation). The reason is that all functions $\rho_{\alpha,\beta}, \mathcal{N}_{\alpha,\beta}, \tau_{\alpha,\beta}, \kappa_\beta$ are not recurrent.

**Example 38.** In this example, we are going to briefly discuss computational complexity of numerical localization of Fučík curves.

1. In the approach described in Chapter 1 (mainly in Example 2), we are solving $2^{n-1}$ subproblems ($2^{n-1}$ comes from the number of all possible sign properties of vectors). For every candidate for sign property of the Fučík eigenvector, we calculate the following:

   We calculate $\det(\mathbf{A}^{\mathsf{D}} - \mathbf{\Lambda})$, where matrix $\Lambda$ has values $\alpha$ and $\beta$ on the diagonal and their position depends on the chosen sign property. Then we find zeros of such determinant (for example in $\beta$). This gives us candidates for Fučík eigenpairs. Then for each of this candidate, we need to calculate the eigenvector. When we have candidates for Fučík eigenvectors, we have to check whether the sign property is satisfied or not.

   That means that we are solving our problem in an exponential time.

2. Now, let us describe how we can numerically show Fučík spectrum of $\mathbf{A}^{\mathsf{D}}$ using the knowledge from Theorems 32, 34 and 35. We need to fix Fučík curve $\mathcal{C}^{\pm}_l$ ($l \in \{0, 1, \ldots, n-1\}$). Since all descriptions are implicit (recurrent and non-trivial), firstly we need to find implicit function

which describes such Fučík curve (using recurrence in descriptions). After that, we are looking for zero contour of that implicit function (if there is a problem of looking for zero contour of implicit function, we can always fix $\alpha \in \mathbb{R}$ and calculate $\beta$ from that implicit function; or the other way around).

That means that were able to convert our problem to the problem of finding a zero contour of an implicit function (in contrast to the exponential time in the original algorithm).

Let us note that even though our descriptions of the Fučík spectrum are for a particular matrix $\mathbf{A}^{\mathtt{D}}$ (Dirichlet matrix), theory in this thesis (and both research articles [25] and [31]) can be extended. The theory was constructed for a difference equation in (P4). In order to describe the Fučík spectrum for a problem with the same difference equation but with different boundary conditions, one would use the same theory only changing aspects related to the boundary conditions. Thus, our results can be generalized also for different boundary conditions (one would need to explore the inadmissible areas for such matrices, since our theory does not include cases ($\alpha > 4$ and $\beta > 4$) and ($\alpha < 0$ or $\beta < 0$)).

In the next three sections, we will introduce three possible ways how to localize Fučík curves of $\Sigma(\mathbf{A}^{\mathtt{D}})$ in such a way, that the description of these bounds is simpler than the description of the Fučík curves themselves – mainly, the description of bounds will not be recurrent and it will not become more complicated when dimension $n$ increases.

## 5.4 Basic bounds of Fučík curves and their consequence

First bounds we are going to discuss are referred to as the “basic bounds”. Probably the simplest way how to get bounds of Fučík curves is to use properties of $\kappa_\beta$ and $\kappa_\alpha$. The following theorem gives us the basic bounds for the Fučík curves of $\Sigma(\mathbf{A}^{\mathtt{D}})$.

**Theorem 39.** *([31], Theorem 13, p. 28)*
*In the domain $D$, we have the following bounds for Fučík curves $\mathcal{C}^{\pm}_l \subset \Theta^{\pm}_l$, $l = 1, \dots, n-1$, where*

$$\begin{aligned} \Theta^{\pm}_{2j-1} &:= \{(\alpha,\beta) \in D : \ 0 \le n+1 - j(\kappa_\alpha+1) - j(\kappa_\beta+1) \le 2j-1\}\,, \\ \Theta^{+}_{2j} &:= \{(\alpha,\beta) \in D : \ 0 \le n+1 - (j+1)(\kappa_\alpha+1) - j(\kappa_\beta+1) \le 2j\}\,, \\ \Theta^{-}_{2j} &:= \{(\alpha,\beta) \in D : \ 0 \le n+1 - j(\kappa_\alpha+1) - (j+1)(\kappa_\beta+1) \le 2j\}\,. \end{aligned}$$

These basic bounds can be used for the first localization of Fučík curves. Thus when we calculate points of the Fučík spectrum numerically, they can give us estimates of areas where to look. This way, the numerical method can output better results in a shorter time. For illustration of basic bounds see Figure 5.5.

Region $D = \big((0,4)\times(0,+\infty)\big) \cup \big((0,+\infty)\times(0,4)\big)$ can be split into subregions by $\kappa_\beta$ and $\kappa_\alpha$ (for illustration, see Figure 5.6). Basic bounds in Theorem 39 are sets of these subregions where Fučík curves are localized.

Let us have a fixed Fučík curve. Such Fučík curve belongs to some basic bound, thus we have a set of subregions where to (numerically) look for values $(\alpha,\beta)$ belonging to the Fučík curve (see again Figure 5.5). Even numerically faster (yet not exact) way is to look for the lowest such subregion (a maximal $\kappa_\beta$ for which the basic bound is still satisfied). We are going to denote such value as $\kappa_\beta^{\mathtt{max}}$ (since it is a maximal $\kappa_\beta$ for which the basic bound is still satisfied). While knowing $\kappa_\beta^{\mathtt{max}}$, we can numerically look for points from Fučík curves in the upper part of $D$ – i.e. in the region $\alpha \in (0,+\infty)$, $\beta > \xi_k$ (and $(\alpha,\beta) \in D$), where $k = \kappa_\beta^{\mathtt{max}} + 2$. E.g., for $\kappa_\beta^{\mathtt{max}} = 0$ we are assuming $\beta > \xi_2$, for $\kappa_\beta^{\mathtt{max}} = 1$ we are assuming $\beta > \xi_3$ and so on. Let us derive these values $\kappa_\beta^{\mathtt{max}}$:

1. Let us assume bound $\Theta^{\pm}_{2j-1}$ from Theorem 39. For a fixed Fučík curve (thus $n, j$ fixed), $\kappa_\beta$ needs to satisfy
$$\frac{2-4j+n-j\kappa_\alpha}{j} \le \kappa_\beta \le \frac{1-2j+n-j\kappa_\alpha}{j}.$$

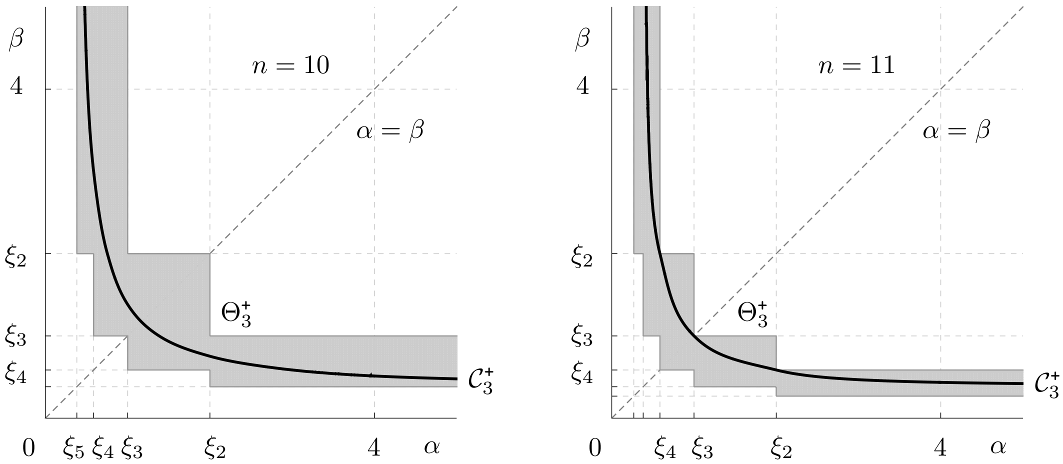



Figure 5.5: The set $\Theta_3^+$ (grey region) as the basic bound for the third non-trivial Fučík curve $\mathcal{C}_3^+ \subset \Theta_3^+$ (black curve) for $n = 10$ (left) and for $n = 11$ (right).

Maximal $\kappa_\beta$ is attained when

$$\kappa_\beta = \left\lfloor \frac{1 - 2j + n - j\kappa_\alpha}{j} \right\rfloor,$$

thus (taking $\kappa_\alpha = 0$ in order to maximize the fraction)

$$\Theta_{2j-1}^{\pm} : \ \kappa_\beta^{\max} = \left\lfloor \frac{1 - 2j + n}{j} \right\rfloor.$$

2. Let us assume bound $\Theta_{2j}^+$ from Theorem 39. For a fixed Fučík curve (thus $n, j$ fixed), $\kappa_\beta$ needs to satisfy
$$\frac{-4j + n - \kappa_\alpha - j\kappa_\alpha}{j} \le \kappa_\beta \le \frac{-2j + n - \kappa_\alpha - j\kappa_\alpha}{j}.$$
Using the same logic as in the first case, we have
$$\Theta_{2j}^+ : \ \kappa_\beta^{\max} = \left\lfloor \frac{-2j + n}{j} \right\rfloor.$$

3. Finally, let us assume bound $\Theta_{2j}^-$ from Theorem 39. For a fixed Fučík curve (thus $n, j$ fixed), $\kappa_\beta$ needs to satisfy
$$\frac{-4j + n - j\kappa_\alpha}{1 + j} \le \kappa_\beta \le \frac{-2j + n - j\kappa_\alpha}{1 + j}.$$
Using the same logic as previously, we have
$$\Theta_{2j}^- : \ \kappa_\beta^{\max} = \left\lfloor \frac{-2j + n}{1 + j} \right\rfloor.$$

In Table 5.1, we can see values of $\kappa_\beta^{\max}$ for different Fučík curves (first column), fixed $j$ (second column) and fixed $n$ (headers of all other columns). For example, for $n = 3$, there exist Fučík curves $\mathcal{C}_1^{\pm}$, $\mathcal{C}_2^+$ and $\mathcal{C}_2^-$ and we have:

- The "lowest subregion" where $\mathcal{C}_1^{\pm}$ is located is $\kappa_\beta = 2$, thus we are looking for values of $(\alpha, \beta) \in \mathcal{C}_1^{\pm}$ in the region $\beta > \xi_4$, $\alpha \in (0, +\infty)$ (and $(\alpha, \beta) \in D$);
- The "lowest subregion" where $\mathcal{C}_2^+$ is located is $\kappa_\beta = 1$, thus we are looking for values of $(\alpha, \beta) \in \mathcal{C}_2^+$ in the region $\beta > \xi_3$, $\alpha \in (0, +\infty)$ (and $(\alpha, \beta) \in D$);

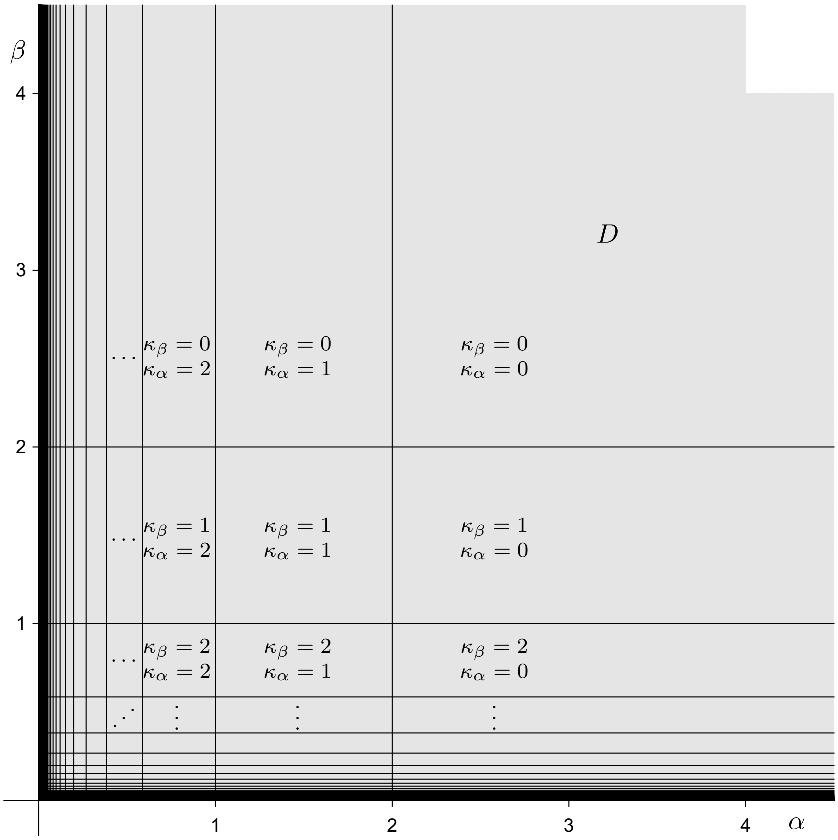



Figure 5.6: Region $D = \big((0,4) \times (0,+\infty)\big) \cup \big((0,+\infty) \times (0,4)\big)$ (grey) split into regions by $\kappa_\beta$ and $\kappa_\alpha$. We did not include boundaries in the figure – for example, region $\kappa_\beta = 2$, $\kappa_\alpha = 1$ is $(\xi_3, \xi_2] \times (\xi_4, \xi_3]$.

- The "lowest subregion" where $\mathcal{C}_2^-$ is located is $\kappa_\beta = 0$, thus we are looking for values of $(\alpha, \beta) \in \mathcal{C}_2^-$ in the region $\beta > \xi_2$, $\alpha \in (0, +\infty)$ (and $(\alpha, \beta) \in D$).

| Fučík curve | $j$ | Value of $\kappa_\beta^{\max}$ for a particular Fučík curve $\mathcal{C}_l^{\pm}$ for fixed $n$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $n$=3 | $n$=4 | $n$=5 | $n$=6 | $n$=7 | $n$=8 | $n$=9 | $n$=10 | $n$=11 | $n$=12 |
| $\mathcal{C}_1^{\pm}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| $\mathcal{C}_2^{+}$ | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $\mathcal{C}_2^{-}$ | 1 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| $\mathcal{C}_3^{\pm}$ | 2 | | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| $\mathcal{C}_4^{+}$ | 2 | | | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 |
| $\mathcal{C}_4^{-}$ | 2 | | | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 2 |
| $\mathcal{C}_5^{\pm}$ | 3 | | | | 0 | 0 | 1 | 1 | 1 | 2 | 2 |
| $\mathcal{C}_6^{+}$ | 3 | | | | | 0 | 0 | 1 | 1 | 1 | 2 |
| $\mathcal{C}_6^{-}$ | 3 | | | | | 0 | 0 | 0 | 1 | 1 | 1 |
| $\mathcal{C}_7^{\pm}$ | 4 | | | | | | 0 | 0 | 0 | 1 | 1 |
| $\mathcal{C}_8^{+}$ | 4 | | | | | | | 0 | 0 | 0 | 1 |
| $\mathcal{C}_8^{-}$ | 4 | | | | | | | 0 | 0 | 0 | 0 |
| $\mathcal{C}_9^{\pm}$ | 5 | | | | | | | | 0 | 0 | 0 |
| $\mathcal{C}_{10}^{+}$ | 5 | | | | | | | | | 0 | 0 |
| $\mathcal{C}_{10}^{-}$ | 5 | | | | | | | | | 0 | 0 |
| $\mathcal{C}_{11}^{\pm}$ | 6 | | | | | | | | | | 0 |
| | | $n$=13 | $n$=14 | $n$=15 | $n$=16 | $n$=17 | $n$=18 | $n$=19 | $n$=20 | $n$=21 | $n$=22 |
| $\mathcal{C}_1^{\pm}$ | 1 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| $\mathcal{C}_2^{+}$ | 1 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $\mathcal{C}_2^{-}$ | 1 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 | 10 |
| $\mathcal{C}_3^{\pm}$ | 2 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 9 |
| $\mathcal{C}_4^{+}$ | 2 | 4 | 5 | 5 | 6 | 6 | 7 | 7 | 8 | 8 | 9 |
| $\mathcal{C}_4^{-}$ | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 5 | 6 |
| $\mathcal{C}_5^{\pm}$ | 3 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 5 | 5 |
| $\mathcal{C}_6^{+}$ | 3 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 5 |
| $\mathcal{C}_6^{-}$ | 3 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| $\mathcal{C}_7^{\pm}$ | 4 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 |
| $\mathcal{C}_8^{+}$ | 4 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 |
| $\mathcal{C}_8^{-}$ | 4 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 |
| $\mathcal{C}_9^{\pm}$ | 5 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| $\mathcal{C}_{10}^{+}$ | 5 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| $\mathcal{C}_{10}^{-}$ | 5 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 |
| $\mathcal{C}_{11}^{\pm}$ | 6 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\mathcal{C}_{12}^{+}$ | 6 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| $\mathcal{C}_{12}^{-}$ | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $\mathcal{C}_{13}^{\pm}$ | 7 | | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| $\mathcal{C}_{14}^{+}$ | 7 | | | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| $\mathcal{C}_{14}^{-}$ | 7 | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $\mathcal{C}_{15}^{\pm}$ | 8 | | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathcal{C}_{16}^{+}$ | 8 | | | | | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathcal{C}_{16}^{-}$ | 8 | | | | | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathcal{C}_{17}^{\pm}$ | 9 | | | | | | 0 | 0 | 0 | 0 | 0 |
| $\mathcal{C}_{18}^{+}$ | 9 | | | | | | | 0 | 0 | 0 | 0 |
| $\mathcal{C}_{18}^{-}$ | 9 | | | | | | | 0 | 0 | 0 | 0 |
| $\mathcal{C}_{19}^{\pm}$ | 10 | | | | | | | | 0 | 0 | 0 |
| $\mathcal{C}_{20}^{+}$ | 10 | | | | | | | | | 0 | 0 |
| $\mathcal{C}_{20}^{-}$ | 10 | | | | | | | | | 0 | 0 |
| $\mathcal{C}_{21}^{\pm}$ | 11 | | | | | | | | | | 0 |

Table 5.1: The value of $\kappa_\beta^{\max}$ (thus the lowest part of $D$ split into subregions by $\kappa_\beta$ based on basic bounds in Theorem 39).

## 5.5 Delta bounds of Fučík curves

Second bounds we are going to discuss are referred to as the "delta bounds". In this part we will restrict ourselves only to $0 < \alpha < 4$ and $0 < \beta < 4$ and denote

$$D_{0,4} := (0,4) \times (0,4).$$

Let us investigate in detail the "gaps" between positive and negative semi-waves – the difference between zero points of two consecutive positive and negative semi-waves. Knowing minimal and maximal such difference allows us to find regions (bounds) for Fučík curves. In the following definition, we define function $\delta_{\alpha,\beta}$ which represents such difference (the length of such "gap" is then equal to the absolute value of function $\delta_{\alpha,\beta}$) – see Figure 5.7.

**Definition 40.** For $0 < \alpha, \beta < 4$, let us define

$$\delta_{\alpha,\beta}(q) := T^{\alpha}(q) - T^{\beta}(q), \quad q < 0.$$

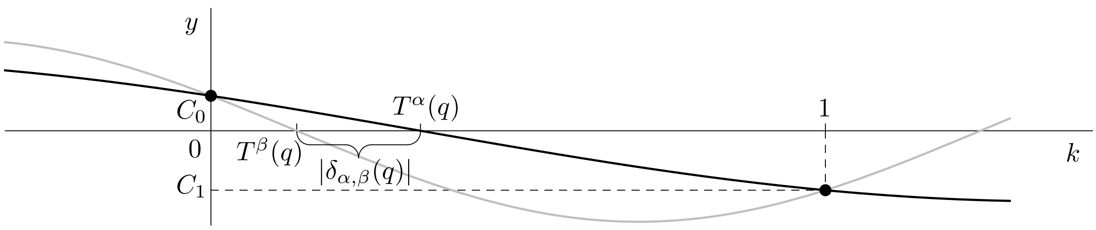



Figure 5.7: The distance between zero point of positive semi-wave (black) and zero point of negative semi-wave (grey) – the distance $|\delta_{\alpha,\beta}|$ for $\alpha = 1.9$, $\beta = 3.9$.

Since function $\delta_{\alpha,\beta}$ is a difference of functions $T^\alpha$ and $T^\beta$, firstly we will investigate function $T^\lambda$ for $\lambda \in (0,4)$ in detail. We will assume only $q < 0$ because otherwise there would not be an anchoring of two consecutive semi-waves. In this section, for the purpose of easy reading, we are going to work with a function $T$ defined as

$$T(q,\lambda) := T^{\lambda}(q), \quad \lambda \in (0,4),\ q < 0. \tag{5.3}$$

**Lemma 41.** *Let $\lambda \in (0,4)$, $q < 0$. Then we have (for function $T$ defined in* (5.3)*)*

$$\frac{\partial T(q,\lambda)}{\partial q} > 0$$

*and*

$$\frac{\partial T(q,\lambda)}{\partial \lambda} \begin{cases} < 0 & \text{for } q < -1, \\ = 0 & \text{for } q = -1, \\ > 0 & \text{for } q \in (-1,0). \end{cases}$$

*Proof.* In the first part of this proof, we will focus only on the derivative of function $T$ with respect to $q$, i.e.

$$\frac{\partial T(q,\lambda)}{\partial q} = \frac{\sin\omega_\lambda}{(q^2 - 2q\cos\omega_\lambda + 1)\omega_\lambda}.$$

Functions $\omega_\lambda > 0$, $\sin\omega_\lambda > 0$ are both positive. Quadratic term is also positive, since its discriminant is negative

$$D = 4\cos^2\omega_\lambda - 4 = (2-\lambda)^2 - 4 = \lambda(\lambda - 4) < 0.$$

This leads towards

$$\frac{\partial T(q,\lambda)}{\partial q} > 0.$$

In the rest of this proof, we will derive a sign of derivative of function $T$ with respect to $\lambda$. Such derivative can be written in the form

$$\frac{\partial T(q,\lambda)}{\partial \lambda} = \frac{\partial}{\partial \omega_\lambda}\left(\frac{1}{\omega_\lambda}\operatorname{arccot}\left(\frac{\cos\omega_\lambda - q}{\sin\omega_\lambda}\right)\right)\frac{\partial \omega_\lambda}{\partial \lambda}. \tag{5.4}$$

Last term in (5.4) is positive since for $\lambda \in (0,4)$ we have

$$\frac{\partial \omega_\lambda}{\partial \lambda} = \frac{1}{\sqrt{\lambda(\lambda-4)}} > 0.$$

Next, we have

$$\begin{aligned}
&\frac{\partial}{\partial \omega_\lambda}\left(\frac{1}{\omega_\lambda}\operatorname{arccot}\left(\frac{\cos\omega_\lambda - q}{\sin\omega_\lambda}\right)\right)\\
&= \frac{1}{\omega_\lambda^2}\left(\frac{-1}{1+\left(\frac{\cos\omega_\lambda - q}{\sin\omega_\lambda}\right)^2}\frac{-\sin^2\omega_\lambda - (\cos\omega_\lambda - q)\cos\omega_\lambda}{\sin^2\omega_\lambda} - \operatorname{arccot}\left(\frac{\cos\omega_\lambda - q}{\sin\omega_\lambda}\right)\right)\\
&= \frac{1}{\omega_\lambda^2}\left(\frac{-\sin^2\omega_\lambda}{\sin^2\omega_\lambda + (\cos\omega_\lambda - q)^2}\frac{q\cos\omega_\lambda - 1}{\sin^2\omega_\lambda}\omega_\lambda - \operatorname{arccot}\left(\frac{\cos\omega_\lambda - q}{\sin\omega_\lambda}\right)\right)\\
&= \frac{1}{\omega_\lambda^2}\left(\frac{\omega_\lambda q\cos\omega_\lambda - \omega_\lambda}{\sin^2\omega_\lambda + (\cos\omega_\lambda - q)^2} - \operatorname{arccot}\left(\frac{\cos\omega_\lambda - q}{\sin\omega_\lambda}\right)\right).
\end{aligned}$$

The derivative in (5.4) can be written as

$$\frac{\partial T(q,\lambda)}{\partial \lambda} = \frac{1}{\omega_\lambda^2}F(\omega_\lambda, q)\frac{1}{\sqrt{\lambda(\lambda-4)}},$$

where

$$F(\omega_\lambda, q) := \frac{\omega_\lambda q\cos\omega_\lambda - \omega_\lambda}{\sin^2\omega_\lambda + (\cos\omega_\lambda - q)^2} - \operatorname{arccot}\left(\frac{\cos\omega_\lambda - q}{\sin\omega_\lambda}\right), \quad \omega_\lambda \in (0,\pi),\ q<0. \tag{5.5}$$

Thus, the sign of $\frac{\partial T(q,\lambda)}{\partial \lambda}$ depends on the sign of function $F$ defined in (5.5) (other terms in (5.4) are positive). Firstly, the right-hand limit of $F$ near zero is

$$\lim_{\omega_\lambda \to 0^+} F(\omega_\lambda, q) = 0.$$

The derivative of $F$ with respect to $\omega_\lambda$ can be simplified as

$$\frac{\partial F(\omega_\lambda, q)}{\partial \omega_\lambda} = \frac{q(q^2-1)(\omega_\lambda \sin\omega_\lambda)}{(1+q^2-2q\cos\omega_\lambda)^2},$$

thus the term $(q^2-1)$ is responsible for the sign of such derivative. For the simplest case of $q=-1$, the derivative of $F$ with respect to $\omega_\lambda$ is zero, thus function $F$ is constant. Since the right-hand limit near zero is zero, then

$$F(\omega_\lambda, -1) \equiv 0, \quad \omega_\lambda \in (0,\pi).$$

For $q \in (-1,0)$, the derivative of $F$ with respect to $\omega_\lambda$ is positive, thus function $F$ is increasing. And having in mind the right-hand limit near zero, we have

$$F(\omega_\lambda, q) > 0, \quad \omega_\lambda \in (0,\pi),\ q \in (-1,0).$$

Similarly, we have

$$F(\omega_\lambda, q) < 0, \quad \omega_\lambda \in (0,\pi),\ q < -1.$$

Knowing the signs of function $F$ leads directly to the signs of $\dfrac{\partial T(q,\lambda)}{\partial \lambda}$. □

In the proof of the following theorem we will find stationary points of function $\delta_{\alpha,\beta}$ with respect to $q$ and we will prove when these points are global minima and global maxima of function $\delta_{\alpha,\beta}$.

**Theorem 42.** *Let $0 < \alpha, \beta < 4$. Function $\delta_{\alpha,\beta}$ has one global minimum and one global maximum (for $q < 0$):*

$$\min_{q<0} \delta_{\alpha,\beta}(q) = \begin{cases} \delta_{\alpha,\beta}(q_1^{\alpha,\beta}) & \textit{for } \alpha > \beta, \\ 0 & \textit{for } \alpha = \beta, \\ \delta_{\alpha,\beta}(q_2^{\alpha,\beta}) & \textit{for } \alpha < \beta, \end{cases} \qquad \max_{q<0} \delta_{\alpha,\beta}(q) = \begin{cases} \delta_{\alpha,\beta}(q_2^{\alpha,\beta}) & \textit{for } \alpha > \beta, \\ 0 & \textit{for } \alpha = \beta, \\ \delta_{\alpha,\beta}(q_1^{\alpha,\beta}) & \textit{for } \alpha < \beta, \end{cases}$$

*where*

$$q_{1,2}^{\alpha,\beta} := J^{\alpha,\beta} \mp \sqrt{(J^{\alpha,\beta})^2 - 1}, \qquad J^{\alpha,\beta} := \frac{\omega_\alpha \sin\omega_\beta \cos\omega_\alpha - \omega_\beta \sin\omega_\alpha \cos\omega_\beta}{\omega_\alpha \sin\omega_\beta - \omega_\beta \sin\omega_\alpha}. \tag{5.6}$$

*Also,*

$$\max_{q<0} \delta_{\alpha,\beta}(q) = -\min_{q<0} \delta_{\alpha,\beta}(q).$$

*Proof.* Let $0 < \alpha, \beta < 4$. Function $\delta_{\alpha,\beta}$ can be written as

$$\delta_{\alpha,\beta}(q) = \frac{1}{\omega_\alpha}\left(\frac{\pi}{2} - \arctan\left(\frac{\cos\omega_\alpha - q}{\sin\omega_\alpha}\right)\right) - \frac{1}{\omega_\beta}\left(\frac{\pi}{2} - \arctan\left(\frac{\cos\omega_\beta - q}{\sin\omega_\beta}\right)\right).$$

For $\alpha = \beta$, we have

$$\delta_{\alpha,\beta}(q) \equiv 0.$$

In the rest of the proof, we will assume $\alpha \neq \beta$. Regarding the boundary points, we have

$$\delta_{\alpha,\beta}(0) = 0, \qquad \lim_{q\to-\infty} \delta_{\alpha,\beta}(q) = 0.$$

The first derivative of $\delta_{\alpha,\beta}$ (with respect to $q$) is

$$\frac{\partial \delta_{\alpha,\beta}(q)}{\partial q} = \frac{\sin\omega_\alpha}{(1 + q^2 - 2q\cos\omega_\alpha)\omega_\alpha} - \frac{\sin\omega_\beta}{(1 + q^2 - 2q\cos\omega_\beta)\omega_\beta}.$$

Zero points of such derivative have to satisfy

$$(\omega_\alpha \sin\omega_\beta - \omega_\beta \sin\omega_\alpha)q^2 - 2(\omega_\alpha \sin\omega_\beta \cos\omega_\alpha - \omega_\beta \sin\omega_\alpha \cos\omega_\beta)q + \omega_\alpha \sin\omega_\beta - \omega_\beta \sin\omega_\alpha = 0.$$

Since $\alpha \neq \beta$, we have

$$q^2 - 2\frac{\omega_\alpha \sin\omega_\beta \cos\omega_\alpha - \omega_\beta \sin\omega_\alpha \cos\omega_\beta}{\omega_\alpha \sin\omega_\beta - \omega_\beta \sin\omega_\alpha} q + 1 = 0.$$

Using definition of $J^{\alpha,\beta}$ in (5.6), we can write

$$q^2 - 2J^{\alpha,\beta}q + 1 = 0,$$

which is a quadratic equation in $q$ with a discriminant

$$\begin{aligned} D &= 4(J^{\alpha,\beta})^2 - 4 = 4((J^{\alpha,\beta})^2 - 1) \\ &= -\frac{4\sin\omega_\alpha \sin\omega_\beta \left(\left(\omega_\alpha^2 + \omega_\beta^2\right)\sin\omega_\alpha \sin\omega_\beta + 2\omega_\alpha\omega_\beta(\cos\omega_\alpha \cos\omega_\beta - 1)\right)}{(\omega_\beta \sin\omega_\alpha - \omega_\alpha \sin\omega_\beta)^2}. \end{aligned} \tag{5.7}$$

The discriminant in (5.7) is positive when function $g$ is negative, where function $g$ is defined as (since $\alpha, \beta \in (0, 4)$, we have $\omega_\alpha, \omega_\beta \in (0, \pi)$ and we can denote $a := \omega_\alpha$, $b := \omega_\beta$)

$$g(a, b) := \left(a^2 + b^2\right)\sin a \sin b + 2ab(\cos a \cos b - 1), \quad (a, b) \in (0, \pi) \times (0, \pi),\ a \neq b. \tag{5.8}$$

For the investigation of stationary points, we will take function $g$ in $[0,\pi]\times[0,\pi]$. Function $g$ has stationary points for $a=b$, where $g(a,a)=0$. There are no other stationary points. For the boundary, we have $g(0,.)\equiv 0$, $g(.,0)\equiv 0$, $g(\pi,.)\leq 0$ and $g(.,\pi)\leq 0$. That means that function $g$ in (5.8) is negative, thus the discriminant in (5.7) is positive.

Thus such quadratic equation has two zero points $q_{1,2}^{\alpha,\beta}$ – they are defined in (5.6).

Using Vieta's formula $q_1^{\alpha,\beta}q_2^{\alpha,\beta}=1$, we have

$$q_2^{\alpha,\beta}=\frac{1}{q_1^{\alpha,\beta}}.$$

Since $J^{\alpha,\beta}<0$, we also have $q_1^{\alpha,\beta}<0$. Thus $q_2^{\alpha,\beta}=\dfrac{1}{q_1^{\alpha,\beta}}<0$. Moreover, $q_1^{\alpha,\beta}<-1$ and $q_2^{\alpha,\beta}\in(-1,0)$. Using properties of function $T^\lambda$ in Lemma 10, we can derive

$$\begin{aligned}
\delta_{\alpha,\beta}(q_2^{\alpha,\beta}) &= T^\alpha(q_2^{\alpha,\beta})-T^\beta(q_2^{\alpha,\beta})=T^\alpha\left(\frac{1}{q_1^{\alpha,\beta}}\right)-T^\beta\left(\frac{1}{q_1^{\alpha,\beta}}\right)\\
&= 1-T^\alpha(q_1^{\alpha,\beta})-\big(1-T^\beta(q_1^{\alpha,\beta})\big)=-\big(T^\alpha(q_1^{\alpha,\beta})-T^\beta(q_1^{\alpha,\beta})\big)\\
&= -\delta_{\alpha,\beta}(q_1^{\alpha,\beta}).
\end{aligned}$$

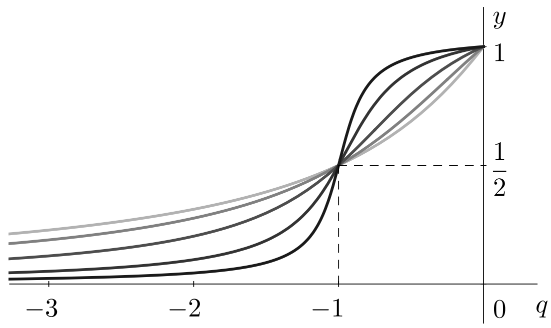



Figure 5.8: Function $T^\lambda$ as a function of $q$ for fixed $\lambda\in\{1.2,2.5,3.5,3.9,3.98\}$ – the darker shade of grey, the larger parameter $\lambda$.

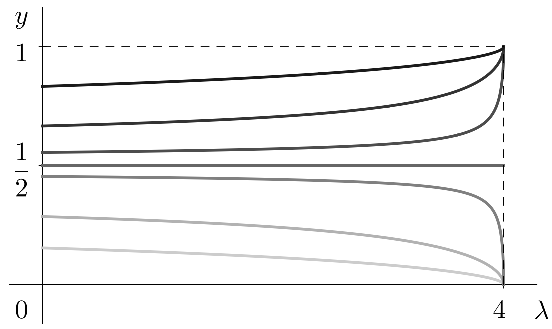



Figure 5.9: Function $T^\lambda$ as a function of $\lambda$ for fixed $q\in\{-5.5,-2.5,-1.2,-1,-0.8,-0.5,-0.2\}$ – the darker shade of grey, the larger parameter $q$.

Since function $\delta_{\alpha,\beta}$ is a difference of $T^\alpha$ and $T^\beta$, we will investigate function $T^\lambda=T^\lambda(q)$, $\lambda\in(0,4)$ in detail in order to clarify, that points $q_{1,2}^{\alpha,\beta}$ are points of global extrema for function

$\delta_{\alpha,\beta}$ and also to distinguish for which parameters $\alpha$ and $\beta$ the points are points of global maximum / minimum.

Let us look at the limits on the boundary of our set (for $q$):

$$\lim_{q\to 0^-} T^\lambda(q) = T^\lambda(0) = 1, \lim_{q\to -\infty} T^\lambda(q) = T^\lambda(\infty) = 0.$$

Function $T^\lambda$ is increasing with respect to $q$, since its first derivative is positive – see Lemma 41. On Figure 5.8, we can see function $T^\lambda$ as a function of $q$ for several fixed values of $\lambda$.

Function $T^\lambda$ is (with respect to $\lambda$) decreasing when $q < -1$, constant for $q = -1$, $T(-1, \lambda) = 1/2$ and increasing for $q \in (-1, 0)$ – see Lemma 41. On Figure 5.9, we have illustrated function $T^\lambda$ as a function of $\lambda$ for several fixed values of $q$.

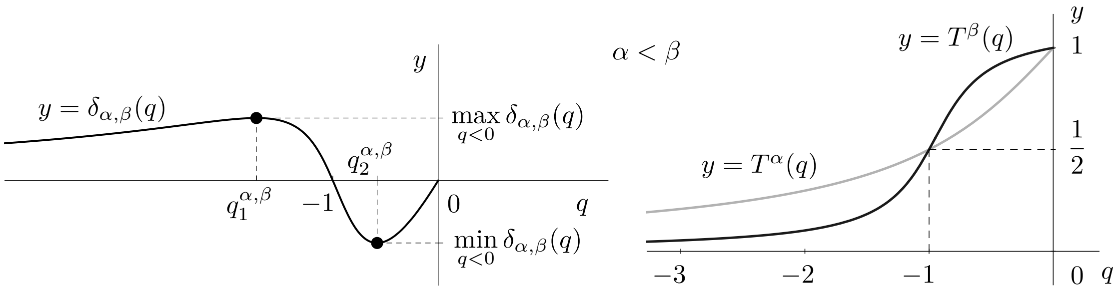



Figure 5.10: Maximum and minimum for $\delta_{\alpha,\beta}$ (on the left) and functions $T^\alpha, T^\beta$ (on the right) for $\alpha < \beta$ ($\alpha = 1.9$, $\beta = 3.9$).

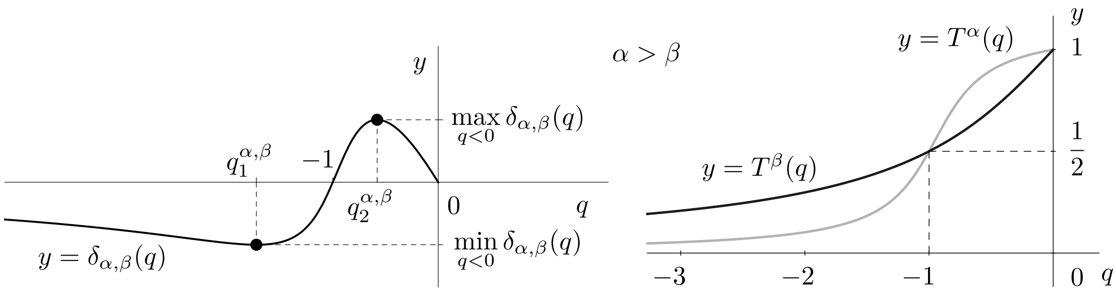



Figure 5.11: Maximum and minimum for $\delta_{\alpha,\beta}$ (on the left) and functions $T^\alpha, T^\beta$ (on the right) for $\alpha > \beta$ ($\alpha = 3.9$, $\beta = 1.9$).

From (2.2) we know that $T^\lambda(-1) = 1/2$, thus this value does not depend of the value of $\lambda$.

Since function $\delta_{\alpha,\beta}$ is a difference of $T^\alpha$ and $T^\beta$, the location of its minimum and maximum will depend on the inequality between $\alpha$ and $\beta$.

Let us assume $\alpha < \beta$. Such case is illustrated on Figure 5.10. We have two stationary points of $\delta_{\alpha,\beta}$ (points $q_1^{\alpha,\beta}$ and $q_2^{\alpha,\beta}$) with the property $q_1^{\alpha,\beta} < -1 < q_2^{\alpha,\beta} < 0$. Limits of function $\delta_{\alpha,\beta}$ are $\delta_{\alpha,\beta}(0) = 0$, $\lim_{q\to-\infty} \delta_{\alpha,\beta}(q) = 0$. Value of $\delta_{\alpha,\beta}$ for $q = -1$ is $\delta_{\alpha,\beta}(-1) = 0 - 0 = 0$. From the investigation of monotonicity of function $T^\lambda$ with respect to $q$ and with respect to $\lambda$, we can conclude, that

$$T^\alpha(q) > T^\beta(q) \text{ for all } q \in (-\infty, -1).$$

That means that there has to be a local maximum for $q = q_1^{\alpha,\beta}$ and $\delta_{\alpha,\beta}(q_1^{\alpha,\beta}) > 0$. On the other hand,

$$T^\alpha(q) < T^\beta(q) \text{ for all } q \in (-1, 0),$$

thus there has to be a local minimum at $q_2^{\alpha,\beta}$ and $\delta_{\alpha,\beta}(q_2^{\alpha,\beta}) < 0$. The absence of other stationary points leads to that both local extrema are global extrema.

For $\alpha > \beta$, the situation is very similar, we have $\min_{q<0} \delta_{\alpha,\beta}(q) = \delta_{\alpha,\beta}(q_1^{\alpha,\beta})$ and $\max_{q<0} \delta_{\alpha,\beta}(q) = \delta_{\alpha,\beta}(q_2^{\alpha,\beta})$ – see illustration on Figure 5.11.

Since $\delta_{\alpha,\beta}(q_2^{\alpha,\beta}) = -\delta_{\alpha,\beta}(q_1^{\alpha,\beta})$, we have

$$\max_{q<0} \delta_{\alpha,\beta}(q) = -\min_{q<0} \delta_{\alpha,\beta}(q).$$

$\square$

**Definition 43.** For $0 < \alpha, \beta < 4$, let us define

$$\begin{aligned} \delta^{\mathtt{min}}_{\alpha,\beta} &:= -|\delta_{\alpha,\beta}(q_1^{\alpha,\beta})|, \\ \delta^{\mathtt{max}}_{\alpha,\beta} &:= +|\delta_{\alpha,\beta}(q_1^{\alpha,\beta})|, \end{aligned}$$

where $q_1^{\alpha,\beta}$ is defined in (5.6).

**Remark 44.** In Definition 43, we have denoted $\delta^{\mathtt{min}}_{\alpha,\beta}$ and $\delta^{\mathtt{max}}_{\alpha,\beta}$ the minimum and maximum length that we need to add in the intervals of anchoring in order to get upper and lower bounds. We have used the "symmetry" of global extrema of function $\delta_{\alpha,\beta}$ derived in Theorem 42, i.e. $\max_{q<0} \delta_{\alpha,\beta}(q) = -\min_{q<0} \delta_{\alpha,\beta}(q)$.

In the following theorem, we derive bounds for Fučík curves $C_l^{\pm}$, $l = 1, \dots, n-1$ using extrema points from the Theorem 42 (and values $\delta^{\mathtt{min}}_{\alpha,\beta}$ and $\delta^{\mathtt{max}}_{\alpha,\beta}$) – for illustration of such bounds, see Figure 5.12.

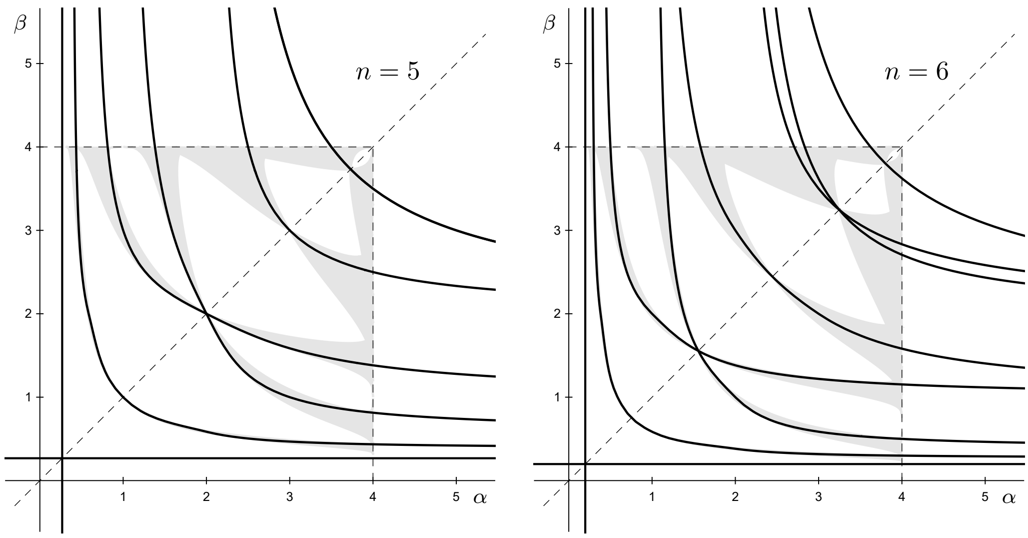



Figure 5.12: Delta bounds for $\Sigma(\mathbf{A}^{\mathtt{D}})$ for $n = 5$ (left) and $n = 6$ (right).

**Theorem 45.** *In the domain $D_{0,4} = (0,4) \times (0,4)$, we have the following "delta bounds" for Fučík curves $\mathcal{C}_l^{\pm}$, $l = 1, \dots, n-1$,*

$$\begin{aligned} \left(\mathcal{C}^{\pm}_{2j-1} \cap D_{0,4}\right) &\subset \Psi_{j,j} &=: \Psi^{\pm}_{2j-1}, \\ \left(\mathcal{C}^{+}_{2j} \cap D_{0,4}\right) &\subset \Psi_{j+1,j} &=: \Psi^{+}_{2j}, \\ \left(\mathcal{C}^{-}_{2j} \cap D_{0,4}\right) &\subset \Psi_{j,j+1} &=: \Psi^{-}_{2j}, \end{aligned}$$

*$j \in \mathbb{N}$, where for $k, s \in \mathbb{N}$, sets $\Psi_{k,s}$ are given by*

$$\Psi_{k,s} \quad := \quad \{(\alpha, \beta) \in D_{0,4} : \; F_{k,s}(\alpha, \beta) \leq n+1 \leq G_{k,s}(\alpha, \beta)\}$$

*and*

$$F_{k,s}(\alpha,\beta) := k\frac{\pi}{\omega_\alpha} + s\frac{\pi}{\omega_\beta} + (k+s-1)\delta^{\mathtt{min}}_{\alpha,\beta}, \quad G_{k,s}(\alpha,\beta) := k\frac{\pi}{\omega_\alpha} + s\frac{\pi}{\omega_\beta} + (k+s-1)\delta^{\mathtt{max}}_{\alpha,\beta}.$$

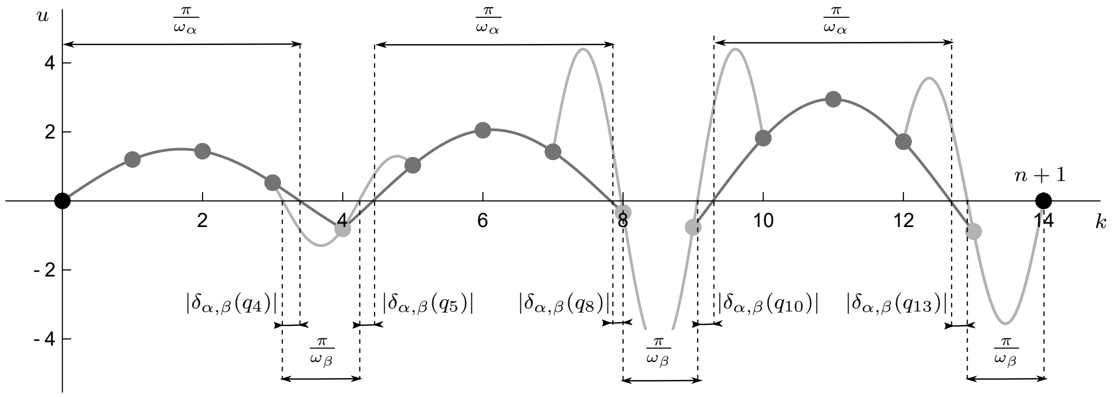



Figure 5.13: Solution $u \in \mathcal{C}^{+}_5$, $n = 13$, $\alpha = 0.8$, $\beta \doteq 3.9369177$). Positive semi-waves are in dark grey color (their length is equal to $\frac{\pi}{\omega_\alpha}$), negative semi-waves are in light grey color (their length is equal to $\frac{\pi}{\omega_\beta}$). This solution consists of 5 anchorings (it has 5 generalized zeros on $\mathbb{T}$) and the "gaps" between zero points of positive and negative semi-waves are given by functions $|\delta_{\alpha,\beta}(q_i)|$, where $i \in (4, 5, 8, 10, 13)$.

*Proof.* Fučík curves $\mathcal{C}^{\pm}_l$ consist of points $(\alpha, \beta)$ such that their corresponding Fučík eigenvectors have certain sign properties. For Fučík curve $\mathcal{C}^{+}_l$, $l = 1, \dots, n-1$, we have $u(1) > 0$ and the solution $u$ has $l$ generalized zeros on $\mathbb{T}$. Since we are assuming $(\alpha, \beta) \in D_{0,4} = (0, 4) \times (0, 4)$, the length of all semi-waves can be expressed as $\frac{\pi}{\omega_\alpha}$ for positive semi-waves and $\frac{\pi}{\omega_\beta}$ for negative semi-waves. In the linear case ($\alpha = \beta$), if we sum appropriate number of lengths of positive and negative semi-waves, such number has to be equal to the length of the solution plus the boundary points, i.e. $n+1$. In our case ($\alpha \neq \beta$ in general), we have to "correct" this length by the differences between every anchoring of two consecutive semi-waves. For a fixed anchoring, such difference is given by the function $\delta_{\alpha,\beta}$ (either $+\delta_{\alpha,\beta}$ or $-\delta_{\alpha,\beta}$.) If we estimate this difference by $\delta^{\mathtt{max}}_{\alpha,\beta}$, we have an upper bound of the length $n+1$. And if we estimate this difference by $\delta^{\mathtt{min}}_{\alpha,\beta}$, we have a lower bound of the length $n+1$. ☐

See Figure 5.13 for a particular solution $u$ for which $(\alpha, \beta) \in \mathcal{C}^{+}_5$ ($n = 13$). We can construct both boundaries of delta bounds for this fixed solution. The upper boundary is given as ($n+1 = 14$)

$$3\frac{\pi}{\omega_\alpha} + 3\frac{\pi}{\omega_\beta} + 5\delta^{\mathtt{max}}_{\alpha,\beta} = n+1$$

and the lower boundary in given as

$$3\frac{\pi}{\omega_\alpha} + 3\frac{\pi}{\omega_\beta} + 5\delta^{\mathtt{min}}_{\alpha,\beta} = n+1.$$

See Figures 6.4 and 6.5 for detailed illustration how delta bounds for each Fučík curve may look like – shown for $\Sigma(\mathbf{A}^{\mathtt{D}})$, $n = 6$.

## 5.6 Rho bounds of Fučík curves

Finally, third bounds we are going to discuss are referred here as "rho bounds" (originally introduced as "improved bounds" in [31], see Figure 5.14 for illustration). The main idea is based on using extrema of function $\rho_{\alpha,\beta}$. In [31], Section 6, we have explored properties of function $\rho_{\alpha,\beta}$ in detail. We were able to find its minimum and maximum – see Figures 4.7 and 6.6.

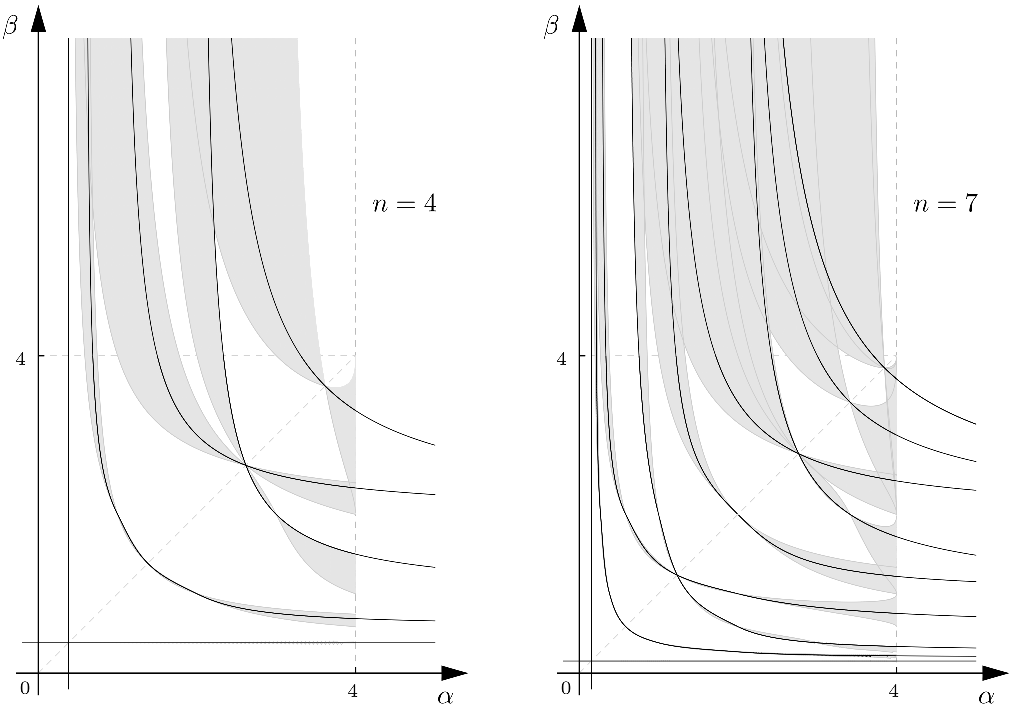



Figure 5.14: Rho bounds $\Upsilon_l^{\pm}$ $(l = 1, \ldots, n-1)$ for the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathtt{D}}\right)$ for $n = 4$ (left) and $n = 7$ (right).

**Theorem 46.** *([31], Theorem 31, p. 47)*
*Let $0 < \alpha < 4$ and $\beta > 0$. Then the function $\rho_{\alpha,\beta}$ attains its global extrema at $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$. More precisely, we have that*

$$\min_{s\in[0,1+\tau_{\alpha,\beta}]} \rho_{\alpha,\beta}(s) = \begin{cases} \rho_{\alpha,\beta}(\mu_{\alpha,\beta}) = 2\mu_{\alpha,\beta} + \kappa_\beta & \text{for } \alpha \le \beta, \\ \rho_{\alpha,\beta}(\eta_{\alpha,\beta}) = 2\eta_{\alpha,\beta} + \kappa_\beta + 1 & \text{for } \alpha > \beta, \end{cases}$$

$$\max_{s\in[0,1+\tau_{\alpha,\beta}]} \rho_{\alpha,\beta}(s) = \begin{cases} \rho_{\alpha,\beta}(\eta_{\alpha,\beta}) = 2\eta_{\alpha,\beta} + \kappa_\beta + 1 & \text{for } \alpha \le \beta, \\ \rho_{\alpha,\beta}(\mu_{\alpha,\beta}) = 2\mu_{\alpha,\beta} + \kappa_\beta & \text{for } \alpha > \beta. \end{cases}$$

In order to have an easily readable text, we denote maximum and minimum of function $\rho_{\alpha,\beta}$ as follows:

**Definition 47.** ([31], Definition 2, p. 11)
For $0 < \alpha < 4$ and $\beta > 0$, let us define

$$\rho^{\mathtt{min}}_{\alpha,\beta} := \begin{cases} 2\mu_{\alpha,\beta} + \kappa_\beta & \alpha \le \beta, \\ 2\eta_{\alpha,\beta} + \kappa_\beta + 1 & \alpha > \beta, \end{cases}$$

$$\rho^{\mathtt{max}}_{\alpha,\beta} := \begin{cases} 2\eta_{\alpha,\beta} + \kappa_\beta + 1 & \alpha \le \beta, \\ 2\mu_{\alpha,\beta} + \kappa_\beta & \alpha > \beta. \end{cases}$$

Finally, rho bounds are such bounds, that we take $\rho^{\mathtt{min}}_{\alpha,\beta}$ and $\rho^{\mathtt{max}}_{\alpha,\beta}$ instead of function $\rho_{\alpha,\beta}$ in the description of Fučík spectrum $\Sigma(\mathbf{A}^{\mathrm{D}})$ in Theorem 35.

**Theorem 48.** *([31], Theorem 3, p. 11)*
*In the domain $\mathcal{D} = (0,4) \times (0,+\infty)$, we have the following "rho bounds" for Fučík curves $\mathcal{C}^{\pm}_{l}$, $l = 1,\dots,n-1$,*

$$\begin{aligned}
\left(\mathcal{C}^{\pm}_{2j-1} \cap \mathcal{D}\right) &\subset \Upsilon_{j,j} &&=: \Upsilon^{\pm}_{2j-1},\\
\left(\mathcal{C}^{+}_{2j} \cap \mathcal{D}\right) &\subset \Upsilon_{j+1,j} &&=: \Upsilon^{+}_{2j},\\
\left(\mathcal{C}^{-}_{2j} \cap \mathcal{D}\right) &\subset \Upsilon_{j,j+1} &&=: \Upsilon^{-}_{2j},
\end{aligned}$$

*$j \in \mathbb{N}$, where for $k, s \in \mathbb{N}$, sets $\Upsilon_{k,s}$ are given by*

$$\Upsilon_{k,s} := \left\{(\alpha,\beta) \in \mathcal{D} :\ \rho^{\mathtt{min}}_{\alpha,\beta} \ \le\ \tfrac{1}{s}\left(n+1-k\tfrac{\pi}{\omega_\alpha}\right) \ \le\ \rho^{\mathtt{max}}_{\alpha,\beta}\right\}.$$

## 5.7 Comparison between delta and rho bounds

In this section, we want to show that delta bounds from Theorem 45 and rho bounds from Theorem 48 are distinct from each other and also, we want to show when which bounds are better to use. From the construction of delta and rho bounds, we can decide which bounds are better in which cases. For illustration of delta and rho bounds of $\Sigma(\mathbf{A}^{\mathrm{D}})$ for $n = 5$, see Figure 5.20.

1. In delta bounds, firstly, we are assuming that all semi-waves are anchored in zero points, thus the total length is equal to the sum of $\frac{\pi}{\omega_\alpha}$ and $\frac{\pi}{\omega_\beta}$ (the correct amount of such lengths). Then, we add negative value $\delta^{\mathtt{min}}_{\alpha,\beta}$ or positive value $\delta^{\mathtt{max}}_{\alpha,\beta}$ – that represents the "correction" of anchorings. It is possible to construct delta bounds only in $D_{0,4}$.

2. In rho bounds, we have fixed positive semi-waves (and their length $\frac{\pi}{\omega_\alpha}$) and we are taking maximal value of distance function $\rho_{\alpha,\beta}$ and minimal value of $\rho_{\alpha,\beta}$ in order to substitute for negative semi-waves. We have constructed rho bounds in $\mathcal{D}$.

Firstly, since rho bounds are constructed even for $\beta > 4$, they are better when we need estimate for such $\beta$. In general, it is better to estimate the whole negative semi-wave in total (rho bounds). Thus, in most of the cases, rho bounds give us better estimates. It is evident in the case of bounds of higher order Fučík curves (we are making a lot of anchorings, thus we are estimating many times) – see Figure 5.16. On the other hand, when we are estimating lower order Fučík curves with first and last semi-wave negative ($\mathcal{C}^{-}_{l}$), delta bounds give us often better results – see Figure 5.17. Starting with negative semi-wave means that the first semi-wave is calculated exactly in case of delta bounds, but is estimated in case of rho bounds. And when we are starting (and ending) with positive semi-wave, rho bounds are usually better – see Figure 5.18. We can compare delta and rho bounds for $\mathcal{C}^{+}_{4}$ and $\mathcal{C}^{-}_{4}$ in Figure 5.19.

In the following example, we are going to select three points from $D_{0,4} = (0,4) \times (0,4)$ close to each other and show which point belongs to which bound.

**Example 49.** Let $n = 5$. We are going to take Fučík curve $\mathcal{C}^{-}_{2}$ and find description of delta and rho bounds using Theorems 45 and 48.

For delta bounds $\Psi^{-}_{2}$ of $\mathcal{C}^{-}_{2}$ we have

$$\Psi^{-}_{2} = \left\{(\alpha,\beta) \in D_{0,4} : \frac{\pi}{\omega_\alpha} + 2\frac{\pi}{\omega_\beta} + 2\delta^{\mathtt{min}}_{\alpha,\beta} \le n+1 \le \frac{\pi}{\omega_\alpha} + 2\frac{\pi}{\omega_\beta} + 2\delta^{\mathtt{max}}_{\alpha,\beta}\right\}. \tag{5.9}$$

And for rho bounds $\Upsilon^{-}_{2}$ of $\mathcal{C}^{-}_{2}$ we have

$$\Upsilon^{-}_{2} = \left\{(\alpha,\beta) \in \mathcal{D} : \rho^{\mathtt{min}}_{\alpha,\beta} \le \frac{1}{2}\left(n+1-\frac{\pi}{\omega_\alpha}\right) \le \rho^{\mathtt{max}}_{\alpha,\beta}\right\}. \tag{5.10}$$

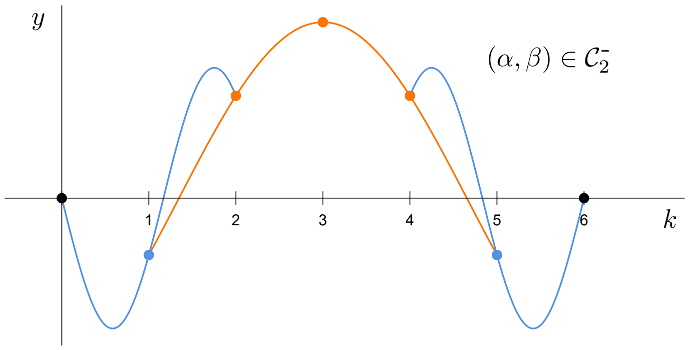



Figure 5.15: Solution $u$ of (P4) for $\alpha \doteq 0.8366$, $\beta = 0.8$. Since $(\alpha, \beta) \in \mathcal{C}_2^-$, solution $u$ has two anchorings – two generalized zeros on $\mathbb{T}$; and it starts with a negative semi-wave.

We have chosen three points $X_1 = (0.6, 3.8)$, $X_2 = (0.73, 3.8)$ and $X_3 = (0.92, 3.8)$ and we are going to calculate if these points belong to delta and rho bounds of $\mathcal{C}_2^-$. In Table 49 we can see that point $X_2$ is in both delta bounds $\Psi_2^-$ and rho bounds $\Upsilon_2^-$. On the other hand, $X_1$ is only in delta bounds $\Psi_2^-$ and $X_3$ is only in rho bounds $\Upsilon_2^-$.

| | (5.9) for $\Psi_2^-$ | $X_i \in \Psi_2^-$ | (5.10) for $\Upsilon_2^-$ | $X_i \in \Upsilon_2^-$ |
|---|---|---|---|---|
| $X_1 = (0.60, 3.8)$ | $5.915 \leq 6 \leq 6.655$ | YES | $1.056 \leq 1.025 \leq 1.506$ | NO |
| $X_2 = (0.73, 3.8)$ | $5.528 \leq 6 \leq 6.26$ | YES | $1.056 \leq 1.221 \leq 1.501$ | YES |
| $X_3 = (0.92, 3.8)$ | $5.116 \leq 6 \leq 5.835$ | NO | $1.057 \leq 1.430 \leq 1.494$ | YES |

Table 5.2: Delta bounds $\Psi_2^-$, rho bounds $\Upsilon_2^-$ and a decision whether points $X_1$, $X_2$ and $X_3$ belong to either $\Psi_2^-$ or $\Upsilon_2^-$.

Moreover, if we fix $\beta = 3.8$, then using either Theorem 32 or Theorem 35, we can calculate approximate value of $\alpha$ in a way, that $(\alpha, \beta) \in \mathcal{C}_2^-$. Such value is $\alpha \doteq 0.8366$. See Figure 5.15 for corresponding solution $u$. And see Figure 5.17 to see a detail of Fučík curve $\mathcal{C}_2^-$ within both bounds.

**Remark 50.** One of the possible ways how to make rho bounds better, is to deal with first and last semi-wave differently. Instead of using $\rho_{\alpha,\beta}^{\mathtt{max}}$ and $\rho_{\alpha,\beta}^{\mathtt{min}}$ as estimates of the the distance $\rho_{\alpha,\beta}$, we calculate these lengths (for the first and last semi-wave) precisely using boundary conditions. For example, when starting with negative semi-wave, we calculate $t_1^-$ precisely, i.e. we use $t_1^- = \rho_{\alpha,\beta}(0)$ instead of $\rho_{\alpha,\beta}^{\mathtt{max}}$ or $\rho_{\alpha,\beta}^{\mathtt{min}}$.

**Remark 51.** Diagram of Fučík spectrum is symmetric along the line $\alpha = \beta$. Thus, we can also "mirror" all bounds. This leads to better results but it is more complicated to calculate as a final bounds for all Fučík curves of $\Sigma(\mathbf{A}^{\mathtt{D}})$.

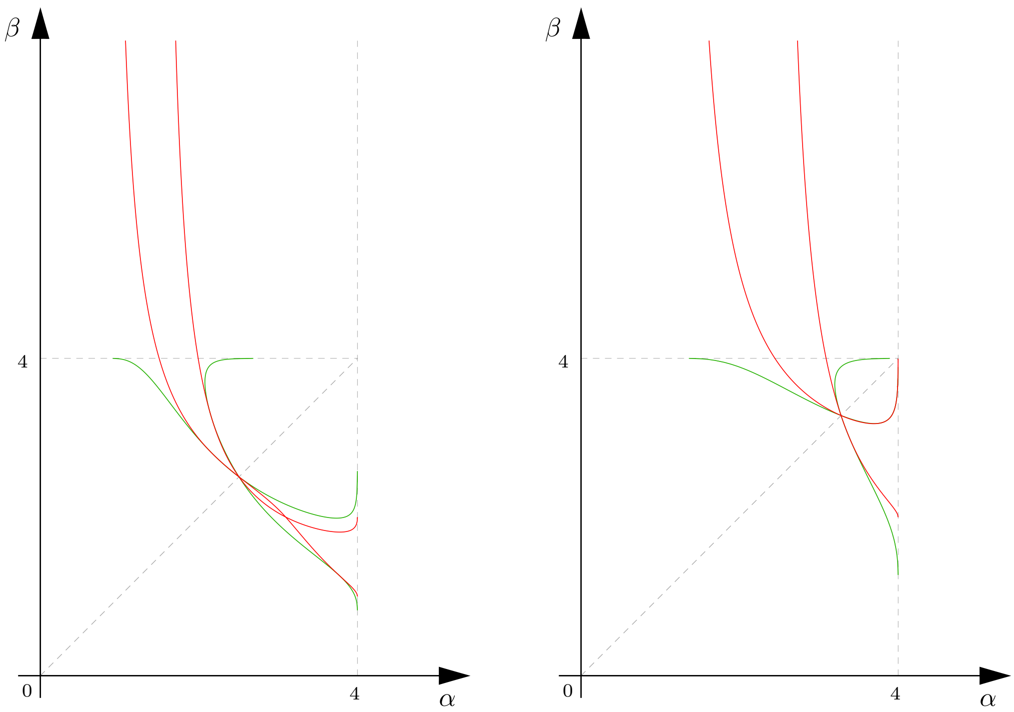



Figure 5.16: Delta bound $\Psi^{+}_{24}$ (green – its boundary) and rho bound $\Upsilon^{+}_{24}$ (red – its boundary) for $\mathcal{C}^{+}_{24}$ (left) and delta bound $\Psi^{+}_{30}$ (green – its boundary) and rho bound $\Upsilon^{+}_{30}$ (red – its boundary) for $\mathcal{C}^{+}_{30}$ (right) of $\Sigma(\mathbf{A}^{\mathtt{D}})$ for $n = 42$.

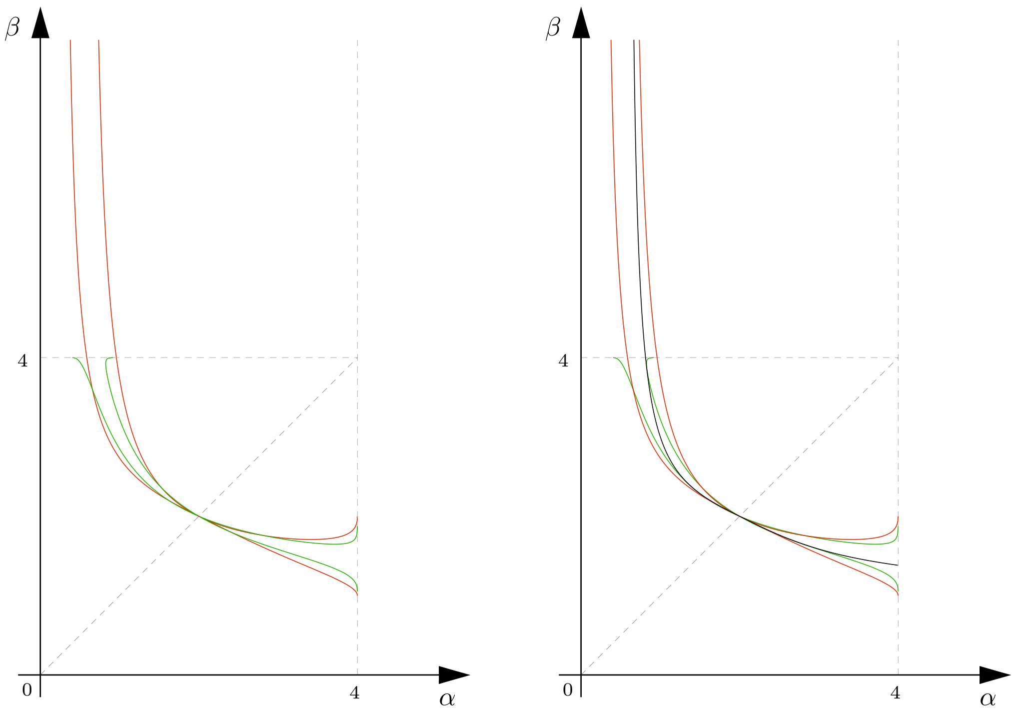



Figure 5.17: Delta bound $\Psi^{-}_{2}$ (green – its boundary) and rho bound $\Upsilon^{-}_{2}$ (red – its boundary) for $\mathcal{C}^{-}_{2}$ (black) of $\Sigma(\mathbf{A}^{\mathtt{D}})$ for $n = 5$.

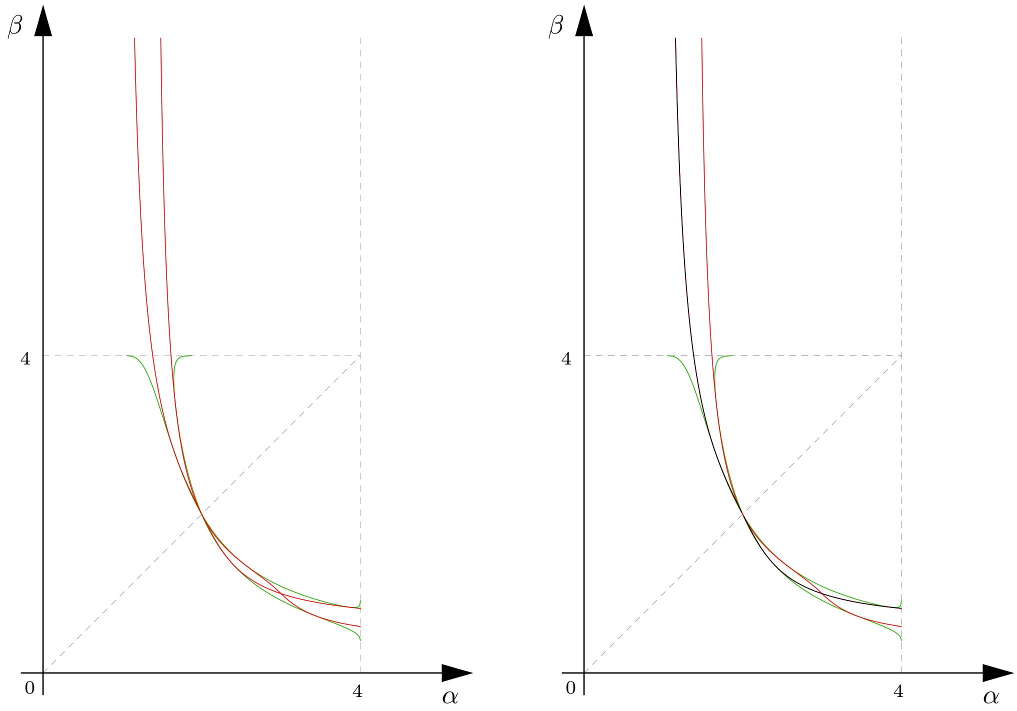



Figure 5.18: Delta bound $\Psi_2^+$ (green – its boundary) and rho bound $\Upsilon_2^+$ (red – its boundary) for $\mathcal{C}_2^+$ (black) of $\Sigma(\mathbf{A}^{\mathrm{D}})$ for $n = 5$.

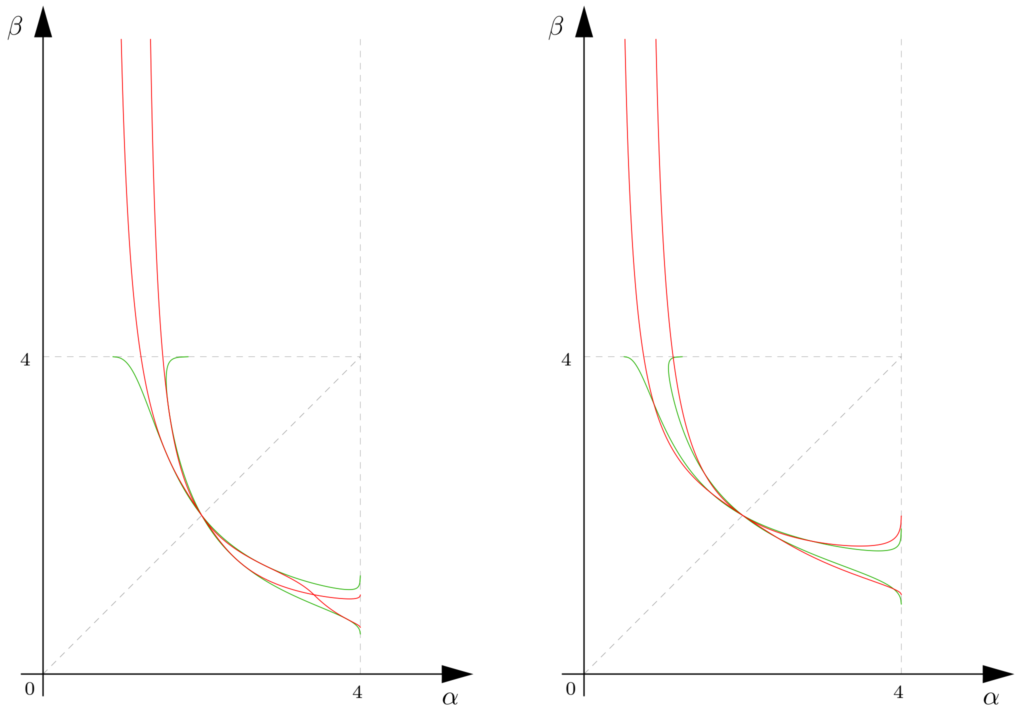



Figure 5.19: Delta bound $\Psi_4^+$ (green – its boundary) and rho bound $\Upsilon_4^+$ (red – its boundary) for $\mathcal{C}_4^+$ (left) and delta bound $\Psi_4^-$ (green – its boundary) and rho bound $\Upsilon_4^-$ (red – its boundary) for $\mathcal{C}_4^-$ (right) of $\Sigma(\mathbf{A}^{\mathrm{D}})$ for $n = 9$.

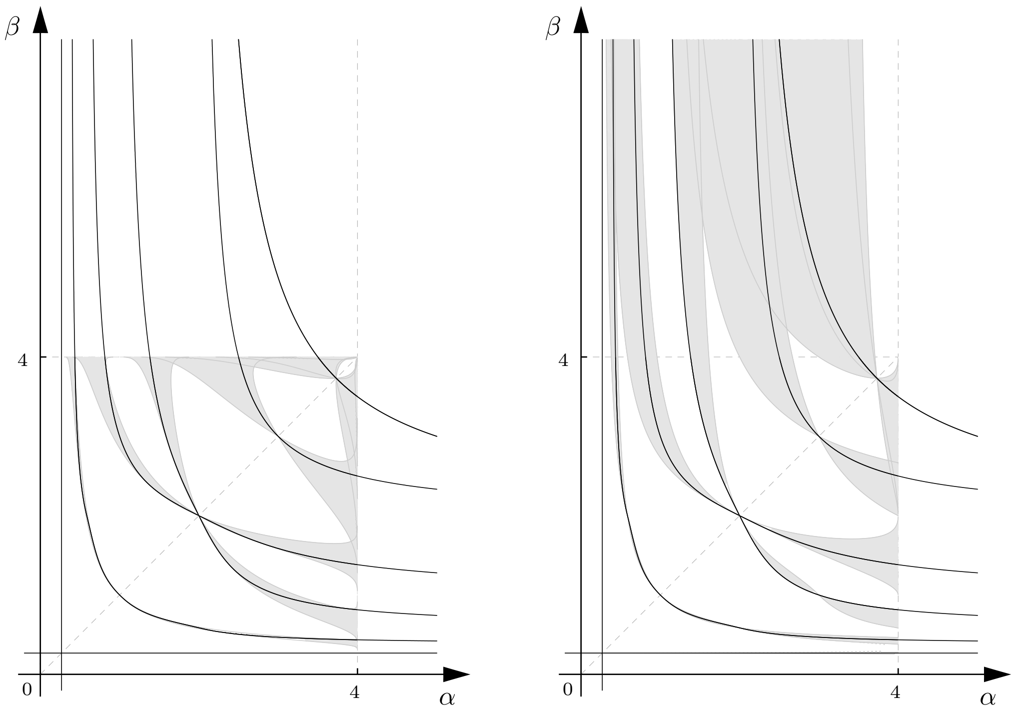



Figure 5.20: Delta bounds $\Psi_l^{\pm}$ (left) and rho bounds $\Upsilon_l^{\pm}$ (right) ($l = 1, \ldots, n-1$) for the Fučík spectrum $\Sigma(\mathbf{A}^{\mathtt{D}})$, $n = 5$.

# Chapter 6

# Conclusion

In this thesis, we provided a complementary text that would be recommended to be read alongside research articles of the author: [25] and [31]. The main purpose of this thesis was to give the reader a more comprehensive understanding and background on the Fučík spectrum for discrete operators.

We studied Fučík spectrum of matrix $\mathbf{A}^{\mathtt{D}}$, i.e. Fučík spectrum for the semi-linear boundary value problem (P4). In order to do that, we explored a linear initial value problem (P1), found its solution (Lemma 3) and defined a continuous extension of the solution.

We also investigated how to retrieve the first non-negative zero of such continuous extension (for $\lambda \in (0,4)$ it is determined by function $T^{\lambda}$ defined in Definition 8) and explored in detail sequence $(q_k)_{k\in\mathbb{Z}}$ of ratios of values of the solution $u$ in two consecutive integers using Chebyshev polynomials of the second kind. We used function $W_k^{\lambda}$ (Definition 13) to find any term of sequence $(q_k)_{k\in\mathbb{Z}}$ (Lemma 15 and Remark 16).

This led to a generalized approach suitable for a semi-linear initial value problem (P3). Another generalization described in the thesis relates to the concept of positive and negative semi-waves introduced in Section 3.1. Therein, we showed a relationship between function $W_k^{\lambda}$ and a solution of a semi-linear problem (P3) – i.e. functions $W_k^{\alpha}$ and $W_k^{\beta}$ allow us to get any term of $(q_k)_{k\in\mathbb{Z}}$ even for semi-linear problem where the role of a continuous extension have positive and negative semi-waves which are "anchored" together.

We introduced sequences of recurrently defined functions $(p_k)_{k\in\mathbb{Z}}$ and $(\vartheta_k)_{k\in\mathbb{Z}}$ in Definition 20. These sequences allowed us to calculate generalized zeros of solution $u$ – Lemma 22. Using such calculation together with the boundary conditions leads to an implicit and recurrent description of the Fučík spectrum of matrix $\mathbf{A}^{\mathtt{D}}$ – see Theorem 32 and Theorem 34 (these theorems differ by the logic of "anchoring" the solutions).

Moreover, we extended our theory of semi-linear initial value problem to the study of the distance function $\rho_{\alpha,\beta}$ – the distance between two consecutive zeros of two different consecutive positive semi-waves. For the definition of function $\rho_{\alpha,\beta}$ (defined in Definition 28) we have explored what is the role of piece-wise linear function $\kappa_{\beta}$ and how it can be used in order to measure the distance function $\rho_{\alpha,\beta}$. This lead to a new (and different) description of the Fučík spectrum of matrix $\mathbf{A}^{\mathtt{D}}$ in Theorem 35.

Finally, we also investigated in detail three different bounds of Fučík curves – basic, delta and rho bounds which are practical applications of the previously introduced theory. Basic bounds were provided in Theorem 39, delta bounds in Theorem 45 and rho bounds in Theorem 48, respectively. Any of these bounds can be used for a numerical localization of Fučík curves even for high dimension matrices $\mathbf{A}^{\mathtt{D}}$, since these bounds will not become more complicated when $n$ is increased.

Even though our descriptions of the Fučík spectrum are for a particular matrix $\mathbf{A}^{\mathtt{D}}$ (Dirichlet matrix), theory in this thesis (and both research articles [25] and [31]) can be extended. The theory was constructed for a difference equation in (P4). In order to describe the Fučík spectrum for a problem with the same difference equation but with a different boundary conditions, one would use the same theory only changing aspects related to the boundary conditions. Thus, our results can be generalized also for different boundary conditions.

We note the newly introduced theory was well received by the academic community – both research articles were published in impacted journals ranked within the first quartile in their respective fields. The main aim of the thesis was to provide a comprehensive overview of the theory, illustrative examples and to make the discrete Fučík spectrum analysis more accessible to a general mathematical audience. Last but not least, we contemplated on further applications of the newly introduced theory – either to generalise results to other boundary conditions or to apply the results on a practical problem (e.g. involving high dimension $\mathbf{A}^{\mathtt{D}}$ matrices). These aspects, among others, are out of scope of the thesis and are left for future research activities.

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# Appendix – Figures

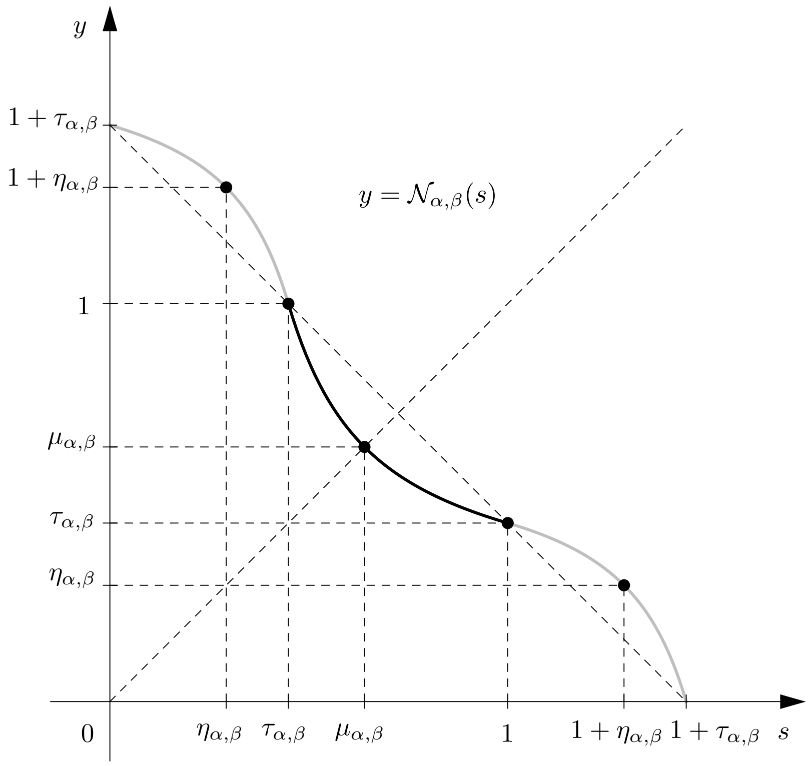


Figure 6.1: Function $\mathcal{N}_{\alpha,\beta}$ for $\alpha < \beta$ ($\alpha = 1.2$, $\beta = 3.2$).

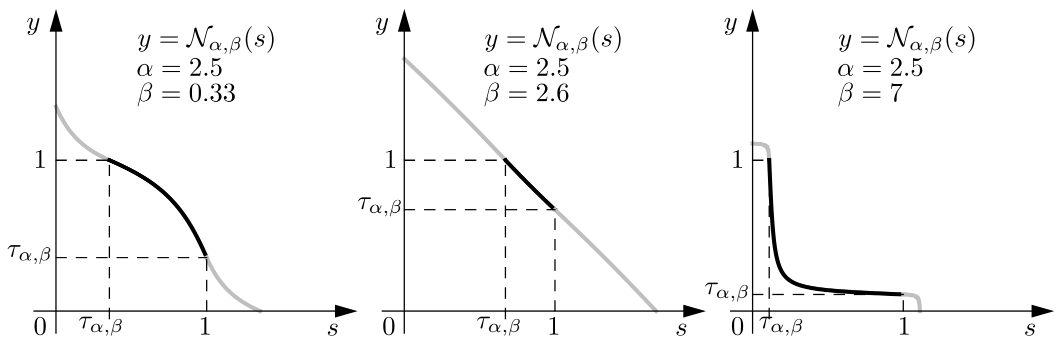


Figure 6.2: Three different shapes of graph of function $\mathcal{N}_{\alpha,\beta}$ when the values $\alpha$ are fixed and values of $\beta$ are changed.

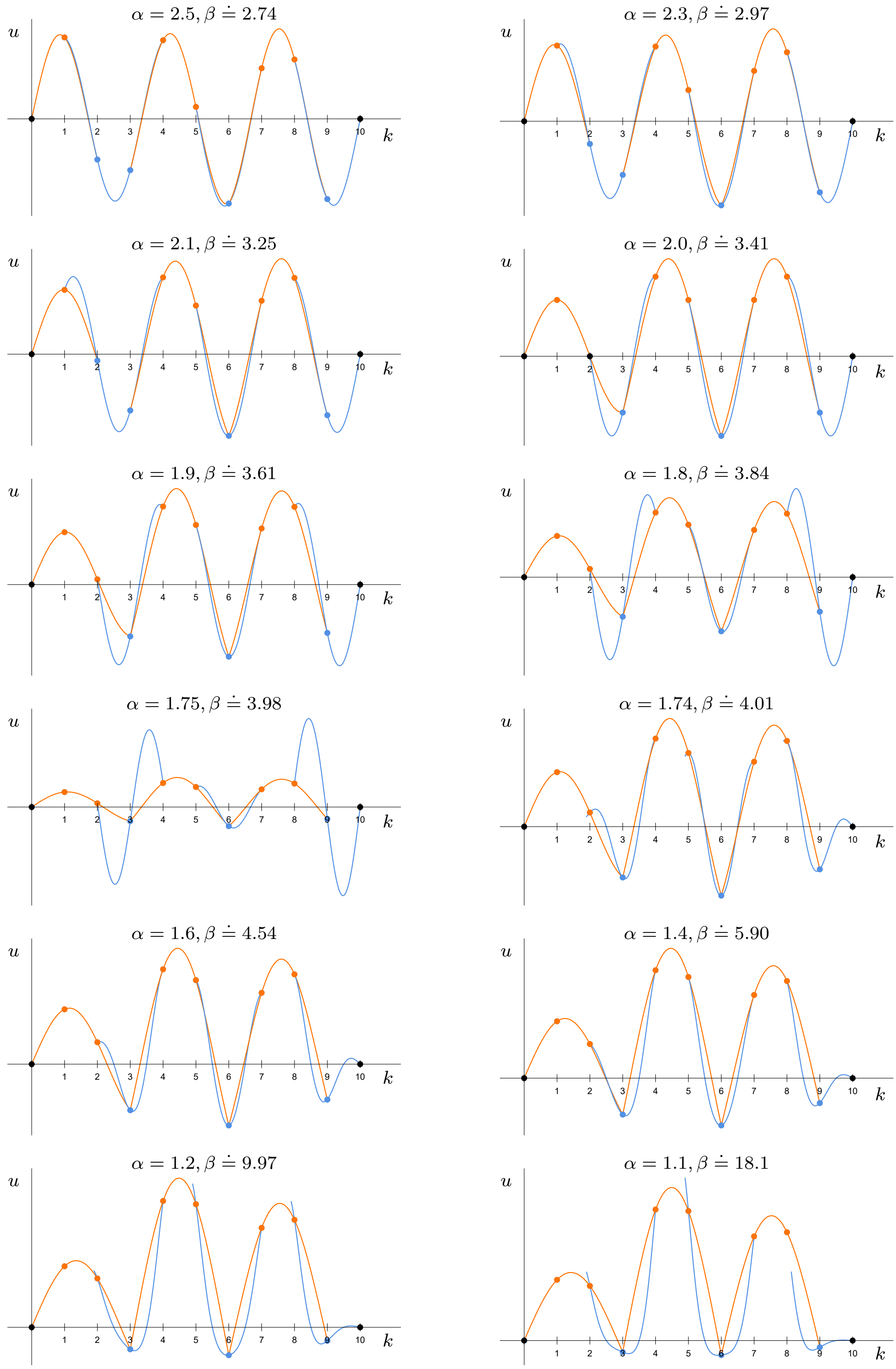


Figure 6.3: Corresponding solutions $u$ for $(\alpha, \beta) \in \mathcal{C}_5^{\pm}$ for $\Sigma(\mathbf{A}^{\mathrm{D}})$ of $n = 9$. Notice, that these solutions have different sign properties – for $\alpha > 2$ we have $u(2) < 0$, for $\alpha = 2$ we have $u(2) = 0$ and for $\alpha < 2$ we have $u(2) > 0$. Also, compare the continuous extensions for $\alpha = 1.75$ versus $\alpha = 1.74$ (for the first one $\beta < 4$ and for the second one $\beta > 4$).

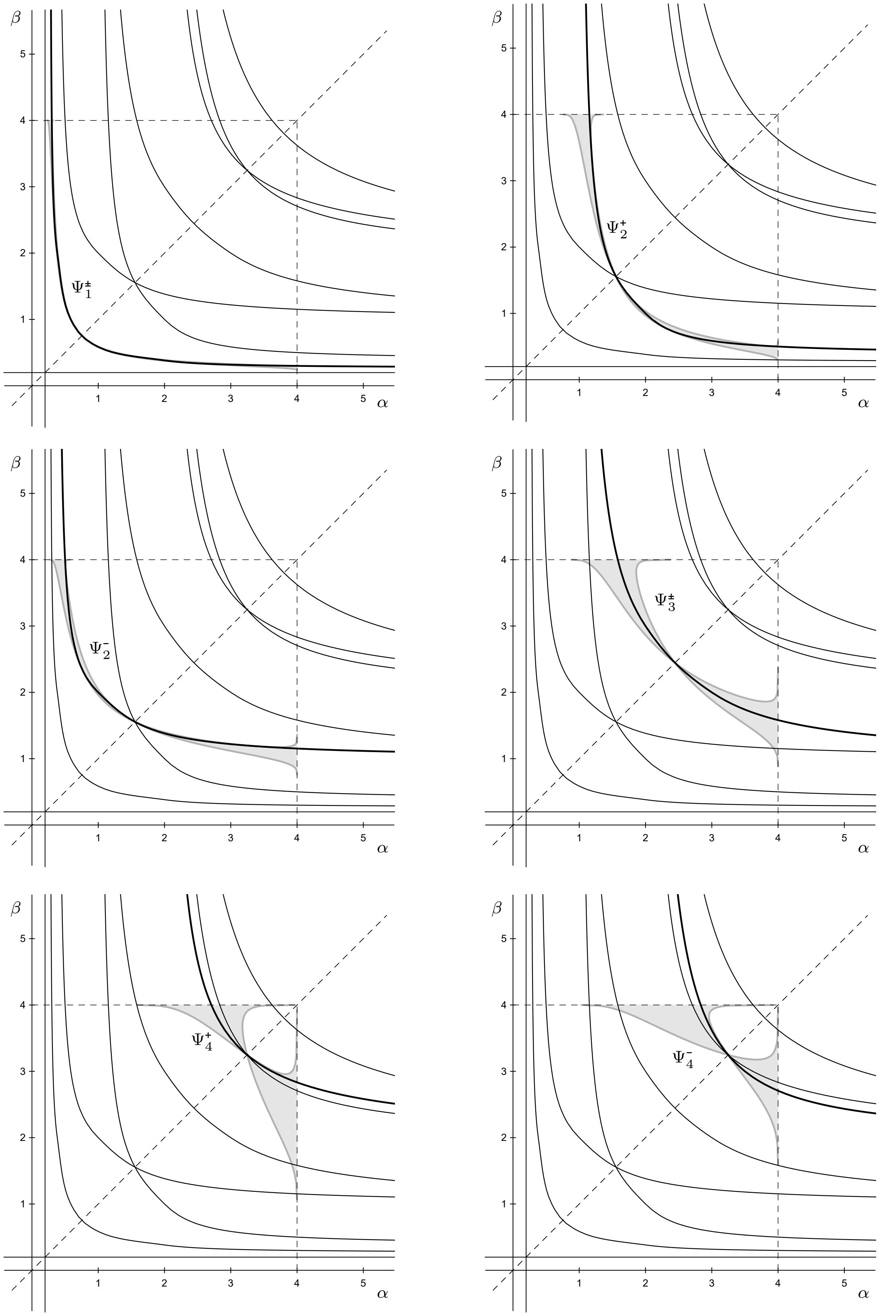


Figure 6.4: Delta bounds for Fučík curves of $\Sigma(\mathbf{A}^{\mathsf{D}})$ for $n = 6$. Bound $\Psi_1^{\pm}$ as a delta bound for $\mathcal{C}_1^{\pm}$ (top, left); bound $\Psi_2^{+}$ as a delta bound for $\mathcal{C}_2^{+}$ (top, right); bound $\Psi_2^{-}$ as a delta bound for $\mathcal{C}_2^{-}$ (middle, left); bound $\Psi_3^{\pm}$ as a delta bound for $\mathcal{C}_3^{\pm}$ (middle, right); bound $\Psi_4^{+}$ as a delta bound for $\mathcal{C}_4^{+}$ (bottom, left) and bound $\Psi_4^{-}$ as a delta bound for $\mathcal{C}_4^{-}$ (bottom, right).

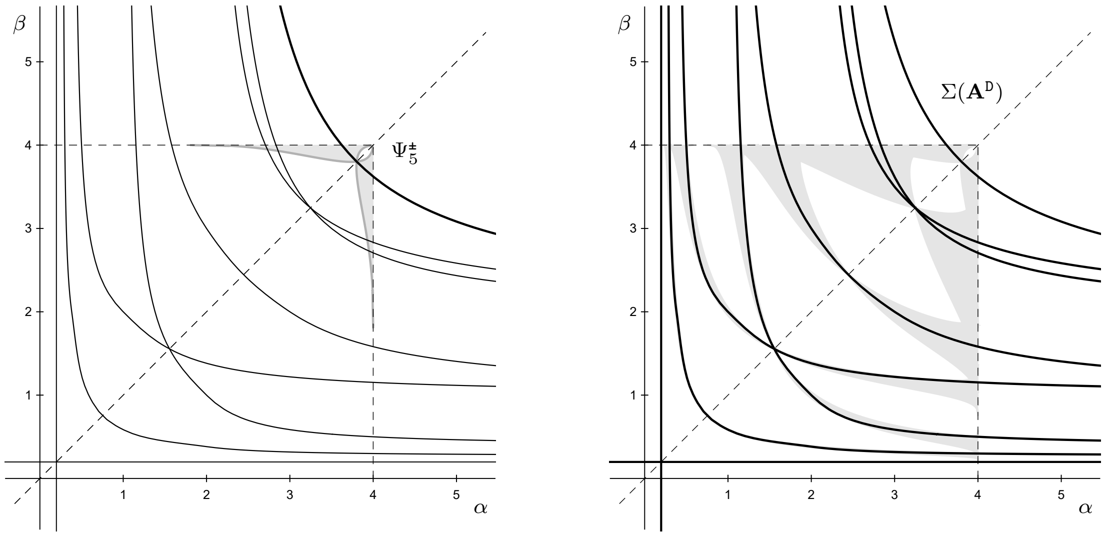


Figure 6.5: Complete illustration of delta bounds for Fučík curves of $\Sigma(\mathbf{A}^{\mathtt{D}})$ for $n = 6$ (right). Bound $\Psi_5^{\pm}$ as a delta bound for $\mathcal{C}_5^{\pm}$ (left).

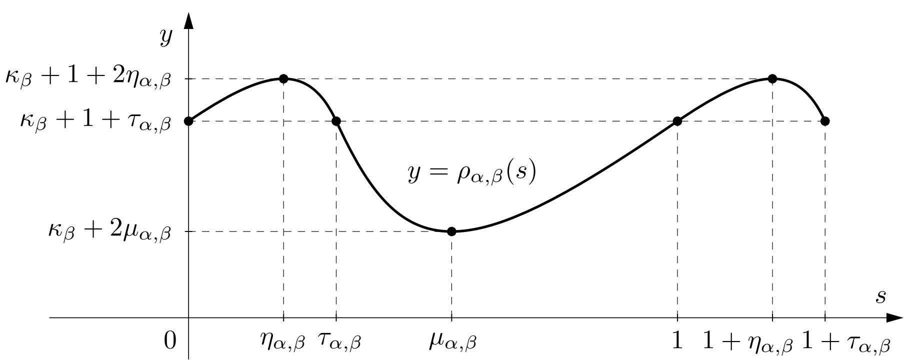


Figure 6.6: The graph of the function $\rho_{\alpha,\beta}$ for $\alpha < \beta$ ($\alpha = 2.6$, $\beta = 3.8$).

# Appendix – Published research articles of the author

In this appendix, we attach the following research articles:

[25] I. Looseová (Sobotková), P. Nečesal, The Fučík spectrum of the discrete Dirichlet operator, Linear Algebra Appl. 553 (2018) 58–103

[31] P. Nečesal, I. Sobotková, Localization of Fučík curves for the second order discrete Dirichlet operator, Bulletin des Sciences Mathématiques 171 (2021) 103014

Linear Algebra and its Applications 553 (2018) 58–103

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# The Fučík spectrum of the discrete Dirichlet operator

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A R T I C L E   I N F O

*Article history:*
Received 31 December 2017
Accepted 16 April 2018
Available online 30 May 2018
Submitted by P. Semrl

*MSC:*
39A10
39A12
39A70
34B15

*Keywords:*
Asymmetric nonlinearities
Chebyshev polynomials of the second kind
Difference operators
Fučík spectrum
Möbius transformation
Möbius matrix
Homogeneous coordinates

A B S T R A C T

In this paper, we deal with the discrete Dirichlet operator of the second order and we investigate its Fučík spectrum, which consists of a finite number of algebraic curves. For each non-trivial Fučík curve, we are able to detect a finite number of its points, which are given explicitely. We provide the exact implicit description of all non-trivial Fučík curves in terms of Chebyshev polynomials of the second kind. Moreover, for each non-trivial Fučík curve, we give several different implicit descriptions, which differ in the level of depth of used nested functions. Our approach is based on the Möbius transformation and on the appropriate continuous extension of solutions of the discrete problem. Let us note that all presented descriptions of Fučík curves have the form of necessary and sufficient conditions. Finally, our approach can be also directly used in the case of difference operators of the second order with other local boundary conditions.



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https://doi.org/10.1016/j.laa.2018.04.017

## 1. Introduction

In this paper, we deal with the following discrete problem with Dirichlet boundary conditions

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, & k \in \mathbb{T}, \\ u(0) = u(n+1) = 0, \end{cases} \tag{1}$$

where $u : \hat{\mathbb{T}} \to \mathbb{R}$, $\hat{\mathbb{T}} := \{0, 1, \ldots, n, n+1\}$, $\mathbb{T} := \{1, \ldots, n\}$, $n \in \mathbb{N}$, $u^+$ and $u^-$ stand for the positive and the negative parts of $u$, respectively, i.e.

$$u^+, u^- : \hat{\mathbb{T}} \to \mathbb{R}, \qquad u^\pm(k) := \max\{\pm u(k), 0\},$$

$\alpha, \beta \in \mathbb{R}$ and the second order forward difference $\Delta^2 u(k-1)$ is given by

$$\Delta^2 u(k-1) := u(k-1) - 2u(k) + u(k+1).$$

The purpose of this paper is to study the set of all pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that the problem (1) has a non-trivial solution $u$, which is equivalent to investigate the set

$$\Sigma\left(\mathbf{A}^{\mathtt{D}}\right) := \big\{(\alpha, \beta) \in \mathbb{R}^2 : \text{ the problem } \mathbf{A}^{\mathtt{D}}\mathbf{u} = \alpha\mathbf{u}^+ - \beta\mathbf{u}^- \text{ has a non-trivial solution } \mathbf{u}\big\},$$

where $\mathbf{u}$, $\mathbf{u}^+$ and $\mathbf{u}^-$ are column vectors with $n$ elements $u(k)$, $u^+(k)$ and $u^-(k)$, respectively, i.e.

$$\mathbf{u} := [u(1), \ldots, u(n)]^t, \qquad \mathbf{u}^\pm := [u^\pm(1), \ldots, u^\pm(n)]^t,$$

and $\mathbf{A}^{\mathtt{D}}$ is the $n$-by-$n$ Dirichlet matrix

$$\mathbf{A}^{\mathtt{D}} := \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

The set $\Sigma\left(\mathbf{A}^{\mathtt{D}}\right)$ is called the Fučík spectrum of the matrix $\mathbf{A}^{\mathtt{D}}$ and its structure has been studied in [5], [6] and [8]. Let us note that all eigenvalues of $\mathbf{A}^{\mathtt{D}}$ are real eigenvalues and thus, each eigenvalue $\lambda$ of $\mathbf{A}^{\mathtt{D}}$ determines a pair $(\lambda, \lambda)$, which belongs to $\Sigma(\mathbf{A}^{\mathtt{D}})$. Indeed, for $\alpha = \beta = \lambda$, the problem $\mathbf{A}^{\mathtt{D}}\mathbf{u} = \alpha\mathbf{u}^+ - \beta\mathbf{u}^-$ is reduced to the linear eigenvalue problem $\mathbf{A}^{\mathtt{D}}\mathbf{u} = \lambda\mathbf{u}$.

Before we recall some known results concerning the set $\Sigma\left(\mathbf{A}^{\mathtt{D}}\right)$, let us introduce the following notation. Let us denote by $\mathcal{C}_k^+$ $(\mathcal{C}_k^-)$ the set of all pairs $(\alpha, \beta) \in \Sigma\left(\mathbf{A}^{\mathtt{D}}\right)$ such

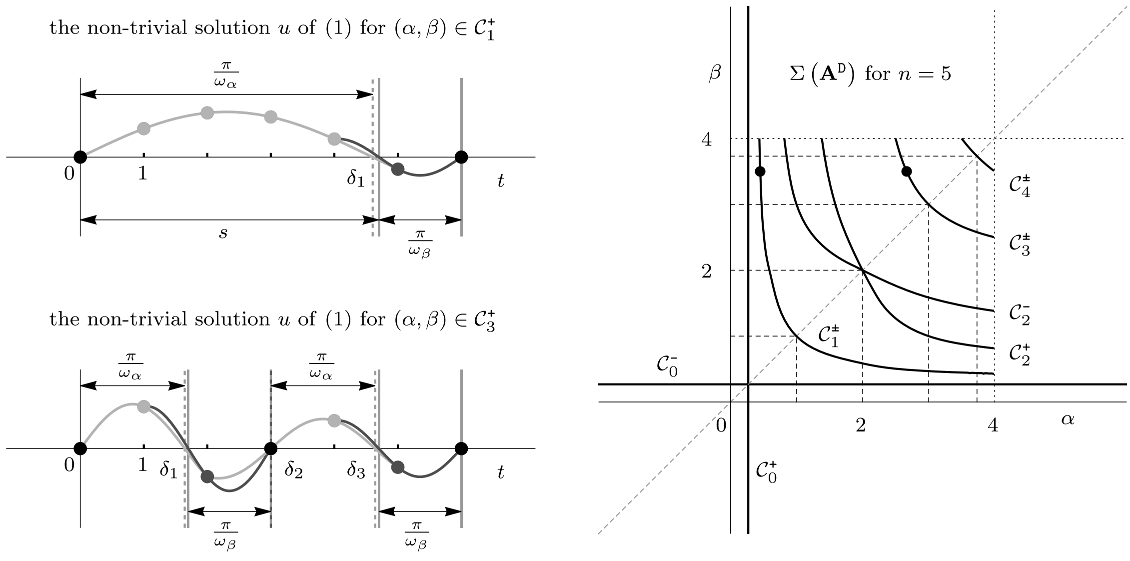


**Fig. 1.** Due to results in [5], it is possible to obtain the numerical approximation of those parts of non-trivial Fučík curves $\mathcal{C}_k^\pm$ of $\Sigma(\mathbf{A}^{\mathtt{D}})$, which are located in the square $(0,4) \times (0,4)$ (right) and also corresponding non-trivial solutions $u$ of (1) for different pairs $(\alpha,\beta) \in \Sigma(\mathbf{A}^{\mathtt{D}}) \cap (0,4) \times (0,4)$ (left).

that the corresponding non-trivial solution $u$ of (1) changes its sign $k$ times on $\mathbb{T}$ and the value $u(1)$ is a positive (negative) one. The sets $\mathcal{C}_0^+$ and $\mathcal{C}_0^-$ are lines, which are parallel to $\beta$ and $\alpha$ axes, respectively, since corresponding non-trivial solutions $u$ do not change sign on $\mathbb{T}$. We call both curves $\mathcal{C}_0^+$ and $\mathcal{C}_0^-$ as trivial Fučík curves.

Firstly, C. Margulies and W. Margulies studied the solvability of mildly nonlinear matrix equations with a general $n$-by-$n$ self-adjoint matrix and recognized the importance of the corresponding Fučík spectrum as a resonance set. Using their general results published in [6] in 1999, we conclude that the Fučík spectrum $\Sigma(\mathbf{A}^{\mathtt{D}})$ is a closed subset of $\mathbb{R}^2$, which does not contain an open set and is made of finitely many algebraic curves. Moreover, in the case of the 2-by-2 Dirichlet matrix, the structure of the Fučík spectrum $\Sigma(\mathbf{A}^{\mathtt{D}})$ is trivial one and is mentioned in [6] as an example.

Secondly, R. Ma, Y. Xu and Ch. Gao published their paper [5] in 2010, where the matching-extension method is introduced to obtain an expression of $\Sigma(\mathbf{A}^{\mathtt{D}})$ similar to the well-known description of the Fučík spectrum in the case of the continuous Dirichlet problem (see [1] or [2])

$$\begin{cases} v''(t) + \alpha v^+(t) - \beta v^-(t) = 0, \quad t \in (0,\pi), \\ v(0) = v(\pi) = 0. \end{cases}$$

Theorem 3.1. in [5] provides an important description of parts of all non-trivial Fučík curves $\mathcal{C}_k^\pm$, which are located in the square $(0,4) \times (0,4)$ (see Fig. 1, right). Namely, for the first non-trivial Fučík curve $\mathcal{C}_1^\pm$, we have that $(\alpha,\beta) \in \mathcal{C}_1^\pm \cap (0,4) \times (0,4)$ if and only if $\alpha, \beta \in (2 - 2\cos\frac{\pi}{n}, 4)$ are such that

$$\sin\left(\omega_\alpha\left[\tfrac{\pi}{\omega_\alpha}\right]\right)\sin\left(\omega_\beta\left(\left[\tfrac{\pi}{\omega_\alpha}\right]+1-s\right)\right)=\sin\left(\omega_\alpha\left(\left[\tfrac{\pi}{\omega_\alpha}\right]+1\right)\right)\sin\left(\omega_\beta\left(\left[\tfrac{\pi}{\omega_\alpha}\right]-s\right)\right), \tag{2}$$

where

$$s+\tfrac{\pi}{\omega_\beta}=n+1, \qquad \left[\tfrac{\pi}{\omega_\alpha}\right]\le s<\left[\tfrac{\pi}{\omega_\alpha}\right]+1. \tag{3}$$

In the previous assertion, we denoted $\omega_\alpha := \arccos\frac{2-\alpha}{2}$ and $\omega_\beta := \arccos\frac{2-\beta}{2}$ and $[\cdot]$ means the integer part function. The corresponding non-trivial solution $u$ of (1) changes its sign exactly ones in $\mathbb{T}$, thus, it consists of one positive and one negative semi-waves (see Fig. 1, left). The positive (negative) semi-wave has its continuous extension with the frequency $\omega_\alpha$ ($\omega_\beta$) and the distance of two consecutive zeros of its continuous extension is exactly $\frac{\pi}{\omega_\alpha}$ ($\frac{\pi}{\omega_\beta}$). The equality in (3) can be rephrased as

$$\frac{\pi}{\omega_\alpha}+\frac{\pi}{\omega_\beta}+\delta_1=n+1,$$

where $\delta_1 = s-\frac{\pi}{\omega_\alpha}\in(-1,1)$ is the difference of zeros of the continuous extensions of negative and positive semi-waves. For the higher non-trivial Fučík curves, we have the following description

$$\mathcal{C}^{\pm}_{2j-1}:\ j\frac{\pi}{\omega_\alpha}+j\frac{\pi}{\omega_\beta}+\sum_{i=1}^{2j-1}\delta_i=n+1, \tag{4}$$

and

$$\mathcal{C}^{+}_{2j}:\ (j+1)\frac{\pi}{\omega_\alpha}+j\frac{\pi}{\omega_\beta}+\sum_{i=1}^{2j}\delta_i=n+1, \qquad \mathcal{C}^{-}_{2j}:\ j\frac{\pi}{\omega_\alpha}+(j+1)\frac{\pi}{\omega_\beta}+\sum_{i=1}^{2j}\delta_i=n+1, \tag{5}$$

where $\delta_i\in(-1,1)$ depends on $\alpha$ and $\beta$ and is given implicitly by a transcendent equation similar to (2). Thus, (4) and (5) represent the particular Fučík curves $\mathcal{C}^{\pm}_k$ as level sets $F^{\pm}_k(\alpha,\beta)=n+1$, where $F^{\pm}_k$ are given implicitly.

Thirdly, P. Stehlík published his paper [8] in 2013, where he studied the qualitative properties of the first non-trivial Fučík curve $\mathcal{C}^{\pm}_1$. He proved that the first non-trivial Fučík curve is decomposable in the following way (see Fig. 2)

$$\mathcal{C}^{\pm}_1={}^{\mathtt{A}}\mathcal{C}^{\pm}_1\cup{}^{\mathtt{P}}\mathcal{C}^{\pm}_1\cup{}^{\mathtt{B}}\mathcal{C}^{\pm}_1\cup{}^{\mathtt{C}}\mathcal{C}^{\pm}_1,$$

where two continuous curves ${}^{\mathtt{A}}\mathcal{C}^{\pm}_1$ and ${}^{\mathtt{C}}\mathcal{C}^{\pm}_1$ and the set ${}^{\mathtt{P}}\mathcal{C}^{\pm}_1$ of finite number of points are given explicitly as

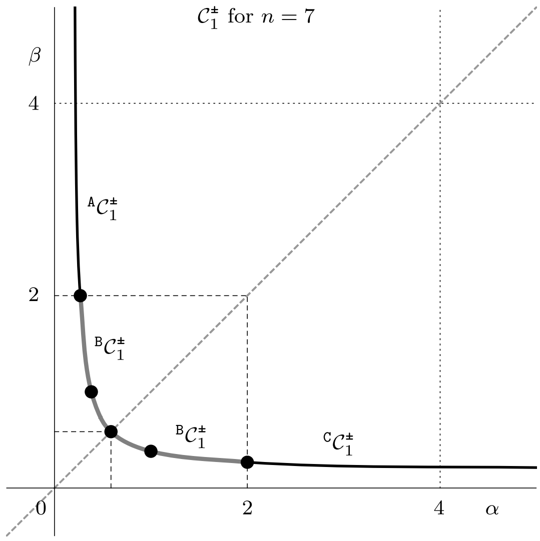


**Fig. 2.** The first non-trivial Fučík curve $\mathcal{C}_1^{\pm}$ and its decomposition $\mathcal{C}_1^{\pm} = {}^{\mathtt{A}}\mathcal{C}_1^{\pm} \cup {}^{\mathtt{P}}\mathcal{C}_1^{\pm} \cup {}^{\mathtt{B}}\mathcal{C}_1^{\pm} \cup {}^{\mathtt{C}}\mathcal{C}_1^{\pm}$ due to results in [8] (black points represents the set ${}^{\mathtt{P}}\mathcal{C}_1^{\pm}$).

$$\begin{aligned}
{}^{\mathtt{A}}\mathcal{C}_1^{\pm} &= \left\{(\alpha,\beta): \quad \alpha = 4\sin^2\frac{\pi}{2(n+1-t)}, \quad \beta = 2 - \frac{\sin\frac{(n-1)\pi}{n+1-t}}{\sin\frac{n\pi}{n+1-t}}, \quad t \in (1,2)\right\},\\
{}^{\mathtt{P}}\mathcal{C}_1^{\pm} &= \left\{(\alpha,\beta): \quad \alpha = 4\sin^2\frac{\pi}{2(n+1-k)}, \quad \beta = 4\sin^2\frac{\pi}{2k}, \quad k = 2,\dots,n-1\right\},\\
{}^{\mathtt{C}}\mathcal{C}_1^{\pm} &= \left\{(\alpha,\beta): \quad \alpha = 2 - \frac{\sin\frac{(n-1)\pi}{t}}{\sin\frac{n\pi}{t}}, \quad \beta = 4\sin^2\frac{\pi}{2t}, \quad t \in (n-1,n)\right\}.
\end{aligned} \tag{6}$$

The set ${}^{\mathtt{B}}\mathcal{C}_1^{\pm}$ is the part of $\mathcal{C}_1^{\pm}$, which belongs to the square $(0,2)\times(0,2)$ and has empty intersection with the set ${}^{\mathtt{P}}\mathcal{C}_1^{\pm}$. Moreover, in [8] is proved the following necessary condition: if $(\alpha,\beta) \in {}^{\mathtt{B}}\mathcal{C}_1^{\pm}$ then

$$\sin\left(\omega_\alpha\left\lfloor\frac{\pi}{\omega_\alpha}\right\rfloor\right)\sin\left(\omega_\beta\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor\right) = \sin\left(\omega_\alpha\left\lceil\frac{\pi}{\omega_\alpha}\right\rceil\right)\sin\left(\omega_\beta\left\lceil\frac{\pi}{\omega_\beta}\right\rceil\right),$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote floor and ceiling functions, respectively. Finally, the last part of [8] is devoted to the elementariness of ${}^{\mathtt{B}}\mathcal{C}_1^{\pm}$. A conjecture is stated that ${}^{\mathtt{B}}\mathcal{C}_1^{\pm}$ has no elementary parametrization and also possible ways how to prove it are discussed.

In this paper, we prove the following main results.

1. Theorem 22 provides the complete description of the Fučík spectrum $\Sigma(\mathbf{A}^{\mathtt{D}})$, the particular Fučík curves are described as level sets $F_k^{\pm}(\alpha,\beta) = n+1$, where the functions $F_k^{\pm}$ are given explicitly. In addition, Theorem 22 contains the description

of all Fučík curves as algebraic curves in terms of Chebyshev polynomials of the second kind in prescribed regions (see Figs. 15, 16 and Example 24).

2. In Theorem 26, we have another description of all Fučík curves as level curves $F^{\pm}_{i,j}(\alpha,\beta) = n+1$, where functions $F^{\pm}_{i,j}$ are given explicitly and differ in the level of depth of used nested functions. Moreover, using this description, we proved Theorem 27 containing the representation of all Fučík curves as algebraic curves in prescribed regions, which are different from regions given by Theorem 22 (see Figs. 18 to 24 and Example 29).
3. In Corollary 25, we provide finite number of points $(\xi_i, \xi_j)$, which belong to particular Fučík curves and contain all points of ${}^{\mathsf{P}}\mathcal{C}^{\pm}_1$ given by (6).
4. In Corollary 30, we give the implicit description of the first non-trivial Fučík curve $\mathcal{C}^{\pm}_1$ in terms of Chebyshev polynomials of the second kind. More precisely, we show that the first non-trivial Fučík curve $\mathcal{C}^{\pm}_1$ consist of parts of algebraic curves in prescribed rectangles. As a consequence of Corollary 30, it is straightforward to verify that the part ${}^{\mathsf{B}}\mathcal{C}^{\pm}_1$ of the first non-trivial Fučík curve $\mathcal{C}^{\pm}_1$ has an elementary parametrization for $n = 4, 5, 6, 7$ (note that ${}^{\mathsf{B}}\mathcal{C}^{\pm}_1$ is the empty set for $n = 1, 2, 3$).

We show that the first non-trivial Fučík curve $\mathcal{C}^{\pm}_1$ of the Fučík spectrum consists of parts of algebraic curves (see Corollary 30)

$$U_{n-i}\left(\tfrac{2-\alpha}{2}\right) U_i\left(\tfrac{2-\beta}{2}\right) - U_{n-i-1}\left(\tfrac{2-\alpha}{2}\right) U_{i-1}\left(\tfrac{2-\beta}{2}\right) = 0, \qquad i = 1, \ldots, n-1, \tag{7}$$

where $U_k = U_k(x)$ are the Chebyshev polynomials of the second kind of degree $k$. Let us note that for $\alpha = \beta = \lambda$, each equation in (7) simplifies in polynomial equation

$$U_n\left(\tfrac{2-\lambda}{2}\right) = 0,$$

which has roots made of eigenvalues $\lambda^{\mathsf{D}}_0 < \lambda^{\mathsf{D}}_1 < \cdots < \lambda^{\mathsf{D}}_{n-1}$ of $\mathbf{A}^{\mathsf{D}}$ since $U_n\left(\tfrac{2-\lambda}{2}\right) = \det\left(\mathbf{A}^{\mathsf{D}} - \lambda \mathbf{I}\right)$. The Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$ consists of particular Fučík curves $\mathcal{C}^{\pm}_k$, $k = 0, \ldots, n-1$, which emanate from points $(\lambda^{\mathsf{D}}_k, \lambda^{\mathsf{D}}_k)$ on the diagonal $\alpha = \beta$ (see Fig. 3). We have the following description of the Fučík curves $\mathcal{C}^{\pm}_k$. For $k$ odd, the Fučík curve $\mathcal{C}^{+}_k$ coincides with the curve $\mathcal{C}^{-}_k$ and consists of parts of algebraic curves (see Theorem 22)

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \left[W^{\beta}_{p_{k+1}}\right] \cdot \left[W^{\alpha}_{p_k}\right] \cdots \left[W^{\beta}_{p_2}\right] \cdot \left[W^{\alpha}_{p_1}\right] \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0, \tag{8}$$

where $\left[W^{\lambda}_j\right]$ is $2 \times 2$ matrix given by

$$\left[W^{\lambda}_j\right] = \begin{bmatrix} U_j\left(\tfrac{2-\lambda}{2}\right) & -U_{j-1}\left(\tfrac{2-\lambda}{2}\right) \\ U_{j-1}\left(\tfrac{2-\lambda}{2}\right) & -U_{j-2}\left(\tfrac{2-\lambda}{2}\right) \end{bmatrix}, \quad j \in \mathbb{Z},\ \lambda \in \mathbb{R},$$

and $p_1, \ldots, p_{k+1}$ are integers, which depend on $\alpha$ and $\beta$, and are given by explicit

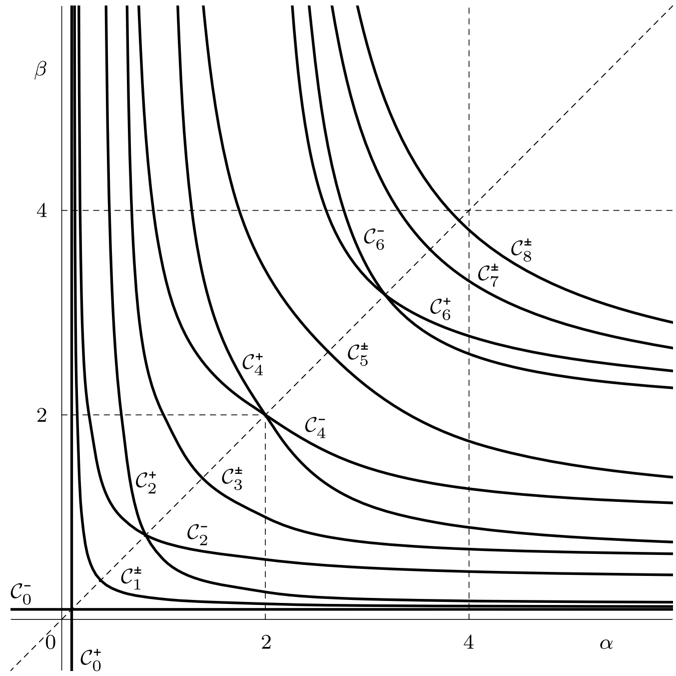


**Fig. 3.** The Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$ of the Dirichlet matrix $\mathbf{A}^{\mathsf{D}}$ of order $n = 9$, the particular Fučík curves $\mathcal{C}_k^\pm$ are described as level sets $F_k^\pm(\alpha, \beta) = n + 1$ due to Theorem 22.

analytic formulas (see Definition 17). Moreover, algebraic curves (8) are of the form $\det\left(\mathbf{A}^{\mathsf{D}} - \mathbf{\Lambda}\right) = 0$, where $\mathbf{\Lambda}$ is the diagonal matrix

$$\mathbf{\Lambda} = \operatorname{diag}(\underbrace{\alpha, \dots, \alpha}_{p_1\text{–times}}, \underbrace{\beta, \dots, \beta}_{p_2\text{–times}}, \dots, \underbrace{\alpha, \dots, \alpha}_{p_k\text{–times}}, \underbrace{\beta, \dots, \beta}_{p_{k+1}\text{–times}}).$$

Integers $p_1, \dots, p_{k+1}$ determine, which components of the corresponding non-trivial solution $\mathbf{u}$ of the problem $\mathbf{A}^{\mathsf{D}}\mathbf{u} = \alpha\mathbf{u}^+ - \beta\mathbf{u}^-$ are non-negative and which are non-positive, i.e.

$$\begin{aligned}
\mathbf{u}^+ &= (\overbrace{\blacksquare, \dots, \blacksquare}^{p_1\text{–times}}, \overbrace{0, \dots, 0}^{p_2\text{–times}}, \dots, \overbrace{\blacksquare, \dots, \blacksquare}^{p_k\text{–times}}, \overbrace{0, \dots, 0}^{p_{k+1}\text{–times}}),\\
\mathbf{u}^- &= (\underbrace{0, \dots, 0}_{p_1\text{–times}}, \underbrace{\blacksquare, \dots, \blacksquare}_{p_2\text{–times}}, \dots, \underbrace{0, \dots, 0}_{p_k\text{–times}}, \underbrace{\blacksquare, \dots, \blacksquare}_{p_{k+1}\text{–times}}).
\end{aligned}$$

Let us note that for $k = 1$, algebraic curves (8) can be rewritten in the form of (7). For $k$ even, the Fučík curve $\mathcal{C}_k^+$ consists of parts of algebraic curves

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} W_{p_{k+1}}^{\alpha} \end{bmatrix} \cdot \begin{bmatrix} W_{p_k}^{\beta} \end{bmatrix} \cdots \begin{bmatrix} W_{p_2}^{\beta} \end{bmatrix} \cdot \begin{bmatrix} W_{p_1}^{\alpha} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0,$$

and finally, the Fučík curve $\mathcal{C}_k^-$ consists of parts of algebraic curves

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} W^{\beta}_{p_{k+1}} \end{bmatrix} \cdot \begin{bmatrix} W^{\alpha}_{p_k} \end{bmatrix} \cdots \begin{bmatrix} W^{\alpha}_{p_2} \end{bmatrix} \cdot \begin{bmatrix} W^{\beta}_{p_1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$$

This paper is organized in the following way. Firstly, we review some standard facts on linear initial value problem (9) and discuss properties of its solution in Section 2. We look more closely at continuous extension and zero points of such extension. We introduce Möbius transformation $W^{\lambda}_{k}$ (see Definition 5) which allows us to get ratio of any consecutive elements of the solution. At the end of this section, we discuss the case of linear problem (24) and the developed theory will be useful in the following parts of this paper. In Section 3, we investigate in detail continuous extension of a solution of semi-linear initial value problem (29). We introduce several sequences of functions which allow us to locate all generalized zeros of the solution of the problem (29). Finally, in Section 4, our main results are stated and proved. We give several descriptions of the Fučík spectrum of the Dirichlet matrix in the form of necessary and sufficient conditions.

## 2. The continuous extension of a solution of the linear problem

In this section, we look more closely at the linear initial value problem

$$\begin{cases} \Delta^2 u(k-1) + \lambda u(k) = 0, & k \in \mathbb{Z}, \\ u(0) = C_0,\ u(1) = C_1, \end{cases} \tag{9}$$

where $\lambda$ and $C_0, C_1 \in \mathbb{R}$ are constants such that $C_0^2 + C_1^2 \neq 0$.

The characteristic equation for the difference equation in (9) has the form

$$r^2 + (\lambda - 2)r + 1 = 0$$

with the roots

$$r_{1,2} = \begin{cases} \frac{2-\lambda}{2} \pm \sqrt{\left(\frac{2-\lambda}{2}\right)^2 - 1} & \text{for } |\lambda - 2| \geq 2, \\ \frac{2-\lambda}{2} \pm \mathrm{i}\sqrt{1 - \left(\frac{2-\lambda}{2}\right)^2} & \text{for } |\lambda - 2| < 2. \end{cases}$$

Let us point out that $r_1 r_2 = 1$. For $\lambda = 0$ ($\lambda = 4$), we get $r_1 = r_2 = 1$ ($r_1 = r_2 = -1$). For $|\lambda - 2| \leq 2$, roots $r_1$ and $r_2$ are complex conjugate such that $|r_1| = |r_2| = 1$. For $\lambda \leq 0$ ($\lambda \geq 4$), both roots $r_1$ and $r_2$ are real and positive (negative). For given $\lambda \in \mathbb{R}$ and $C_0, C_1 \in \mathbb{R}$, the initial value problem (9) has a unique solution of the form

$$u(k) = \begin{cases} C_0(1-k) + C_1(k) & \text{for } \lambda = 0, \\ -C_0(1-k)(-1)^{1-k} - C_1(k)(-1)^k & \text{for } \lambda = 4, \\ C_0 \dfrac{r_1 r_2^k - r_2 r_1^k}{r_1 - r_2} + C_1 \dfrac{r_1^k - r_2^k}{r_1 - r_2} & \text{for } \lambda \in \mathbb{R} \setminus \{0, 4\}. \end{cases}$$

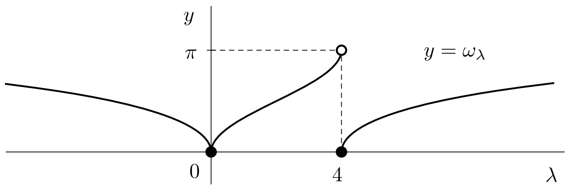


**Fig. 4.** The graph of $\omega_\lambda$ as a function of $\lambda$.

In the following lemma, we provide a different formula for the discrete solution $u$ of the initial value problem (9), which allows us to extent this discrete solution $u$ to the whole real line.

**Lemma 1.** *For given $\lambda \in \mathbb{R}$ and $C_0, C_1 \in \mathbb{R}$, the initial value problem (9) has a unique solution of the form*

$$u(k) = C_0 F^\lambda(1-k) + C_1 F^\lambda(k), \quad k \in \mathbb{Z}, \tag{10}$$

*where the function $F^\lambda : \mathbb{R} \to \mathbb{R}$ is defined as*

$$F^\lambda(t) := \begin{cases} \sinh(\omega_\lambda t)/\sinh\omega_\lambda & \textit{for } \lambda < 0, \\ t & \textit{for } \lambda = 0, \\ \sin(\omega_\lambda t)/\sin\omega_\lambda & \textit{for } \lambda \in (0,4), \\ -t\cos(\pi t) & \textit{for } \lambda = 4, \\ -\cos(\pi t)\sinh(\omega_\lambda t)/\sinh\omega_\lambda & \textit{for } \lambda > 4, \end{cases}$$

$$\omega_\lambda := \begin{cases} \operatorname{arcosh}\frac{2-\lambda}{2} & \textit{for } \lambda \le 0, \\ \arccos\frac{2-\lambda}{2} & \textit{for } \lambda \in (0,4), \\ \operatorname{arcosh}\frac{\lambda-2}{2} & \textit{for } \lambda \ge 4. \end{cases}$$

**Proof.** It is straightforward to verify that (10) holds for $\lambda = 0$ and for $\lambda = 4$ where we used the fact that $(-1)^k = \cos(k\pi)$ for $k \in \mathbb{Z}$. Now, let us assume $\lambda < 0$. If we take into account that $r_1 r_2 = 1$, we conclude that $r_1 r_2^k - r_2 r_1^k = -(r_1^{k-1} - r_2^{k-1})$. Since

$$\begin{aligned} \ln r_1 &= \ln\left(\cosh\omega_\lambda + \sqrt{\cosh^2\omega_\lambda - 1}\right) = \ln(\cosh\omega_\lambda + \sinh\omega_\lambda) \\ &= \ln\left(\frac{\mathrm{e}^{\omega_\lambda} + \mathrm{e}^{-\omega_\lambda}}{2} + \frac{\mathrm{e}^{\omega_\lambda} - \mathrm{e}^{-\omega_\lambda}}{2}\right) = \omega_\lambda, \end{aligned}$$

we obtain that

$$r_1^k - r_2^k = r_1^k - r_1^{-k} = \mathrm{e}^{k\ln r_1} - \mathrm{e}^{-k\ln r_1} = \mathrm{e}^{k\omega_\lambda} - \mathrm{e}^{-k\omega_\lambda} = 2\sinh(\omega_\lambda k).$$

Thus, we finally get

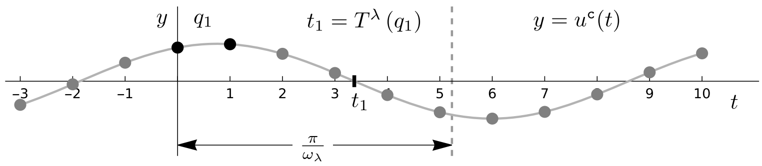


**Fig. 5.** The continuous extension $u^c$ of the solution $u$ of the discrete problem (9) and the first non-negative zero $t_1$ of $u^c$.

$$u(k) = C_0 \frac{r_1 r_2^k - r_2 r_1^k}{r_1 - r_2} + C_1 \frac{r_1^k - r_2^k}{r_1 - r_2} = C_0 F^\lambda(1-k) + C_1 F^\lambda(k).$$

Cases $\lambda \in (0,4)$ and $\lambda > 4$ are very similar to the case $\lambda < 0$. We have

$$\ln r_1 = \begin{cases} \mathrm{i}\,\omega_\lambda & \text{for } \lambda \in (0,4), \\ -\omega_\lambda + \pi\,\mathrm{i} & \text{for } \lambda > 4, \end{cases}$$

and

$$r_1^k - r_2^k = \begin{cases} 2\,\mathrm{i}\sin(\omega_\lambda k) & \text{for } \lambda \in (0,4), \\ -2\sinh(\omega_\lambda k)\cos(\pi k) & \text{for } \lambda > 4, \end{cases}$$

and thus (10) holds. □

See Fig. 4 for the graph of function $\lambda \mapsto \omega_\lambda$ and note its point of discontinuity of the first kind at $\lambda = 4$.

For the solution $u$ of the discrete problem (9), let us define its *continuous extension* on $\mathbb{R}$ (see Fig. 5) as

$$u^c(t) := C_0 F^\lambda(1-t) + C_1 F^\lambda(t), \quad t \in \mathbb{R}.$$

Moreover, for the non-trivial solution $u$ of the discrete problem (9) (let us remind that we have $C_0^2 + C_1^2 \neq 0$), we define the bi-infinite sequence $(q_k)_{k\in\mathbb{Z}}$ of ratios of values of $u$ in two consecutive integers in the following way

$$q_k := \frac{u(k)}{u(k-1)}, \quad k \in \mathbb{Z}. \tag{11}$$

Let us note that the sequence $(q_k)$ is a mapping from $\mathbb{Z}$ to $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$ (the one-point compactification of $\mathbb{R}$). We say that the solution $u$ of the discrete problem (9) has a *generalized zero* at $k \in \mathbb{Z}$ if

$$u(k) = 0 \quad \text{or} \quad u(k)u(k-1) < 0.$$

Let us point out that $u$ has a generalized zero at $k \in \mathbb{Z}$ if and only if $q_k \leq 0$ and $q_k \neq \infty$. For $\lambda \leq 0$, the solution $u$ of (9) has no generalized zero if $q_1 = \frac{C_1}{C_0} \in [\mathrm{e}^{-\omega_\lambda}, \mathrm{e}^{\omega_\lambda}]$ and has

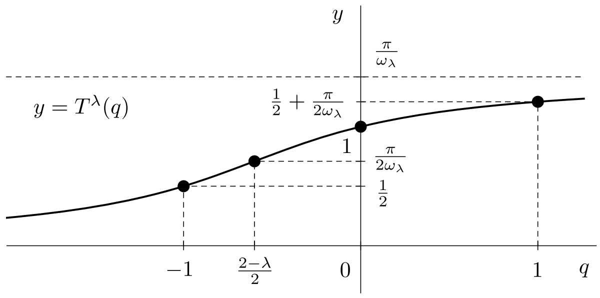


**Fig. 6.** The graph of $T^\lambda$ as a function of $q$.

exactly one generalized zero for $q_1 \notin [e^{-\omega_\lambda}, e^{\omega_\lambda}]$. For $\lambda > 0$, the solution $u$ of (9) has infinitely many generalized zeros. Let us consider $\lambda \in (0, 4)$. In this case, $0 < \omega_\lambda < \pi$ and the continuous extension $u^c$ of the solution $u$ of the initial value problem (9) is $\frac{2\pi}{\omega_\lambda}$-periodic function. Let us denote the first non-negative zero of $u^c$ by $t_1$ (see Fig. 5). Then all zeros of $u^c$ are $t_k = t_1 + (k-1)\frac{\pi}{\omega_\lambda}$, $k \in \mathbb{Z}$. If $C_0 = 0$ then $u^c(t) = C_1 \frac{\sin(\omega_\lambda t)}{\sin \omega_\lambda}$ and $q_1 = \infty$, $t_1 = 0$. If $C_0 \neq 0$ then for $t_1$, we have that

$$\sin(\omega_\lambda(1 - t_1)) + q_1 \sin(\omega_\lambda t_1) = 0, \qquad 0 < t_1 < \frac{\pi}{\omega_\lambda},$$

which gives us

$$q_1 = \cos \omega_\lambda - \sin \omega_\lambda \cdot \cot(\omega_\lambda t_1), \qquad t_1 = \frac{1}{\omega_\lambda} \operatorname{arccot} \frac{\cos \omega_\lambda - q_1}{\sin \omega_\lambda}.$$

Let us point out that function arccotangent has the usual principal values, thus it is defined for all real numbers and its range is interval $(0, \pi)$. In the following definition, we introduce the function $T^\lambda$ such that $t_1 = T^\lambda(q_1)$.

**Definition 2.** For $\lambda \in (0, 4)$, let us define the function $T^\lambda : \mathbb{R}^* \to \mathbb{R}$ as

$$\operatorname{Dom}\left(T^\lambda\right) := \mathbb{R}^*, \qquad T^\lambda(\infty) := 0, \qquad T^\lambda(q) := \frac{1}{\omega_\lambda} \operatorname{arccot} \frac{\cos \omega_\lambda - q}{\sin \omega_\lambda} \quad \text{for } q \in \mathbb{R}.$$

Let us note that $T^\lambda$ is a strictly increasing function on $\mathbb{R}$ and maps $\mathbb{R}^*$ onto $\left[0, \frac{\pi}{\omega_\lambda}\right)$ and it is straightforward to verify that (see Fig. 6)

$$T^\lambda(0) = 1, \qquad T^\lambda(-1) = \tfrac{1}{2}, \qquad T^\lambda(1) = \tfrac{1}{2} + \tfrac{\pi}{2\omega_\lambda}, \qquad T^\lambda\left(\tfrac{2-\lambda}{2}\right) = \tfrac{\pi}{2\omega_\lambda}.$$

Moreover, if we take into account that the difference equation in (9) is autonomous, we realize that

$$t_1 = j + T^\lambda(q_{1+j}), \qquad j = \lceil t_0 \rceil, \ldots, 0, \ldots, \lfloor t_1 \rfloor.$$

Finally, in the following lemma, let us introduce a useful formula for $T^\lambda$.

**Lemma 3.** *For $q = \infty$ and $q \le 0$, we have*

$$T^\lambda(q) + T^\lambda\left(\tfrac{1}{q}\right) = 1. \tag{12}$$

**Proof.** For $q = \infty$, (12) is trivially satisfied since $T^\lambda(\infty) = 0$ and $T^\lambda(0) = 1$. For $q \le 0$, let us prove (12) in the following way. Let us denote the inverse function of $T^\lambda$ by $Q^\lambda$:

$$Q^\lambda : \left[0, \tfrac{\pi}{\omega_\lambda}\right) \to \mathbb{R}^*, \qquad Q^\lambda(0) = \infty, \qquad Q^\lambda(t) = -\frac{\sin(\omega_\lambda(1-t))}{\sin(\omega_\lambda t)} \quad \text{for } 0 < t < \tfrac{\pi}{\omega_\lambda}.$$

Thus, we obtain that $Q^\lambda(1-t) = 1/Q^\lambda(t)$ for all $t \in [0,1]$ and using $t = T^\lambda(q)$ and $q = Q^\lambda(t)$ we obtain

$$1 - T^\lambda(q) = 1 - t = T^\lambda(Q^\lambda(1-t)) = T^\lambda\left(\frac{1}{Q^\lambda(t)}\right) = T^\lambda\left(\tfrac{1}{q}\right). \quad \square$$

Now, for all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, let us denote

$$V_k^\lambda := U_k\left(\tfrac{2-\lambda}{2}\right),$$

where $U_k = U_k(x)$ are the Chebyshev polynomials of the second kind of degree $k$. For all $\lambda \in \mathbb{R}$, polynomials $V_k^\lambda$ satisfy the three terms recurrence formula

$$V_{k-1}^\lambda - (2-\lambda)V_k^\lambda + V_{k+1}^\lambda = 0, \quad k \in \mathbb{Z}, \tag{13}$$

and moreover, we have that $V_{-k}^\lambda = -V_{k-2}^\lambda$ for all $k \in \mathbb{Z}$. Since $F^\lambda(k) = V_{k-1}^\lambda$ for all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, the solution $u$ of (9) can be written as

$$u(k) = -C_0 V_{k-2}^\lambda + C_1 V_{k-1}^\lambda. \tag{14}$$

In the following lemma, we introduce an identity for Chebyshev polynomials of the second kind (also known as the special form of Turán inequality).

**Lemma 4.** *For all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, we have the following identity*

$$\left(V_k^\lambda\right)^2 - V_{k+1}^\lambda V_{k-1}^\lambda = 1. \tag{15}$$

**Proof.** For $k = 0$, the equality (15) is trivially satisfied since we have that

$$V_{-1}^\lambda = 0, \qquad V_0^\lambda = 1, \qquad V_1^\lambda = 2 - \lambda.$$

For $k \in \mathbb{Z}$, using (13), we obtain that

$$
\begin{aligned}
\left(V_{k+1}^{\lambda}\right)^2 - V_{k+2}^{\lambda}V_k^{\lambda} &= \left(V_{k+1}^{\lambda}\right)^2 - \left((2-\lambda)V_{k+1}^{\lambda} - V_k^{\lambda}\right)V_k^{\lambda} \\
&= V_{k+1}^{\lambda}\left(V_{k+1}^{\lambda} - (2-\lambda)V_k^{\lambda}\right) + \left(V_k^{\lambda}\right)^2 \\
&= -V_{k+1}^{\lambda}V_{k-1}^{\lambda} + \left(V_k^{\lambda}\right)^2.
\end{aligned}
$$

But it implies that the equality

$$\left(V_{k+1}^{\lambda}\right)^2 - V_{k+2}^{\lambda}V_k^{\lambda} = \left(V_k^{\lambda}\right)^2 - V_{k+1}^{\lambda}V_{k-1}^{\lambda}$$

holds for all $k \in \mathbb{Z}$, which leads to (15) using induction. □

Let us introduce the function $W_k^{\lambda}$ which determines the value of $k$-th element $q_k$ defined by (11) by the value of $q_0$.

**Definition 5.** For all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, let us define the function $W_k^{\lambda} : \mathbb{R}^* \to \mathbb{R}^*$ in the following way

$$W_k^{\lambda}(q) := \begin{cases} \dfrac{q \cdot V_k^{\lambda} - V_{k-1}^{\lambda}}{q \cdot V_{k-1}^{\lambda} - V_{k-2}^{\lambda}} & \text{for } q \in \mathbb{R}, \\ \dfrac{V_k^{\lambda}}{V_{k-1}^{\lambda}} & \text{for } q = \infty. \end{cases}$$

Let us recall that a Möbius transformation is given by $(a, b, c, d \in \mathbb{C},\ ad - bc \neq 0)$

$$f : \mathbb{C}^* \to \mathbb{C}^* : z \mapsto \frac{a \cdot z + b}{c \cdot z + d}.$$

Thus, using (15), we conclude that $W_k^{\lambda}$ is the restriction of a Möbius transformation on $\mathbb{R}^*$.

**Lemma 6.** *For all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, we have*

$$q_k = W_k^{\lambda}(q_0). \tag{16}$$

**Proof.** Firstly, we claim that

$$W_{k+1}^{\lambda}(q) = 2 - \lambda - \frac{1}{W_k^{\lambda}(q)}, \quad q \in \mathbb{R}^*,\ k \in \mathbb{Z},\ \lambda \in \mathbb{R}. \tag{17}$$

Indeed, using (13), we obtain the following relation

$$
\begin{aligned}
\frac{1}{W_k^{\lambda}(q)} &= \frac{q \cdot V_{k-1}^{\lambda} - V_{k-2}^{\lambda}}{q \cdot V_k^{\lambda} - V_{k-1}^{\lambda}} = \frac{q\left((2-\lambda)\cdot V_k^{\lambda} - V_{k+1}^{\lambda}\right) - \left((2-\lambda)\cdot V_{k-1}^{\lambda} - V_k^{\lambda}\right)}{q \cdot V_k^{\lambda} - V_{k-1}^{\lambda}} \\
&= 2 - \lambda - W_{k+1}^{\lambda}(q)
\end{aligned}
$$

for all $q \in \mathbb{R}$ such that $W_k^\lambda(q) \neq 0$. On the other hand, for all $q \in \mathbb{R}$ such that $W_k^\lambda(q) = 0$, we get $q \cdot V_k^\lambda - V_{k-1}^\lambda = 0$, which implies that $W_{k+1}^\lambda(q) = \infty$. In the case of $q = \infty$, we have

$$\frac{1}{W_k^\lambda(\infty)} = \frac{V_{k-1}^\lambda}{V_k^\lambda} = \frac{(2-\lambda) \cdot V_k^\lambda - V_{k+1}^\lambda}{V_k^\lambda} = 2 - \lambda - W_{k+1}^\lambda(\infty),$$

provided $W_k^\lambda(\infty) \neq 0$. If $W_k^\lambda(\infty) = 0$ then we have that $V_k^\lambda = 0$, which implies that $W_{k+1}^\lambda(\infty) = \infty$.

Now, for $k = 0$, the equality in (16) holds since $W_0^\lambda$ is the identity on $\mathbb{R}^*$. For $k \in \mathbb{Z}$, the difference equation in (9) can be written in the form

$$\frac{u(k+1)}{u(k)} = 2 - \lambda - \frac{u(k-1)}{u(k)},$$

provided $u(k) \neq 0$, which means $q_k \neq 0$, $q_{k+1} \neq \infty$ and that

$$q_{k+1} = 2 - \lambda - \frac{1}{q_k}. \tag{18}$$

If $u(k) = 0$ then $q_k = 0$ and $q_{k+1} = \infty$. Thus, the equality (18) holds for all $k \in \mathbb{Z}$ and for any sequence $(q_k)$ defined by (11). Finally, if we assume that (16) holds for fixed $k = j \in \mathbb{Z}$, using (17) and (18), we obtain

$$q_{j+1} = 2 - \lambda - \frac{1}{q_j} = 2 - \lambda - \frac{1}{W_j^\lambda(q_0)} = W_{j+1}^\lambda(q_0)$$

and

$$q_{j-1} = \frac{1}{2 - \lambda - q_j} = \frac{1}{2 - \lambda - W_j^\lambda(q_0)} = W_{j-1}^\lambda(q_0). \quad \square$$

Every Möbius transformation can be associated with its Möbius matrix. In our case it is useful to simplify proofs of certain identities (see Lemma 9).

**Definition 7.** For all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, we associate a corresponding $2 \times 2$ *Möbius matrix* $\left[W_k^\lambda\right]$ with the Möbius transformation $W_k^\lambda(q)$ (see [7], page 156):

$$\left[W_k^\lambda\right] := \begin{bmatrix} V_k^\lambda & -V_{k-1}^\lambda \\ V_{k-1}^\lambda & -V_{k-2}^\lambda \end{bmatrix}.$$

Moreover, let us define the *homogeneous coordinates* of $q \in \mathbb{R}$ and of $\infty$ as $2 \times 1$ matrices

$$[q] := \begin{bmatrix} q \\ 1 \end{bmatrix}, \qquad [\infty] := \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus, for all $q \in \mathbb{R}$, we have

$$\left[W_k^\lambda\right] \cdot [q] = \begin{bmatrix} qV_k^\lambda - V_{k-1}^\lambda \\ qV_{k-1}^\lambda - V_{k-2}^\lambda \end{bmatrix} \quad \text{and} \quad \frac{qV_k^\lambda - V_{k-1}^\lambda}{qV_{k-1}^\lambda - V_{k-2}^\lambda} = W_k^\lambda(q),$$

$$\left[W_k^\lambda\right] \cdot [\infty] = \begin{bmatrix} V_k^\lambda \\ V_{k-1}^\lambda \end{bmatrix} \quad \text{and} \quad \frac{V_k^\lambda}{V_{k-1}^\lambda} = W_k^\lambda(\infty).$$

**Remark 8.**

1. Using (15), we conclude that $\det\left[W_k^\lambda\right] = 1$. Moreover, for the inverse matrix, we obtain

$$\left[W_k^\lambda\right]^{-1} = \begin{bmatrix} -V_{k-2}^\lambda & V_{k-1}^\lambda \\ -V_{k-1}^\lambda & V_k^\lambda \end{bmatrix} = \begin{bmatrix} V_{-k}^\lambda & -V_{-k-1}^\lambda \\ V_{-k-1}^\lambda & -V_{-k-2}^\lambda \end{bmatrix} = \left[W_{-k}^\lambda\right]. \tag{19}$$

2. The composition of Möbius transformations corresponds to the multiplication of Möbius matrices (see [7], page 157)

$$\left[W_{k_2}^{\lambda_2} \circ W_{k_1}^{\lambda_1}\right] = \left[W_{k_2}^{\lambda_2}\right] \cdot \left[W_{k_1}^{\lambda_1}\right]. \tag{20}$$

In the following lemma, let us introduce some useful properties of $W_k^\lambda$.

**Lemma 9.** *For all $k, l \in \mathbb{Z}$ and $q \in \mathbb{R}^*$, we have that*

$$W_l^\lambda(W_k^\lambda(q)) = W_{k+l}^\lambda(q), \tag{21}$$

$$W_{-k}^\lambda(W_k^\lambda(q)) = q, \tag{22}$$

$$W_{-k}^\lambda(q) = \frac{1}{W_k^\lambda\left(\frac{1}{q}\right)}. \tag{23}$$

**Proof.** Firstly, we prove that $W_k^\lambda \circ W_l^\lambda = W_{k+l}^\lambda$. Using (20), we obtain

$$\begin{aligned}
\left[W_l^\lambda \circ W_k^\lambda\right] &= \left[W_l^\lambda\right] \cdot \left[W_k^\lambda\right] \\
&= \begin{bmatrix} V_l^\lambda & -V_{l-1}^\lambda \\ V_{l-1}^\lambda & -V_{l-2}^\lambda \end{bmatrix} \cdot \begin{bmatrix} V_k^\lambda & -V_{k-1}^\lambda \\ V_{k-1}^\lambda & -V_{k-2}^\lambda \end{bmatrix} \\
&= \begin{bmatrix} V_k^\lambda V_l^\lambda - V_{k-1}^\lambda V_{l-1}^\lambda & -(V_k^\lambda V_{l-1}^\lambda - V_{k-1}^\lambda V_{l-2}^\lambda) \\ V_{k-1}^\lambda V_l^\lambda - V_{k-2}^\lambda V_{l-1}^\lambda & -(V_{k-1}^\lambda V_{l-1}^\lambda - V_{k-2}^\lambda V_{l-2}^\lambda) \end{bmatrix} \\
&= \begin{bmatrix} V_{k+l}^\lambda & -V_{k+l-1}^\lambda \\ V_{k+l-1}^\lambda & -V_{k+l-2}^\lambda \end{bmatrix} \\
&= \left[W_{k+l}^\lambda\right],
\end{aligned}$$

where we used the fact that $V_k^\lambda V_l^\lambda - V_{k-1}^\lambda V_{l-1}^\lambda = V_{k+l}^\lambda$, which follows from

$$V_k^\lambda V_l^\lambda = \sum_{j=0}^{l} V_{k-l+2j}^\lambda = V_{k-l}^\lambda + V_{k-l+2}^\lambda + V_{k-l+4}^\lambda + \cdots + V_{k+l}^\lambda.$$

Secondly, the equality (22) follows directly from (21) for $l = -k$ (recall that $W_0^\lambda(q) = q$). Finally, using (19), we get

$$\left[W_{-k}^\lambda\right] \cdot [q] = \begin{bmatrix} -V_{k-2}^\lambda & V_{k-1}^\lambda \\ -V_{k-1}^\lambda & V_k^\lambda \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} q_2 V_{k-1}^\lambda - q_1 V_{k-2}^\lambda \\ q_2 V_k^\lambda - q_1 V_{k-1}^\lambda \end{bmatrix},$$

$$\left[W_k^\lambda\right] \cdot \left[\tfrac{1}{q}\right] = \begin{bmatrix} V_k^\lambda & -V_{k-1}^\lambda \\ V_{k-1}^\lambda & -V_{k-2}^\lambda \end{bmatrix} \cdot \begin{bmatrix} q_2 \\ q_1 \end{bmatrix} = \begin{bmatrix} q_2 V_k^\lambda - q_1 V_{k-1}^\lambda \\ q_2 V_{k-1}^\lambda - q_1 V_{k-2}^\lambda \end{bmatrix},$$

which justifies (23). $\square$

**Remark 10.** Let us assume that we have some element of bi-infinite sequence $(q_k)$ (for example $q_1 = \frac{C_1}{C_0}$ is given by the initial conditions). If we want to get any other element of such sequence or the first non-negative zero $t_1$ of $u^{\mathrm{c}}$, we can use the following formulas.

1. For $\lambda \in \mathbb{R}$ and $i, j, k \in \mathbb{Z}$ such that $i + j = k$, we have that

$$q_k = W_j^\lambda(q_i) \qquad \text{and} \qquad [q_k] = \left[W_j^\lambda\right] \cdot [q_i].$$

2. For $\lambda \in (0, 4)$, we have for the first non-negative zero $t_1$ of $u^{\mathrm{c}}$ that

$$t_1 = j + T^\lambda(W_j^\lambda(q_1)), \qquad j = \lceil t_0 \rceil, \dots, 0, \dots, \lfloor t_1 \rfloor.$$

The following lemma provides us with the necessary and sufficient condition for $W_k^\lambda$ to be a linear function.

**Lemma 11.** *For $k \in \mathbb{N}$, $k \geq 2$, we have*

$$W_k^\lambda(q) = q \qquad \textit{if and only if} \qquad \exists\, j \in \{1, \dots, k-1\} : \ \frac{\pi}{\omega_\lambda} = \frac{k}{j}.$$

**Proof.** Let us recall that all zeros of the Chebyshev polynomial of the second kind $U_{k-1} = U_{k-1}(x)$ of degree $(k-1)$, $k \in \mathbb{N}$, $k \geq 2$, are given by

$$x_j = \cos\frac{j\pi}{k}, \qquad j = 1, \dots, k-1.$$

Now, we have that $W_k^\lambda(q) = q$ if and only if $V_{k-1}^\lambda = U_{k-1}\left(\frac{2-\lambda}{2}\right) = 0$, which is true if and only if $\lambda = 2 - 2x_j$ for some $j \in \{1, \dots, k-1\}$. Moreover, for $\lambda = 2 - 2x_j$, we have that $0 < \lambda = 4\sin^2\frac{j\pi}{2k} < 4$ and $\omega_\lambda = \arccos\frac{2-\lambda}{2} = \frac{j\pi}{k}$, which leads to $\frac{\pi}{\omega_\lambda} = \frac{k}{j}$. $\square$

In the last part of this section, let us consider the following linear problem

$$\begin{cases} \Delta^2 u(k-1) + a_k \cdot u(k) = 0, \quad k \in \mathbb{N}, \\ u(0) = 0, \quad u(1) = C_1, \end{cases} \tag{24}$$

where $C_1 \in \mathbb{R}$, $C_1 \neq 0$ and the sequence $(a_k)_{k\in\mathbb{N}}$ is given by

$$(a_k)_{k\in\mathbb{N}} = (\underbrace{\lambda_1, \dots, \lambda_1}_{k_1-\text{times}}, \underbrace{\lambda_2, \dots, \lambda_2}_{k_2-\text{times}}, \dots, \underbrace{\lambda_m, \dots, \lambda_m}_{k_m-\text{times}}, \dots),$$

where $(k_j)_{j\in\mathbb{N}}$ is a sequence of natural numbers, $\lambda_j \in \mathbb{R}$, $j \in \mathbb{N}$. For a solution $u$ of (24), let us define the sequence $(q_k)_{k\in\mathbb{N}}$ by (11). We have that

$$\begin{aligned}
q_1 &= \tfrac{C_1}{0} = \infty, && \\
q_{k+1} &= W_k^{\lambda_1}(q_1) && \text{for } 1 \le k \le k_1, \\
q_{k+k_1+1} &= W_k^{\lambda_2}(q_{k_1+1}) && \text{for } 1 \le k \le k_2, \\
&\vdots && \\
q_{k+k_{m-1}+\dots+k_1+1} &= W_k^{\lambda_m}(q_{k_{m-1}+\dots+k_1+1}) && \text{for } 1 \le k \le k_m, \quad m \ge 3, \\
&\vdots &&
\end{aligned}$$

Thus, for all $m \in \mathbb{N}$, we obtain $q_{k_m+\dots+k_2+k_1+1} = \left(W_{k_m}^{\lambda_m} \circ \dots \circ W_{k_2}^{\lambda_2} \circ W_{k_1}^{\lambda_1}\right)(q_1)$, which implies that

$$\left[q_{k_m+\dots+k_2+k_1+1}\right] = \left[W_{k_m}^{\lambda_m}\right] \cdots \left[W_{k_2}^{\lambda_2}\right] \cdot \left[W_{k_1}^{\lambda_1}\right] \cdot \left[q_1\right]. \tag{25}$$

Let us note that if $\lambda_j = \lambda \in \mathbb{R}$ for all $j \in \mathbb{N}$ then (25) simplifies to

$$\left[q_{k_m+\dots+k_2+k_1+1}\right] = \left[W_{k_m+\dots+k_2+k_1}^{\lambda}\right] \cdot \left[q_1\right].$$

**Lemma 12.** *Let $\mathbf{A}^{\mathtt{D}}$ be $n\times n$ Dirichlet matrix, $n \in \mathbb{N}$. Let $m \in \mathbb{N}$ and let $k_1, k_2, \dots, k_m \in \mathbb{N}$, be such that $\sum_{j=1}^{m} k_j = n$. Moreover, let $\mathbf{\Lambda}$ be $n \times n$ diagonal matrix*

$$\mathbf{\Lambda} = \operatorname{diag}(\underbrace{\lambda_1, \dots, \lambda_1}_{k_1-times}, \underbrace{\lambda_2, \dots, \lambda_2}_{k_2-times}, \dots, \underbrace{\lambda_m, \dots, \lambda_m}_{k_m-times}), \qquad \lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}.$$

*Then*

$$\det\left(\mathbf{A}^{\mathtt{D}} - \mathbf{\Lambda}\right) = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \left[W_{k_m}^{\lambda_m}\right] \cdots \left[W_{k_2}^{\lambda_2}\right] \cdot \left[W_{k_1}^{\lambda_1}\right] \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{26}$$

**Proof.** According to (21), it is enough to show by induction that if

$$\mathbf{\Lambda} = \operatorname{diag}(a_1, a_2, \ldots, a_{n-1}, a_n), \qquad a_i \in \mathbb{R}, \quad i = 1, \ldots, n,$$

then

$$\det \mathbf{M}_n = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} W_1^{a_n} \end{bmatrix} \cdot \begin{bmatrix} W_1^{a_{n-1}} \end{bmatrix} \cdots \begin{bmatrix} W_1^{a_2} \end{bmatrix} \cdot \begin{bmatrix} W_1^{a_1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{27}$$

where $\mathbf{M}_n$ is $n \times n$ matrix $\mathbf{M}_n := \mathbf{A}^{\mathsf{D}} - \mathbf{\Lambda}$. It is straightforward to verify that the equality (27) holds for $n = 1$ and $n = 2$. Indeed, we have that

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} W_1^{a_1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 - a_1 = \det \mathbf{M}_1,$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} W_1^{a_2} \end{bmatrix} \cdot \begin{bmatrix} W_1^{a_1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (2 - a_1)(2 - a_2) - 1 = \det \mathbf{M}_2.$$

Now, let us assume that (27) holds for $n$ and $n - 1$, $n \in \mathbb{N}$, $n \geq 2$. Let us denote

$$\mathbf{B} = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} := \begin{bmatrix} W_1^{a_{n-1}} \end{bmatrix} \cdots \begin{bmatrix} W_1^{a_2} \end{bmatrix} \cdot \begin{bmatrix} W_1^{a_1} \end{bmatrix}$$

and expand the determinant $\det \mathbf{M}_{n+1}$ along the last row of $\mathbf{M}_{n+1}$

$$\begin{aligned} \det \mathbf{M}_{n+1} &= (2 - a_{n+1}) \det \mathbf{M}_n - \det \mathbf{M}_{n-1} \\ &= (2 - a_{n+1}) \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} W_1^{a_n} \end{bmatrix} \cdot \mathbf{B} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \mathbf{B} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= (2 - a_{n+1}) \left( (2 - a_n) b_{1,1} - b_{2,1} \right) - b_{1,1}. \end{aligned} \tag{28}$$

Since

$$\begin{bmatrix} W_1^{a_{n+1}} \end{bmatrix} \cdot \begin{bmatrix} W_1^{a_n} \end{bmatrix} = \begin{bmatrix} (2 - a_{n+1})(2 - a_n) - 1 & -(2 - a_{n+1}) \\ 2 - a_n & -1 \end{bmatrix},$$

we obtain using (28) that

$$\begin{aligned} \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} W_1^{a_{n+1}} \end{bmatrix} \cdot \begin{bmatrix} W_1^{a_n} \end{bmatrix} \cdot \mathbf{B} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= ((2 - a_{n+1})(2 - a_n) - 1) b_{1,1} - (2 - a_{n+1}) b_{2,1} \\ &= \det \mathbf{M}_{n+1}, \end{aligned}$$

which implies that (27) holds for $n + 1$. $\square$

**Remark 13.** Let $u$ be the solution of the initial value problem (24). Then using (25) and (26) we deduce that

$$u(n+1) = u(k_m + \cdots + k_1 + 1) = \det\left(\mathbf{A}^{\mathsf{D}} - \mathbf{\Lambda}\right).$$

Moreover, if $\lambda_j = \lambda \in \mathbb{R}$ for $j = 1, \ldots, m$ then (cf. (14))

$$u(n+1) = V_n^\lambda = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} W_n^\lambda \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \det\left(\mathbf{A}^{\mathsf{D}} - \lambda \mathbf{I}\right).$$

## 3. The localization of generalized zeros of a solution of the semi-linear problem

In this section, we deal with the semi-linear initial value problem

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, & k \in \mathbb{Z}, \\ u(0) = 0, \quad u(1) = C_1, \end{cases} \tag{29}$$

where $u^\pm(k) = \max\{\pm u(k), 0\}$, $C_1 \in \mathbb{R}$, $C_1 \neq 0$ and $(\alpha, \beta) \in D$,

$$D := ((0,4) \times (0, +\infty)) \cup ((0, +\infty) \times (0,4)).$$

Let $u$ be a solution of (29). Then $u$ is also the solution of (24) if we take $(a_k)_{k\in\mathbb{N}}$ in the following form

$$a_k = \begin{cases} \alpha & \text{for } u(k) \geq 0, \\ \beta & \text{for } u(k) < 0. \end{cases}$$

Let $i \in \mathbb{Z}$ be a generalized zero of a solution $u$ of the initial value problem (29). Moreover, let $j \in \mathbb{Z} : j > i$, such that for all $k = i, \ldots, j$, $u(k)$ is non-negative (or non-positive) and (see Fig. 7)

$$u(j)u(j+1) < 0 \quad \text{or} \quad u(j) = 0.$$

This means that $i$ and $(j+1)$ are two consecutive generalized zeros of $u$ if $u(j) \neq 0$. In the case of $u(j) = 0$, $i$ and $j$ are two consecutive generalized zeros of $u$. Since for all

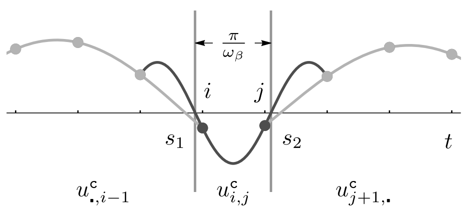


**Fig. 7.** Anchoring of two positive semi-waves by one negative semi-wave $u^{\mathsf{c}}_{i,j}$ of length 3 for $0 < \alpha, \beta < 4$.

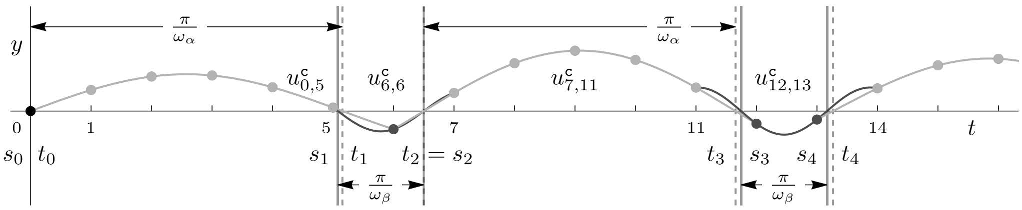


**Fig. 8.** Positive and negative semi-waves of a solution of the initial value problem (29) for $0 < \alpha, \beta < 4$ and $C_1 > 0$.

$k = i, \dots, j$, $u(k)$ is non-negative (or non-positive) then $u$ solves the difference equation in (9) with $\lambda = \alpha$ (or $\lambda = \beta$) if $u(i-1)$ is negative (or $u(i-1)$ is positive). Also, let us note that $u(i-1)u(j+1)$ is strictly positive. Now, we define the *continuous extension* $u^{\mathsf{c}}_{i,j}$ of $u$ on the interval $[i-1, j+1]$ as

$$u^{\mathsf{c}}_{i,j}(t) := \begin{cases} u(i-1)F^{\alpha}(1-(t-i+1)) + u(i)F^{\alpha}(t-i+1) & \text{for } u(i-1) < 0, \\ u(i-1)F^{\beta}(1-(t-i+1)) + u(i)F^{\beta}(t-i+1) & \text{for } u(i-1) > 0, \end{cases}$$

where functions $F^\alpha$ and $F^\beta$ are given by $F^\lambda$ (defined in Lemma 1) for $\lambda = \alpha$ and $\lambda = \beta$, respectively. By the *length of the continuous extension* $u^{\mathsf{c}}_{i,j}$ of $u$ we mean the length of the interval $[i-1, j+1]$, which reads $j - i + 2$.

Finally, we say that the continuous extension $u^{\mathsf{c}}_{i,j}$ of $u$ is a *positive (negative) semi-wave* of $u$ if $u(k)$ is non-negative (or non-positive) for all $k = i, \dots, j$ (see Fig. 7). See also Fig. 8 for two positive semi-waves $u^{\mathsf{c}}_{0,5}$ and $u^{\mathsf{c}}_{7,11}$ on intervals $[-1, 6]$ and $[6, 12]$, and two negative semi-waves $u^{\mathsf{c}}_{6,6}$ and $u^{\mathsf{c}}_{12,13}$ on intervals $[5, 7]$ and $[11, 14]$.

Now, let us restrict ourselves to the case $0 < \alpha, \beta < 4$.

**Lemma 14.** *Let $0 < \alpha, \beta < 4$ and let $u$ be a solution of the initial value problem (29). Moreover, let $u^{\mathsf{c}}_{i,j}$ be a negative semi-wave of $u$ defined on $[i-1, j+1]$. Then*

$$j = i + \left\lfloor T^{\beta}(q_i) + \frac{\pi}{\omega_\beta} \right\rfloor - 1,$$

*where $q_i = \frac{u(i)}{u(i-1)}$. The length of the negative semi-wave $u^{\mathsf{c}}_{i,j}$ is given by $\left\lfloor T^{\beta}(q_i) + \frac{\pi}{\omega_\beta} \right\rfloor + 1$.*

**Proof.** The continuous extension $u^{\mathsf{c}}_{i,j}$ has exactly two zeros $s_1$ and $s_2$ (see Fig. 7):

$$s_1 = i - 1 + T^{\beta}(q_i), \qquad s_2 = s_1 + \frac{\pi}{\omega_\beta},$$

where $q_i = \frac{u(i)}{u(i-1)} \le 0$. Since $j = \lfloor s_2 \rfloor$, we get

$$j = \left\lfloor s_1 + \tfrac{\pi}{\omega_\beta} \right\rfloor = \left\lfloor i - 1 + T^{\beta}(q_i) + \tfrac{\pi}{\omega_\beta} \right\rfloor = i + \left\lfloor T^{\beta}(q_i) + \tfrac{\pi}{\omega_\beta} \right\rfloor - 1. \quad \square$$

Now, let $0 < \alpha, \beta < 4$ and $C_1 > 0$ and let $u$ be a solution $u$ of initial value problem (29) (see Fig. 8). We show how to describe all positive generalized zeros of $u$. For this purpose, let us define recurrently two sequences of functions $(p_j)_{j\in\mathbb{N}}$ and $(\vartheta_j)_{j\in\mathbb{N}}$ defined on $(0,4)\times(0,4)$ in the following way

$$\begin{aligned}
p_1(\alpha,\beta) &:= \left\lfloor \tfrac{\pi}{\omega_\alpha} \right\rfloor, & \vartheta_1(\alpha,\beta) &:= W^{\alpha}_{p_1(\alpha,\beta)}(\infty),\\
p_2(\alpha,\beta) &:= \left\lfloor T^{\beta}(\vartheta_1(\alpha,\beta)) + \tfrac{\pi}{\omega_\beta} \right\rfloor, & \vartheta_2(\alpha,\beta) &:= W^{\beta}_{p_2(\alpha,\beta)}(\vartheta_1(\alpha,\beta)),\\
p_3(\alpha,\beta) &:= \left\lfloor T^{\alpha}(\vartheta_2(\alpha,\beta)) + \tfrac{\pi}{\omega_\alpha} \right\rfloor, & \vartheta_3(\alpha,\beta) &:= W^{\alpha}_{p_3(\alpha,\beta)}(\vartheta_2(\alpha,\beta)),\\
p_4(\alpha,\beta) &:= \left\lfloor T^{\beta}(\vartheta_3(\alpha,\beta)) + \tfrac{\pi}{\omega_\beta} \right\rfloor, & \vartheta_4(\alpha,\beta) &:= W^{\beta}_{p_4(\alpha,\beta)}(\vartheta_3(\alpha,\beta)),\\
&\vdots & &\vdots
\end{aligned} \tag{30}$$

Since $C_1 > 0$, we have the positive semi-wave $u^{\mathsf{c}}_{0,p_1}$ of $u$, which is defined on $[-1, p_1+1]$ and has two zeros $t_0 = 0$ and $t_1 = \frac{\pi}{\omega_\alpha}$. Moreover, we have $p_1 = \lfloor t_1 \rfloor \le t_1 < p_1 + 1$ and $\vartheta_1 = q_{p_1+1} < 0$ if $u(p_1) > 0$ or $\vartheta_1 = q_{p_1+1} = \infty$ if $u(p_1) = 0$. Thus, we obtain

$$t_1 = \lfloor t_1 \rfloor + T^{\alpha}\left(q_{\lfloor t_1 \rfloor+1}\right) = p_1 + T^{\alpha}(q_{p_1+1}) = p_1 + T^{\alpha}(\vartheta_1)$$

and the first positive generalized zero of $u$ is $z_1 = p_1 + 1$ if $\vartheta_1 < 0$ or $z_1 = p_1 = t_1$ if $\vartheta_1 = \infty$. The next semi-wave of $u$ is the negative semi-wave $u^{\mathsf{c}}_{\lceil s_1 \rceil, \lfloor s_2 \rfloor}$, which has two zeros $s_1$ and $s_2$ and is defined on $[\lceil s_1 \rceil - 1, \lfloor s_2 \rfloor + 1]$. Moreover, we have

$$s_1 = \lfloor t_1 \rfloor + T^{\beta}\left(q_{\lfloor t_1 \rfloor+1}\right) = p_1 + T^{\beta}(\vartheta_1) \quad \text{and} \quad s_2 = s_1 + \tfrac{\pi}{\omega_\beta}$$

and thus, we obtain $\lfloor s_2 \rfloor = \left\lfloor s_1 + \frac{\pi}{\omega_\beta} \right\rfloor = \left\lfloor p_1 + T^{\beta}(\vartheta_1) + \frac{\pi}{\omega_\beta} \right\rfloor = p_1 + p_2$. This implies that

$$q_{p_2+p_1+1} = W^{\beta}_{p_2}(p_1+1) = W^{\beta}_{p_2}(\vartheta_1) = W^{\beta}_{p_2}(W^{\alpha}_{p_1}(\infty)) = \vartheta_2$$

and that

$$s_2 = \lfloor s_2 \rfloor + T^{\beta}\left(q_{\lfloor s_2 \rfloor+1}\right) = p_1 + p_2 + T^{\beta}\left(q_{p_2+p_1+1}\right) = p_1 + p_2 + T^{\beta}(\vartheta_2).$$

The second positive generalized zero of $u$ is $z_2 = p_1 + p_2 + 1$ if $\vartheta_2 < 0$ or $z_2 = p_1 + p_2 = s_2$ if $\vartheta_2 = \infty$. The next semi-wave of $u$ is the positive semi-wave $u^{\mathsf{c}}_{\lceil t_2 \rceil, \lfloor t_3 \rfloor}$, which has two zeros $t_2$ and $t_3$ and is defined on $[\lceil t_2 \rceil - 1, \lfloor t_3 \rfloor + 1]$. We have that $t_3 - t_2 = \frac{\pi}{\omega_\alpha}$ and

$$\begin{aligned}
t_2 &= \lfloor s_2 \rfloor + T^{\alpha}\left(q_{\lfloor s_2 \rfloor+1}\right) = p_1 + p_2 + T^{\alpha}(\vartheta_2),\\
\vartheta_3 &= q_{p_3+p_2+p_1+1} = W^{\alpha}_{p_3}(W^{\beta}_{p_2}(W^{\alpha}_{p_1}(\infty))),\\
t_3 &= \lfloor t_3 \rfloor + T^{\alpha}\left(q_{\lfloor t_3 \rfloor+1}\right) = p_1 + p_2 + p_3 + T^{\alpha}(\vartheta_3).
\end{aligned}$$

The third positive generalized zero of $u$ is $z_3 = p_1 + p_2 + p_3 + 1$ if $\vartheta_3 < 0$ or $z_3 = p_1 + p_2 + p_3 = t_3$ if $\vartheta_3 = \infty$.

**Lemma 15.** *Let $0 < \alpha, \beta < 4$ and let $u$ be a solution of (29) with $C_1 > 0$. For $k \geq 0$, $u = u(k)$ consists of infinitely many positive and negative semi-waves. The $m$-th semi-wave, $m \in \mathbb{N}$, is positive (negative) one with zero points $t_{m-1}$ and $t_m$ ($s_{m-1}$ and $s_m$) if $m$ is odd (even), where*

$$t_0 = 0, \qquad t_m = \sum_{j=1}^{m} p_j(\alpha,\beta) + T^\alpha(\vartheta_m(\alpha,\beta)), \qquad s_m = \sum_{j=1}^{m} p_j(\alpha,\beta) + T^\beta(\vartheta_m(\alpha,\beta)).$$

*Moreover, all positive generalized zeros of $u$ form a sequence $(z_m)_{m\in\mathbb{N}}$ such that*

$$z_m = \sum_{j=1}^{m} p_j(\alpha,\beta) + 1 \quad \text{if } \vartheta_m(\alpha,\beta) \neq \infty, \qquad z_m = \sum_{j=1}^{m} p_j(\alpha,\beta) \quad \text{if } \vartheta_m(\alpha,\beta) = \infty.$$

**Proof.** We proceed via an induction. Thus, let $u^{\mathsf{c}}_{\lceil t_{m-1}\rceil, \lfloor t_m \rfloor}$ be a positive semi-wave of $u$, $m \in \mathbb{N}$ is odd, which has two zero points $t_{m-1}$ and $t_m$ such that

$$t_{m-1} = \sum_{j=1}^{m-1} p_j + T^\alpha(\vartheta_{m-1}), \qquad t_m = \sum_{j=1}^{m} p_j + T^\alpha(\vartheta_m).$$

The next semi-wave of $u$ is the negative semi-wave $u^{\mathsf{c}}_{\lceil s_m\rceil, \lfloor s_{m+1}\rfloor}$, which has two zeros $s_m$ and $s_{m+1}$ and is defined on $[\lceil s_m\rceil - 1, \lfloor s_{m+1}\rfloor + 1]$. Moreover, we have

$$s_m = \lfloor t_m \rfloor + T^\beta\left(q_{\lfloor t_m\rfloor+1}\right) = \sum_{j=1}^{m} p_j + T^\beta\left(\vartheta_m\right) \quad \text{and} \quad s_{m+1} = s_m + \tfrac{\pi}{\omega_\beta}$$

and thus, we obtain $\lfloor s_{m+1}\rfloor = \left\lfloor s_m + \frac{\pi}{\omega_\beta}\right\rfloor = \left\lfloor p_1 + \cdots + p_m + T^\beta\left(\vartheta_m\right) + \frac{\pi}{\omega_\beta}\right\rfloor = p_1 + \cdots + p_m + p_{m+1}$. This implies that

$$q_{p_{m+1}+p_m+\cdots+p_1+1} = W^\beta_{p_{m+1}}(p_m + \cdots + p_1 + 1) = W^\beta_{p_{m+1}}(\vartheta_m) = \vartheta_{m+1}$$

and that

$$s_{m+1} = \lfloor s_{m+1}\rfloor + T^\beta\left(q_{\lfloor s_{m+1}\rfloor+1}\right) = \sum_{j=1}^{m+1} p_j + T^\beta\left(q_{p_{m+1}+\cdots+p_1+1}\right) = \sum_{j=1}^{m+1} p_j + T^\beta\left(\vartheta_{m+1}\right).$$

The next positive generalized zero of $u$ is $z_{m+1} = p_1 + \cdots + p_{m+1} + 1$ if $\vartheta_{m+1} < 0$ or $z_{m+1} = p_1 + \cdots + p_{m+1} = s_{m+1}$ if $\vartheta_{m+1} = \infty$.

For positive generalized zeros of $u$, we obtain

$$z_m = \lceil t_m \rceil = \lceil s_m \rceil = \sum_{j=1}^{m} p_j + 1 \qquad \text{if } \vartheta_m \neq \infty,$$

$$z_m = t_m = s_m = \sum_{j=1}^{m} p_j \qquad \text{if } \vartheta_m = \infty. \quad \square$$

Now, let us turn our attention to the case of $0 < \alpha < 4$ and $\beta \geq 4$.

**Lemma 16.** *Let $0 < \alpha < 4 \leq \beta$ and let $u$ be a solution of the initial value problem (29). Moreover, let $u^{\mathrm{c}}_{i,j}$ be a negative semi-wave of $u$ defined on $[i-1, j+1]$. Then*

$$j = i + \lfloor T^{\alpha}(q_i) + T^{\alpha}(2-\beta) \rfloor, \tag{31}$$

*where $q_i = \frac{u(i)}{u(i-1)}$. The length of the negative semi-wave $u^{\mathrm{c}}_{i,j}$ is given by $\lfloor T^{\alpha}(q_i) + T^{\alpha}(2-\beta) \rfloor + 2$.*

**Proof.** Recall that

$$q_{i+1} = 2 - \beta - \frac{1}{q_i}.$$

Now, let us distinguish the following four disjoint cases:

1. If $q_i = 0$ then $q_{i+1} = \infty$ and $q_{i+2} = 2 - \beta - \frac{1}{q_{i+1}} = 2 - \beta < 0$. Thus, in this case, we have that $j = i + 1$ (see Fig. 9, left).
2. If $q_i < 0$ and $q_{i+1} = 0$ then $q_{i+2} = \infty$, $u(j+1) > 0$ and thus, we get $j = i + 1$ (see Fig. 9, right).
3. If $q_i < 0$ and $q_{i+1} < 0$ then $j = i$ (see Fig. 10, left).
4. If $q_i < 0$ and $q_{i+1} > 0$ then $q_{i+2} = 2 - \beta - \frac{1}{q_{i+1}} < 0$ and thus, $j = i + 1$ (see Fig. 10, right).

Now, observe that $q_{i+1} = 0$ if and only if $2 - \beta = \frac{1}{q_i}$, i.e., if and only if $T^{\alpha}(2-\beta) = T^{\alpha}\left(\frac{1}{q_i}\right) = 1 - T^{\alpha}(q_i)$. Thus, for $q_i < 0$, we have that

1. $q_{i+1} = 0$ if and only if $\quad T^{\alpha}(q_i) + T^{\alpha}(2-\beta) = 1$,
2. $q_{i+1} < 0$ if and only if $0 < T^{\alpha}(q_i) + T^{\alpha}(2-\beta) < 1$,
3. $q_{i+1} > 0$ if and only if $1 < T^{\alpha}(q_i) + T^{\alpha}(2-\beta) < 2$.

Which implies that for $q_i < 0$, we obtain

$$j = i + \lfloor T^{\alpha}(q_i) + T^{\alpha}(2-\beta) \rfloor. \tag{32}$$

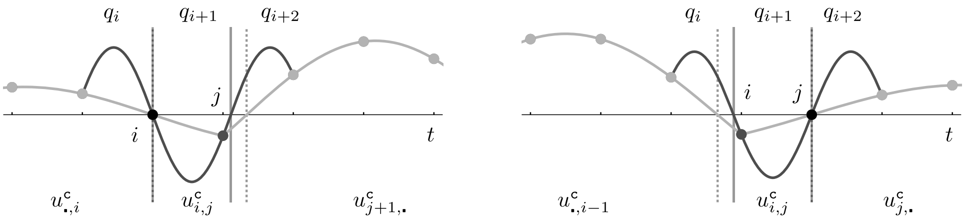


**Fig. 9.** Different anchoring of two positive semi-waves with one negative semi-wave $u^{c}_{i,j}$ for $0 < \alpha < 4 \leq \beta$: $q_i = 0$, $q_{i+1} = \infty$, $q_{i+2} < 0$ (left) and $q_i < 0$, $q_{i+1} = 0$, $q_{i+2} = \infty$ (right).

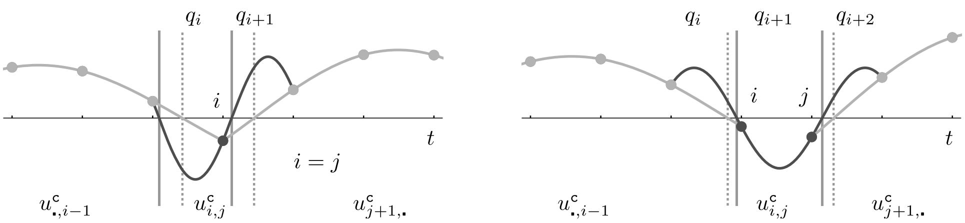


**Fig. 10.** Different anchoring of two positive semi-waves with one negative semi-wave $u^{c}_{i,j}$ for $0 < \alpha < 4 \leq \beta$: $q_i < 0$, $q_{i+1} < 0$ (left) and $q_i < 0$, $q_{i+1} > 0$, $q_{i+2} < 0$ (right).

In the case of $q_i = 0$, we have $\lfloor T^{\alpha}(q_i) + T^{\alpha}(2-\beta) \rfloor = \lfloor 1 + T^{\alpha}(2-\beta) \rfloor = 1$, since $2 - \beta < 0$. Thus, (32) holds also in this case. □

For $0 < \alpha < 4 \leq \beta$ and $C_1 > 0$, we have to extend the definition of $p_{2k}$, $k \in \mathbb{N}$. If $\vartheta_{2k-1} < 0$ then we have one positive semi-wave $u^{c}_{\bullet,i}$ and one negative semi-wave $u^{c}_{i,j}$ such that (see Fig. 9, right, and Fig. 10)

$$q_i = \vartheta_{2k-1}.$$

We define $p_{2k}$ to be equal to $j - i + 1$ (see (31))

$$p_{2k}(\alpha, \beta) := \lfloor T^{\alpha}(\vartheta_{2k-1}(\alpha, \beta)) + T^{\alpha}(2-\beta) \rfloor + 1. \tag{33}$$

If $\vartheta_{2k-1} = \infty$ then we have one positive semi-wave $u^{c}_{\bullet,i}$ and one negative semi-wave $u^{c}_{i,j}$ such that (see Fig. 9, left)

$$q_{i+1} = \vartheta_{2k-1} = \infty, \qquad q_i = 0.$$

In such a case, we define $p_{2k}$ to be equal to $j - i$ (see (31))

$$\begin{aligned} p_{2k}(\alpha, \beta) &:= \lfloor T^{\alpha}(q_i) + T^{\alpha}(2-\beta) \rfloor = \lfloor T^{\alpha}(0) + T^{\alpha}(\infty) + T^{\alpha}(2-\beta) \rfloor \\ &= \lfloor T^{\alpha}(\vartheta_{2k-1}(\alpha, \beta)) + T^{\alpha}(2-\beta) \rfloor + 1, \end{aligned}$$

since $T^{\alpha}(0) = 1$ and $T^{\alpha}(\infty) = 0$. Thus, in this case, we define $p_{2k}$ as in (33).

For $0 < \alpha < 4 \leq \beta$ and $C_1 < 0$, we have to extend the definition of $p_{2k+1}$, $k \in \mathbb{N} \cup \{0\}$, in a similar way as in the case of $C_1 > 0$ (cf. (33)):

$$p_{2k+1}(\alpha,\beta) := \lfloor T^\alpha(\vartheta_{2k}(\alpha,\beta)) + T^\alpha(2-\beta) + 1 \rfloor. \tag{34}$$

In the following definition, we collect all partial definitions (30), (33) and (34) and we extend them also to the case of $0 < \beta < 4 \le \alpha$.

**Definition 17.** For all $j \in \mathbb{Z}$, let us denote

$$\phi_j := \begin{cases} \alpha & \text{for } j \text{ odd}, \\ \beta & \text{for } j \text{ even}. \end{cases}$$

On the domain $D = ((0,4) \times (0,+\infty)) \cup ((0,+\infty) \times (0,4))$, let us define sequences of functions $(p_i)$ and $(\vartheta_i)$, which are given recurrently for $i \in \mathbb{N}$ in the following way

$$\vartheta_0(\alpha,\beta) := \infty,$$

$$p_i(\alpha,\beta) := \begin{cases} \left\lfloor T^{\phi_i}(\vartheta_{i-1}(\alpha,\beta)) + \frac{\pi}{\omega_{\phi_i}} \right\rfloor & \text{for } \phi_i < 4, \\ \left\lfloor T^{\phi_{i+1}}(\vartheta_{i-1}(\alpha,\beta)) + T^{\phi_{i+1}}(2-\phi_i) + 1 \right\rfloor & \text{for } \phi_i \ge 4, \end{cases}$$

$$\vartheta_i(\alpha,\beta) := W^{\phi_i}_{p_i(\alpha,\beta)}(\vartheta_{i-1}(\alpha,\beta)).$$

Moreover, for all $k \in \mathbb{N}$, let us define function $P_k : D \to \mathbb{N}$ and composite functions $\mathcal{W}^\pm_k : \mathbb{R}^* \to \mathbb{R}^*$ as

$$P_k(\alpha,\beta) := \sum_{i=1}^{k} p_i(\alpha,\beta), \qquad \begin{aligned} \mathcal{W}^+_k &:= W^{\phi_k}_{p_k(\alpha,\beta)} \circ \cdots \circ W^{\phi_2}_{p_2(\alpha,\beta)} \circ W^{\phi_1}_{p_1(\alpha,\beta)}, \\ \mathcal{W}^-_k &:= W^{\phi_{k+1}}_{p_k(\beta,\alpha)} \circ \cdots \circ W^{\phi_3}_{p_2(\beta,\alpha)} \circ W^{\phi_2}_{p_1(\beta,\alpha)}. \end{aligned}$$

**Remark 18.** Using notation in Definition 17, we end up with the following relations

$$\vartheta_k(\alpha,\beta) = \mathcal{W}^+_k(\infty), \qquad \vartheta_k(\beta,\alpha) = \mathcal{W}^-_k(\infty).$$

**Lemma 19.** *Let $(\alpha,\beta) \in D$ and let $u$ be a solution of the initial value problem (29) with $C_1 > 0$. All generalized zeros of $u$ form a sequence $(z_m)_{m\in\mathbb{Z}}$, where*

$$z_{-i} = -P_i(\beta,\alpha), \qquad z_i = \begin{cases} P_i(\alpha,\beta) + 1 & \text{if } \vartheta_i(\alpha,\beta) \ne \infty, \\ P_i(\alpha,\beta) & \text{if } \vartheta_i(\alpha,\beta) = \infty, \end{cases} \qquad i \in \mathbb{N}.$$

*Moreover, the solution $u$ consists of infinitely many positive and negative semi-waves.*

*For $0 < \alpha < 4$ and $\beta > 0$, all zero points of all positive semi-waves form a sequence $(t_m)_{m\in\mathbb{Z}}$, where*

$$t_{-i} = -P_i(\beta,\alpha) - T^\alpha(\vartheta_i(\beta,\alpha)), \qquad t_0 = 0, \qquad t_i = P_i(\alpha,\beta) + T^\alpha(\vartheta_i(\alpha,\beta)), \qquad i \in \mathbb{N}.$$

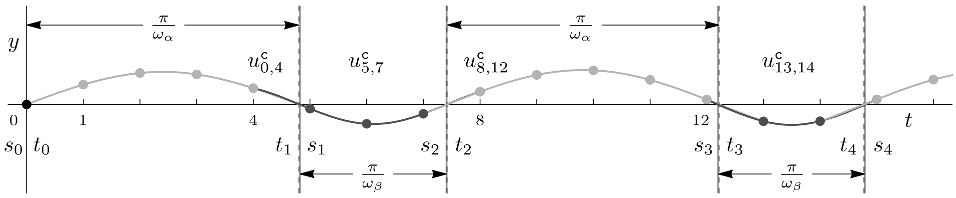


**Fig. 11.** Positive and negative semi-waves of a solution of the initial value problem (29) for $0 < \alpha, \beta < 4$ and $C_1 > 0$ ($0 = s_0 = t_0 < t_1 < s_1 < s_2 < t_2 < s_3 < t_3 < t_4 < s_4$).

*The $m$-th semi-wave, $m \in \mathbb{Z} \setminus \{0\}$, is positive one if and only if $m > 0$ is odd or $m < 0$ is even and it has exactly two zero points $t_{m-1}$ and $t_m$ for $0 < \alpha < 4$ and $\beta > 0$.*

*For $\alpha > 0$ and $0 < \beta < 4$, all zero points of all negative semi-waves form a sequence $(s_m)_{m \in \mathbb{Z}}$, where*

$$s_{-i} = -P_i(\beta,\alpha) - T^\beta(\vartheta_i(\beta,\alpha)), \qquad s_0 = 0, \qquad s_i = P_i(\alpha,\beta) + T^\beta(\vartheta_i(\alpha,\beta)), \qquad i \in \mathbb{N}.$$

*The $m$-th semi-wave, $m \in \mathbb{Z} \setminus \{0\}$, is negative one if and only if $m > 0$ is even or $m < 0$ is odd and it has exactly two zero points $s_{m-1}$ and $s_m$ for $\alpha > 0$ and $0 < \beta < 4$.*

**Proof.** If $u$ is a solution of (29) with $u(1) = C_1 > 0$ then $v(k) := -u(-k)$, $k \geq 0$, solves the initial value problem

$$\begin{cases} \Delta^2 v(k-1) + \beta v^+(k) - \alpha v^-(k) = 0, & k \in \mathbb{N}, \\ v(0) = 0, \quad v(1) = C_1. \end{cases}$$

For $0 < \alpha < 4$ and $\beta > 0$, negative zeros $t_{-i}$ of positive semi-waves of $u$ are determined by positive zeros $\tilde{s}_i$ of negative semi-waves of $v$, thus

$$t_{-i} = -\tilde{s}_i, \qquad \tilde{s}_i = P_i(\beta,\alpha) + T^\alpha(\vartheta_i(\beta,\alpha)), \qquad i \in \mathbb{N}.$$

Moreover, in this case, we have that

$$\begin{aligned} z_i &= \lceil t_i \rceil = \lceil P_i(\alpha,\beta) + T^\alpha(\vartheta_i(\alpha,\beta)) \rceil = \begin{cases} P_i(\alpha,\beta) + 1 & \text{if } \vartheta_i(\alpha,\beta) \neq \infty, \\ P_i(\alpha,\beta) & \text{if } \vartheta_i(\alpha,\beta) = \infty, \end{cases} \\ z_{-i} &= \lceil t_{-i} \rceil = \lceil -P_i(\beta,\alpha) - T^\alpha(\vartheta_i(\beta,\alpha)) \rceil = -P_i(\beta,\alpha). \end{aligned}$$

For $\alpha > 0$ and $0 < \beta < 4$, negative zeros $s_{-i}$ of negative semi-waves of $u$ are determined by positive zeros $\tilde{t}_i$ of positive semi-waves of $v$, thus

$$s_{-i} = -\tilde{t}_i, \qquad \tilde{t}_i = P_i(\beta,\alpha) + T^\beta(\vartheta_i(\beta,\alpha)), \qquad i \in \mathbb{N}.$$

Finally, in this case, we obtain

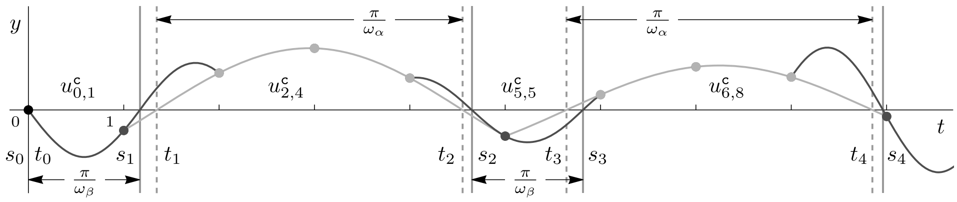


**Fig. 12.** Positive and negative semi-waves of a solution of the initial value problem (29) for $0 < \alpha, \beta < 4$ and $C_1 < 0$.

$$z_i = \lceil s_i \rceil = \lceil P_i(\alpha,\beta) + T^\beta(\vartheta_i(\alpha,\beta)) \rceil = \begin{cases} P_i(\alpha,\beta)+1 & \text{if } \vartheta_i(\alpha,\beta) \neq \infty, \\ P_i(\alpha,\beta) & \text{if } \vartheta_i(\alpha,\beta) = \infty, \end{cases}$$

$$z_{-i} = \lceil s_{-i} \rceil = \lceil -P_i(\beta,\alpha) - T^\beta(\vartheta_i(\beta,\alpha)) \rceil = -P_i(\beta,\alpha). \quad \square$$

**Remark 20.** If $u$ is a solution of (29) with $u(1) = C_1 < 0$ (see Fig. 12) then $v(k) := -u(k)$ solves the initial value problem

$$\begin{cases} \Delta^2 v(k-1) + \beta v^+(k) - \alpha v^-(k) = 0, & k \in \mathbb{Z}, \\ v(0) = 0, \quad v(1) = -C_1 > 0. \end{cases}$$

For $\alpha > 0$ and $0 < \beta < 4$, zeros $\tilde{t}_i$ of positive semi-waves of $v$ are zeros $s_i$ of negative semi-waves of $u$

$$s_{-i} = \tilde{t}_{-i} = -P_i(\alpha,\beta) - T^\beta(\vartheta_i(\alpha,\beta)), \qquad s_0 = \tilde{t}_0 = 0,$$
$$s_i = \tilde{t}_i = P_i(\beta,\alpha) + T^\beta(\vartheta_i(\beta,\alpha)), \qquad i \in \mathbb{N}.$$

For $0 < \alpha < 4$ and $\beta > 0$, zeros $\tilde{s}_i$ of negative semi-waves of $v$ are zeros $t_i$ of positive semi-waves of $u$

$$t_{-i} = \tilde{s}_{-i} = -P_i(\alpha,\beta) - T^\alpha(\vartheta_i(\alpha,\beta)), \qquad t_0 = \tilde{s}_0 = 0,$$
$$t_i = \tilde{s}_i = P_i(\beta,\alpha) + T^\alpha(\vartheta_i(\beta,\alpha)), \qquad i \in \mathbb{N}.$$

**Example 21.** In this example, we show how to obtain all pairs $(\alpha,\beta) \in (0,4) \times (0,4)$ such that the corresponding solution of the initial value problem (29) satisfies the following sign conditions (see Fig. 11)

$$u(k) \geq 0 \qquad \text{for } k = 1,2,3,4 \text{ and } k = 8,9,10,11,12, \tag{35}$$
$$u(k) \leq 0 \qquad \text{for } k = 5,6,7 \text{ and } k = 13. \tag{36}$$

These sign conditions mean that $p_1(\alpha,\beta) = 4$, $p_2(\alpha,\beta) = 3$, $p_3(\alpha,\beta) = 5$. For $0 < \alpha$, $\beta < 4$, we have that

$$p_1(\alpha,\beta) = \left\lfloor \tfrac{\pi}{\omega_\alpha} \right\rfloor, \qquad p_2(\alpha,\beta) = \left\lfloor \tfrac{\pi}{\omega_\beta} + T^\beta(\mathcal{W}_1^+(\infty)) \right\rfloor,$$
$$p_3(\alpha,\beta) = \left\lfloor \tfrac{\pi}{\omega_\alpha} + T^\alpha(\mathcal{W}_2^+(\infty)) \right\rfloor,$$

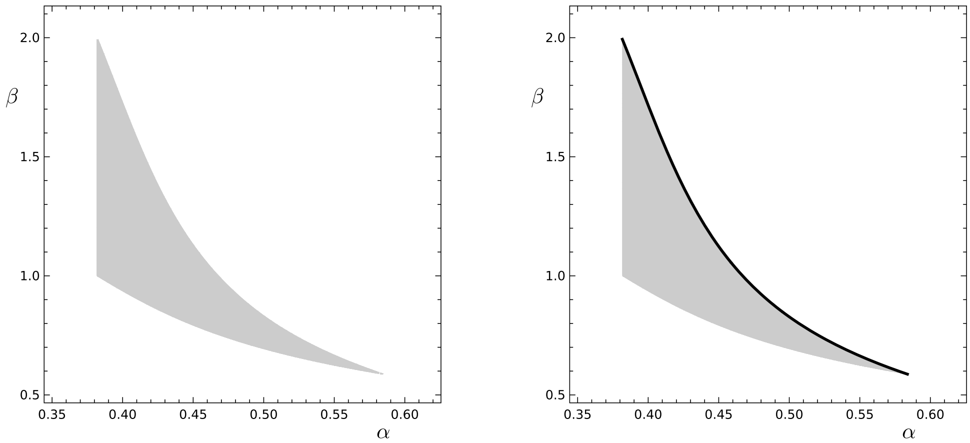


**Fig. 13.** Set of all pairs $(\alpha,\beta) \in (0,4) \times (0,4)$ for which the corresponding solution $u$ of the initial value problem (29) satisfies sign conditions (35) and (36) (left) and those pairs $(\alpha,\beta)$ from this set for which $u(12) = 0$ (black curve, right).

and

$$\mathcal{W}_1^{+}(\infty) = \frac{V_4^\alpha}{V_3^\alpha}, \qquad \mathcal{W}_2^{+}(\infty) = \frac{V_4^\alpha V_3^\beta - V_3^\alpha V_2^\beta}{V_4^\alpha V_2^\beta - V_3^\alpha V_1^\beta},$$

since

$$\begin{aligned}
\left[\mathcal{W}_1^{+}\right] &= \left[W_4^\alpha\right] = \begin{bmatrix} V_4^\alpha & -V_3^\alpha \\ V_3^\alpha & -V_2^\alpha \end{bmatrix}, \\
\left[\mathcal{W}_2^{+}\right] &= \left[W_3^\beta\right] \cdot \left[W_4^\alpha\right] = \begin{bmatrix} V_3^\beta & -V_2^\beta \\ V_2^\beta & -V_1^\beta \end{bmatrix} \cdot \begin{bmatrix} V_4^\alpha & -V_3^\alpha \\ V_3^\alpha & -V_2^\alpha \end{bmatrix} \\
&= \begin{bmatrix} V_4^\alpha V_3^\beta - V_3^\alpha V_2^\beta & V_2^\alpha V_2^\beta - V_3^\alpha V_3^\beta \\ V_4^\alpha V_2^\beta - V_3^\alpha V_1^\beta & V_2^\alpha V_1^\beta - V_3^\alpha V_2^\beta \end{bmatrix}.
\end{aligned}$$

Thus, for $0 < \alpha, \beta < 4$, the sign conditions (35) and (36) read (see Fig. 13, left)

$$\left\lfloor \tfrac{\pi}{\omega_\alpha} \right\rfloor = 4 \quad \wedge \quad \left\lfloor \tfrac{\pi}{\omega_\beta} + T^\beta\left(\tfrac{V_4^\alpha}{V_3^\alpha}\right) \right\rfloor = 3 \quad \wedge \quad \left\lfloor \tfrac{\pi}{\omega_\alpha} + T^\alpha\left(\tfrac{V_4^\alpha V_3^\beta - V_3^\alpha V_2^\beta}{V_4^\alpha V_2^\beta - V_3^\alpha V_1^\beta}\right) \right\rfloor = 5.$$

Moreover, the second zero $t_3$ of the second positive semi-wave $u_{8,12}^{\mathfrak{c}}$ has the following form

$$\begin{aligned}
t_3 &= p_1(\alpha,\beta) + p_2(\alpha,\beta) + p_3(\alpha,\beta) + T^\alpha(\mathcal{W}_3^{+}(\infty)) \\
&= 12 + T^\alpha\left( \frac{V_5^\alpha V_4^\alpha V_3^\beta - (V_5^\alpha V_3^\alpha + V_4^\alpha V_4^\beta) V_2^\beta + V_4^\alpha V_3^\alpha V_1^\beta}{V_4^\alpha V_4^\alpha V_3^\beta - 2 V_4^\alpha V_3^\alpha V_2^\beta + V_3^\alpha V_3^\alpha V_1^\beta} \right),
\end{aligned}$$

since

$$\left[\mathcal{W}_3^{+}\right] = \left[W_5^{\alpha}\right] \cdot \left[W_3^{\beta}\right] \cdot \left[W_4^{\alpha}\right] = \begin{bmatrix} V_5^{\alpha} & -V_4^{\alpha} \\ V_4^{\alpha} & -V_3^{\alpha} \end{bmatrix} \cdot \left[\mathcal{W}_2^{+}\right].$$

In addition, the condition $u(12) = 0$ gives us that $t_3 = 12$, which means that $\mathcal{W}_3^{+}(\infty) = \infty$ or that (see the black curve in Fig. 13, right)

$$V_4^{\alpha} V_4^{\alpha} V_3^{\beta} - 2 V_4^{\alpha} V_3^{\alpha} V_2^{\beta} + V_3^{\alpha} V_3^{\alpha} V_1^{\beta} = 0.$$

## 4. The Fučík spectrum of the Dirichlet matrix

In this section, we provide the description of the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathtt{D}}\right)$ of the Dirichlet matrix $\mathbf{A}^{\mathtt{D}}$, i.e. we describe the set of all pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that the problem

$$\mathbf{A}^{\mathtt{D}} \mathbf{u} = \alpha \mathbf{u}^{+} - \beta \mathbf{u}^{-}, \tag{37}$$

has a non-trivial solution $\mathbf{u} = [u(1), \dots, u(n)]^t$, $n \in \mathbb{N}$, $n \geq 2$. The eigenvalues of $\mathbf{A}^{\mathtt{D}}$ are of the form

$$\lambda_j^{\mathtt{D}} = 4 \sin^2 \frac{(j+1)\pi}{2(n+1)}, \qquad j = 0, \dots, n-1.$$

Thus, all pairs $(\lambda_j^{\mathtt{D}}, \lambda_j^{\mathtt{D}})$, $j = 0, \dots, n-1$, belong to the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathtt{D}}\right)$ since for $\alpha = \beta = \lambda$, the problem (37) is linear one $\mathbf{A}^{\mathtt{D}} \mathbf{u} = \lambda \mathbf{u}$. Now, we apply general results in [4] in the case of the Dirichlet matrix $\mathbf{A}^{\mathtt{D}}$. Due to the symmetry of the matrix $\mathbf{A}^{\mathtt{D}}$, the inadmissible set $\Pi\left(\mathbf{A}^{\mathtt{D}}\right)$ for the Fučík spectrum $\Sigma(\mathbf{A}^{\mathtt{D}})$ (i.e. $\Pi\left(\mathbf{A}^{\mathtt{D}}\right) \cap \Sigma\left(\mathbf{A}^{\mathtt{D}}\right) = \emptyset$) has the following form (see Corollary 4.7 in [4])

$$\Pi\left(\mathbf{A}^{\mathtt{D}}\right) = \bigcup_{i=0}^{n} S_i,$$

where $S_0 := (-\infty, \lambda_0^{\mathtt{D}}) \times (-\infty, \lambda_0^{\mathtt{D}})$, $S_i := (\lambda_{i-1}^{\mathtt{D}}, \lambda_i^{\mathtt{D}}) \times (\lambda_{i-1}^{\mathtt{D}}, \lambda_i^{\mathtt{D}})$ for $i = 1, \dots, n-1$, $S_n := (\lambda_{n-1}^{\mathtt{D}}, +\infty) \times (\lambda_{n-1}^{\mathtt{D}}, +\infty)$. Moreover, $\lambda_0^{\mathtt{D}}$ is a principal eigenvalue of $\mathbf{A}^{\mathtt{D}}$, which implies that

$$\left\{(\alpha, \beta) \in \mathbb{R}^2 : \; \left(\alpha - \lambda_0^{\mathtt{D}}\right)\left(\beta - \lambda_0^{\mathtt{D}}\right) < 0\right\} \cap \Sigma\left(\mathbf{A}^{\mathtt{D}}\right) = \emptyset,$$

i.e. both shifted quadrants are inadmissible sets for the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathtt{D}}\right)$ (see Fig. 14). Thus, it is enough to investigate the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathtt{D}}\right)$ only on the set $D = ((0, 4) \times (0, +\infty)) \cup ((0, +\infty) \times (0, 4))$.

Recall that a solution $u$ of (1) has a generalized zero at $k \in \mathbb{T}$ if $u(k) = 0$ or $u(k-1)u(k) < 0$. Since the boundary value problem (1) is equivalent to the Fučík spectrum problem (37), we conclude that

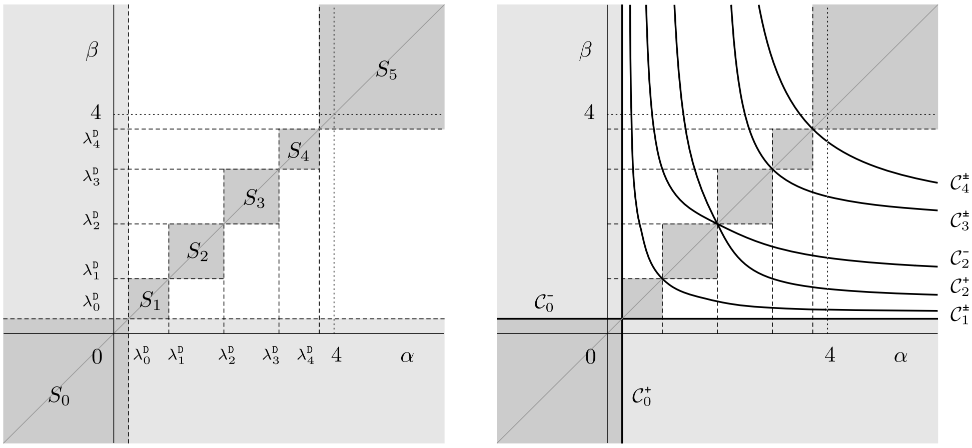


**Fig. 14.** Inadmissible areas for the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathrm{D}}\right)$ (left, $n = 5$) and the particular Fučík curves $\mathcal{C}_k^\pm$ (black curves) of the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathrm{D}}\right)$ (right, $n = 5$).

$$\Sigma\left(\mathbf{A}^{\mathrm{D}}\right) = \bigcup_{k=0}^{n-1} \left(\mathcal{C}_k^+ \cup \mathcal{C}_k^-\right),$$

where

$$\mathcal{C}_k^\pm := \big\{(\alpha,\beta) \in \mathbb{R}^2 : \text{ the problem (1) has a non-trivial solution } u \\ \text{with exactly } k \text{ generalized zeros on } \mathbb{T} \text{ and } u(1) \gtrless 0\big\}.$$

The Fučík curves $\mathcal{C}_0^\pm$ are trivial ones

$$\mathcal{C}_0^+ = \left\{(\alpha,\beta) : \ \alpha = \lambda_0^{\mathrm{D}}\right\}, \qquad \mathcal{C}_0^- = \left\{(\alpha,\beta) : \ \beta = \lambda_0^{\mathrm{D}}\right\},$$

since the corresponding non-trivial solutions $u(k) = C \sin \frac{k\pi}{n+1}$, $C \neq 0$, do not change sign in $\mathbb{T}$. According to Remark 20, we deduce that

$$\mathcal{C}_k^- = \{(\alpha,\beta) \in D : \ (\beta,\alpha) \in \mathcal{C}_k^+\},$$

and thus, it is enough to focus only on Fučík curves $\mathcal{C}_k^+$ for $k = 1, \dots, n-1$. The following theorem provides us with the first two possibilities how to describe these curves $\mathcal{C}_k^+$ (see Figs. 15 and 16).

**Theorem 22.** *For $k = 1, \dots, n-1$, $n \in \mathbb{N}$, $n \geq 2$, we have that*

$$\begin{aligned}\mathcal{C}_k^+ = \ &\{(\alpha,\beta) \in (0,4) \times (0,+\infty) : \ P_{k+1}(\alpha,\beta) + T^{\alpha}(\vartheta_{k+1}(\alpha,\beta)) = n+1\} \ \cup \\ &\big\{(\alpha,\beta) \in (0,+\infty) \times (0,4) : \ P_{k+1}(\alpha,\beta) + T^{\beta}(\vartheta_{k+1}(\alpha,\beta)) = n+1\big\}.\end{aligned} \tag{38}$$

*Moreover, if we denote*

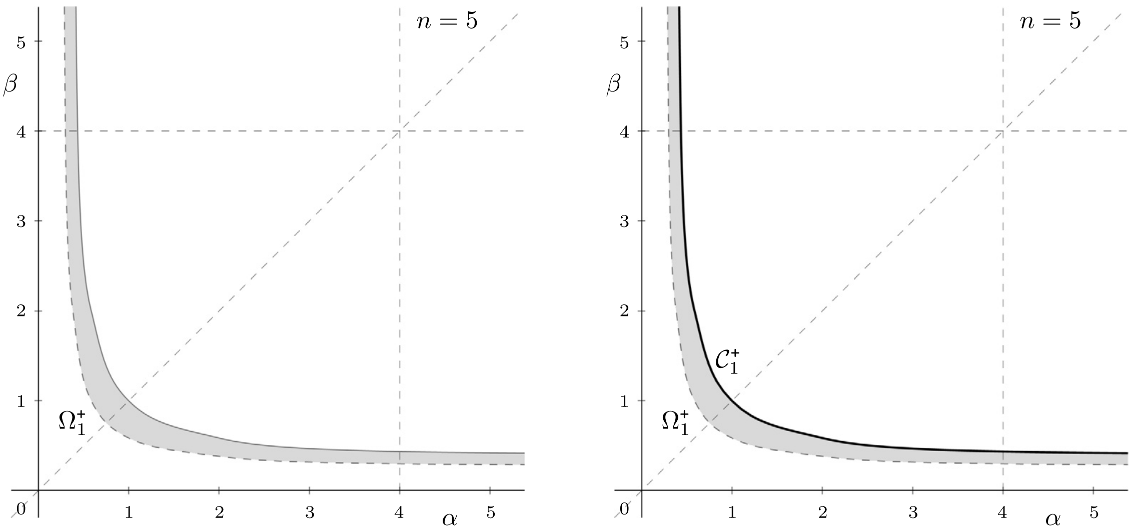


**Fig. 15.** The set $\Omega_1^+$ as the grey region for $n = 5$ (left) and the first non-trivial Fučík curve $\mathcal{C}_1^+$ as the black curve (right) due to Theorem 22.

$$\Omega_k^+ := \{(\alpha, \beta) \in D : \; P_{k+1}(\alpha, \beta) = n + 1\}, \qquad k = 1, \ldots, n-1,$$

*then we have that*

$$\mathcal{C}_k^+ = \left\{(\alpha, \beta) \in \Omega_k^+ : \; \mathcal{W}_{k+1}^+(\infty) = \infty\right\}. \tag{39}$$

**Proof.** Let $(\alpha, \beta) \in D$ and let $u$ be a non-trivial solution of the initial value problem (29) with $C_1 > 0$. Using Lemma 19, we conclude that $u$ has $k$ generalized zeros $z_i$ for $i = 1, \ldots, k$. Moreover, for $0 < \alpha < 4$ and $\beta > 0$, we have that $t_{k+1} = P_{k+1}(\alpha, \beta) + T^{\alpha}(\vartheta_{k+1}(\alpha, \beta))$ is the smallest zero of a positive semi-wave, which is greater than $k$-th generalized zero $z_k$. Thus, the solution $u$ has exactly $k$ generalized zeros on $\mathbb{T}$ and $u(n + 1) = 0$ if and only if $t_{k+1} = n + 1$, i.e.

$$P_{k+1}(\alpha, \beta) + T^{\alpha}(\vartheta_{k+1}(\alpha, \beta)) = n + 1. \tag{40}$$

The equation (40) is satisfied if and only if $P_{k+1}(\alpha, \beta) = n + 1$ and $T^{\alpha}(\vartheta_{k+1}(\alpha, \beta)) = 0$, which implies that $\mathcal{W}_{k+1}^+(\infty) = \vartheta_{k+1}(\alpha, \beta) = \infty$. On the other hand, for $\alpha > 0$ and $0 < \beta < 4$, we have that $s_{k+1} = P_{k+1}(\alpha, \beta) + T^{\beta}(\vartheta_{k+1}(\alpha, \beta))$ is the smallest zero of a negative semi-wave, which is greater than $k$-th generalized zero $z_k$. Thus, the solution $u$ has exactly $k$ generalized zeros on $\mathbb{T}$ and $u(n + 1) = 0$ if and only if $s_{k+1} = n + 1$, i.e.

$$P_{k+1}(\alpha, \beta) + T^{\beta}(\vartheta_{k+1}(\alpha, \beta)) = n + 1. \tag{41}$$

The equation (41) is satisfied if and only if $P_{k+1}(\alpha, \beta) = n + 1$ and $T^{\beta}(\vartheta_{k+1}(\alpha, \beta)) = 0$, which implies that $\mathcal{W}_{k+1}^+(\infty) = \vartheta_{k+1}(\alpha, \beta) = \infty$. $\square$

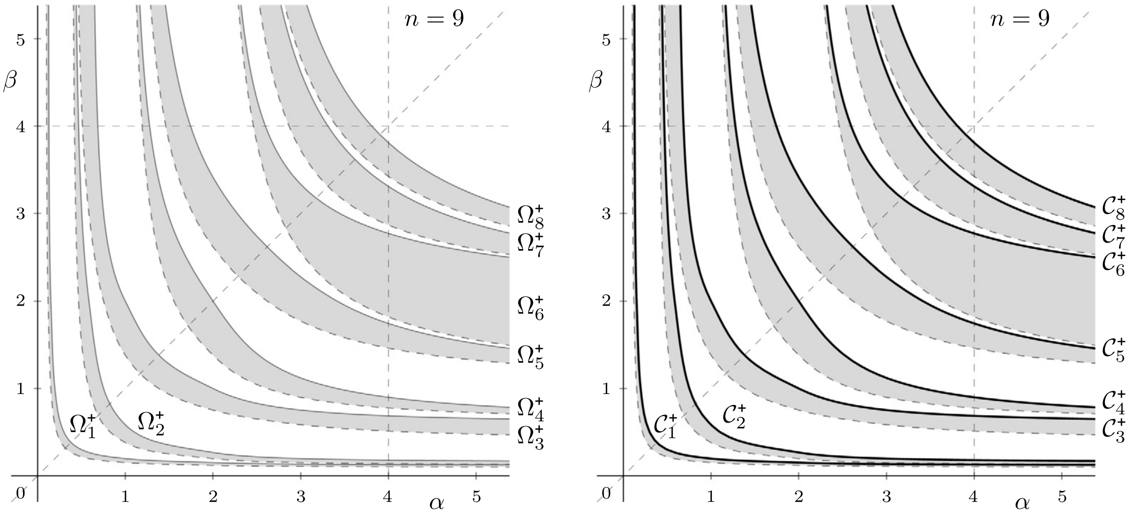


**Fig. 16.** The sets $\Omega_k^+$ as grey regions for $n = 9$ (left) and the Fučík curves $\mathcal{C}_k^+$ as black curves (right) due to Theorem 22.

**Remark 23.**

1. The description (38) is suitable for the numerical approximation of $\mathcal{C}_k^+$ (see Fig. 16, right). For fixed $\beta \geq 4$, we determine numerically $\alpha \in (0, 4)$ such that (40) is satisfied. Then for fixed $\beta \in (0, 4)$, we determine $\alpha > 0$ such that (41) is satisfied.
2. The condition $\mathcal{W}_{k+1}^+(\infty) = \infty$ in the description (39) can be equivalently written as

$$\det\left(\mathbf{A}^{\mathrm{D}} - \mathbf{\Lambda}\right) = 0, \tag{42}$$

where

$$\mathbf{\Lambda} = \operatorname{diag}(\overbrace{\alpha, \alpha, \ldots\ldots, \alpha}^{p_1(\alpha,\beta)\text{–times}}, \overbrace{\beta, \beta, \ldots\ldots, \beta}^{p_2(\alpha,\beta)\text{–times}}, \ldots, \overbrace{\alpha, \alpha, \ldots\ldots, \alpha}^{p_k(\alpha,\beta)\text{–times}}, \overbrace{\beta, \beta, \ldots\ldots\ldots, \beta}^{(p_{k+1}(\alpha,\beta)-1)\text{–times}})$$

for $k$ odd and

$$\mathbf{\Lambda} = \operatorname{diag}(\underbrace{\alpha, \alpha, \ldots\ldots, \alpha}_{p_1(\alpha,\beta)\text{–times}}, \underbrace{\beta, \beta, \ldots\ldots, \beta}_{p_2(\alpha,\beta)\text{–times}}, \ldots, \underbrace{\beta, \beta, \ldots\ldots, \beta}_{p_k(\alpha,\beta)\text{–times}}, \underbrace{\alpha, \alpha, \ldots\ldots\ldots, \alpha}_{(p_{k+1}(\alpha,\beta)-1)\text{–times}})$$

for $k$ even. Indeed, for $k$ odd, the condition $\mathcal{W}_{k+1}^+(\infty) = \infty$ reads

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \left[W^{\beta}_{p_{k+1}(\alpha,\beta)}\right] \cdot \left[W^{\alpha}_{p_k(\alpha,\beta)}\right] \cdots \left[W^{\beta}_{p_2(\alpha,\beta)}\right] \cdot \left[W^{\alpha}_{p_1(\alpha,\beta)}\right] \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0,$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \left[W^{\beta}_{p_{k+1}(\alpha,\beta)-1}\right] \cdot \left[W^{\alpha}_{p_k(\alpha,\beta)}\right] \cdots \left[W^{\beta}_{p_2(\alpha,\beta)}\right] \cdot \left[W^{\alpha}_{p_1(\alpha,\beta)}\right] \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0,$$

which is exactly (42) due to Lemma 12. See Figs. 15 and 16 for the sets $\Omega_k^+$ which contain particular Fučík curves $\mathcal{C}_k^+$ according to the description (39).

**Example 24.** In this example, we focus on the first non-trivial Fučík curve $\mathcal{C}_1^+$. Using Theorem 22, we conclude that (see Fig. 15, left)

$$\Omega_1^+ = \{(\alpha,\beta) \in D: \; p_1(\alpha,\beta) + p_2(\alpha,\beta) = n+1\},$$

where

$$p_1(\alpha,\beta) = \begin{cases} \left\lfloor \frac{\pi}{\omega_\alpha} \right\rfloor & \text{for } 0 < \alpha < 4, \; \beta > 0, \\ \left\lfloor T^\beta(2-\alpha) + 1 \right\rfloor & \text{for } \alpha \geq 4, \; 0 < \beta < 4, \end{cases}$$

$$p_2(\alpha,\beta) = \begin{cases} \left\lfloor T^\beta \left( W^\alpha_{p_1(\alpha,\beta)}(\infty) \right) + \frac{\pi}{\omega_\beta} \right\rfloor & \text{for } 0 < \beta < 4, \; \alpha > 0, \\ \left\lfloor T^\alpha \left( W^\alpha_{p_1(\alpha,\beta)}(\infty) \right) + T^\alpha(2-\beta) + 1 \right\rfloor & \text{for } \beta \geq 4, \; 0 < \alpha < 4. \end{cases}$$

Let us note that for $\alpha \geq 4$ and $0 < \beta < 4$, $p_1(\alpha,\beta) = 1$ and thus, the condition $p_1(\alpha,\beta) + p_2(\alpha,\beta) = n+1$ simplifies into

$$1 + \left\lfloor T^\beta(2-\alpha) + \frac{\pi}{\omega_\beta} \right\rfloor = n+1.$$

Moreover, for the first non-trivial Fučík curve $\mathcal{C}_1^+$, we obtain that (see Fig. 15, right)

$$\begin{aligned} \mathcal{C}_1^+ = \; & \{(\alpha,\beta) \in (0,4) \times (0,+\infty): \; p_1(\alpha,\beta) + p_2(\alpha,\beta) + T^\alpha(\mathcal{W}_2^+(\infty)) = n+1\} \; \cup \\ & \left\{(\alpha,\beta) \in (0,+\infty) \times (0,4): \; p_1(\alpha,\beta) + p_2(\alpha,\beta) + T^\beta(\mathcal{W}_2^+(\infty)) = n+1\right\} \\ = \; & \{(\alpha,\beta) \in \Omega_1^+: \; \mathcal{W}_2^+(\infty) = \infty\}. \end{aligned}$$

Finally, since $\mathcal{W}_2^+ = W^\beta_{p_2(\alpha,\beta)} \circ W^\alpha_{p_1(\alpha,\beta)}$, the condition $\mathcal{W}_2^+(\infty) = \infty$ can be reformulated as (see Remark 23)

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \left[ W^\beta_{p_2(\alpha,\beta)} \right] \cdot \left[ W^\alpha_{p_1(\alpha,\beta)} \right] \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0,$$

$$V^\alpha_{p_1(\alpha,\beta)} V^\beta_{p_2(\alpha,\beta)-1} - V^\alpha_{p_1(\alpha,\beta)-1} V^\beta_{p_2(\alpha,\beta)-2} = 0.$$

In the following corollary we introduce analytical description of some points belonging to the particular Fučík curves $\mathcal{C}_k^\pm$.

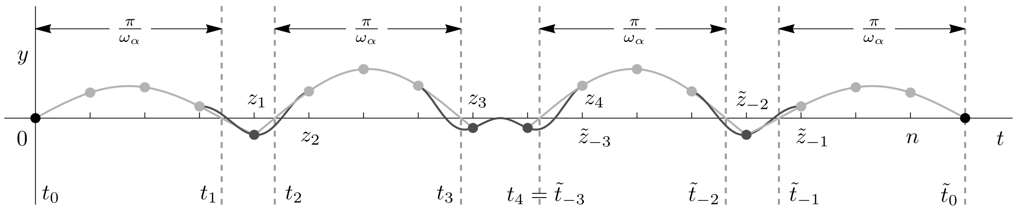


**Fig. 17.** The non-trivial solution $u$ of (1) for $(\alpha, \beta) \in \mathcal{C}_6^+$ ($n = 16$, $i = 4$, $j = 3$, $k = 6$) with six generalized zeros of $u$ on $\mathbb{T}$ ($z_1 < z_2 < z_3 < z_4 = \tilde{z}_{-3} < \tilde{z}_{-2} < \tilde{z}_{-1}$) and six zeros of positive semi-waves strictly between 0 and $n + 1$ ($t_1 < t_2 < t_3 < t_4 = \tilde{t}_{-3} < \tilde{t}_{-2} < \tilde{t}_{-1}$).

**Corollary 25.** *Let $k, n \in \mathbb{N}$ be such that $k \le n - 1$, $n \ge 3$. Moreover, let $k_1, k_2 \in \mathbb{N}$, $k_1, k_2 \ge 2$ and denote*

$$\xi_{k_1} := 4 \sin^2 \frac{\pi}{2k_1}, \qquad \xi_{k_2} := 4 \sin^2 \frac{\pi}{2k_2}.$$

1. *If $k$ is odd and $\frac{k+1}{2}k_1 + \frac{k+1}{2}k_2 = n + 1$ then $(\xi_{k_1}, \xi_{k_2}) \in \mathcal{C}_k^+ = \mathcal{C}_k^-$.*
2. *If $k$ is even and $\left(\frac{k}{2} + 1\right) k_1 + \frac{k}{2}k_2 = n + 1$ then $(\xi_{k_1}, \xi_{k_2}) \in \mathcal{C}_k^+$.*
3. *If $k$ is even and $\frac{k}{2}k_1 + \left(\frac{k}{2} + 1\right) k_2 = n + 1$ then $(\xi_{k_1}, \xi_{k_2}) \in \mathcal{C}_k^-$.*

**Proof.** For $\alpha = \xi_{k_1}$ and $\beta = \xi_{k_2}$, we have $0 < \alpha, \beta < 4$ and using Lemma 11, we obtain

$$\frac{\pi}{\omega_\alpha} = k_1, \qquad W_{k_1}^\alpha(q) = q, \qquad \frac{\pi}{\omega_\beta} = k_2, \qquad W_{k_2}^\beta(q) = q.$$

Moreover, for $i = 1, \dots, k + 1$, we have that

$$\vartheta_i(\alpha, \beta) = \infty, \qquad p_i(\alpha, \beta) = \begin{cases} k_1 & \text{for } i \text{ odd}, \\ k_2 & \text{for } i \text{ even}. \end{cases}$$

Indeed, $T^\alpha(\infty) = T^\beta(\infty) = 0$ and $W_{k1}^\alpha(\infty) = W_{k2}^\beta(\infty) = \infty$. Thus, we obtain that

$$P_{k+1}(\alpha, \beta) = \sum_{i=1}^{k+1} p_i(\alpha, \beta) = \begin{cases} \frac{k+1}{2}(k_1 + k_2) & \text{for } k \text{ odd}, \\ \frac{k}{2}(k_1 + k_2) + k_1 & \text{for } k \text{ even}. \end{cases}$$

The statement now follows from Theorem 22. $\square$

The next theorem provides a different description of Fučík curves $\mathcal{C}_k^\pm$ than Theorem 22. We reconstruct the non-trivial solution of (1) from both end points of $\hat{\mathbb{T}}$: from $t = 0$ to the right and from $t = n + 1$ to the left. Thus, we consider solutions of two initial value problems at $t = 0$ and at $t = n + 1$ and we require that their selected zero points of positive (or negative) semi-waves coincide (see Fig. 17 and note that $t_4 = \tilde{t}_{-3}$).

**Theorem 26.** *Let $k, n \in \mathbb{N}$ be such that $k \le n - 1$, $n \ge 2$. Moreover, let $i, j \in \mathbb{N}$ be such that $i + j = k + 1$.*

1. *If $k$ is odd then*

$$\begin{aligned}\mathcal{C}_k^+ = \big\{(\alpha,\beta) \in (0,4) \times (0,+\infty) : \; & P_i(\alpha,\beta) + P_j(\beta,\alpha) \\ & + T^\alpha(\vartheta_i(\alpha,\beta)) + T^\alpha(\vartheta_j(\beta,\alpha)) = n+1\big\} \cup \\ \big\{(\alpha,\beta) \in (0,+\infty) \times (0,4) : \; & P_i(\alpha,\beta) + P_j(\beta,\alpha) \\ & + T^\beta(\vartheta_i(\alpha,\beta)) + T^\beta(\vartheta_j(\beta,\alpha)) = n+1\big\}.\end{aligned}$$

2. *If $k$ is even then*

$$\begin{aligned}\mathcal{C}_k^+ = \big\{(\alpha,\beta) \in (0,4) \times (0,+\infty) : \; & P_i(\alpha,\beta) + P_j(\alpha,\beta) \\ & + T^\alpha(\vartheta_i(\alpha,\beta)) + T^\alpha(\vartheta_j(\alpha,\beta)) = n+1\big\} \cup \\ \big\{(\alpha,\beta) \in (0,+\infty) \times (0,4) : \; & P_i(\alpha,\beta) + P_j(\alpha,\beta) \\ & + T^\beta(\vartheta_i(\alpha,\beta)) + T^\beta(\vartheta_j(\alpha,\beta)) = n+1\big\}.\end{aligned}$$

**Proof.** Let $(\alpha,\beta) \in D$, let $u$ be the solution of the initial value problem (29) with $C_1 > 0$ and let $v$ be the solution of the following initial value problem

$$\begin{cases} \Delta^2 v(k-1) + \alpha v^+(k) - \beta v^-(k) = 0, & k \in \mathbb{Z}, \\ v(n+1) = 0, \quad v(n) = C_2, \end{cases}$$

where $C_2 \neq 0$.

Firstly, let us consider $0 < \alpha < 4$ and $\beta > 0$. According to Lemma 19, we conclude that

$$t_i = P_i(\alpha,\beta) + T^\alpha(\vartheta_i(\alpha,\beta)) \tag{43}$$

is the zero point of a positive semi-wave of $u$ such that $u$ has exactly $(i-1)$ generalized zeros

$$z_m = \begin{cases} P_m(\alpha,\beta) + 1 & \text{if } \vartheta_m(\alpha,\beta) \neq \infty, \\ P_m(\alpha,\beta) & \text{if } \vartheta_m(\alpha,\beta) = \infty, \end{cases} \qquad m = 1,\dots,i-1,$$

which are strictly between 0 and $z_i = \lceil t_i \rceil$: $0 < z_1 < \cdots < z_{i-1} < t_i \le z_i$. Moreover,

$$\tilde{t}_{-j} = \begin{cases} n+1 - P_j(\beta,\alpha) - T^\alpha(\vartheta_j(\beta,\alpha)) & \text{for } C_2 > 0, \\ n+1 - P_j(\alpha,\beta) - T^\alpha(\vartheta_j(\alpha,\beta)) & \text{for } C_2 < 0, \end{cases} \tag{44}$$

is the zero point of a positive semi-wave of $v$ such that $v$ has exactly $j$ generalized zeros

$$\tilde{z}_{-m} = n+1 - P_m(\alpha,\beta), \qquad m = 1,\dots,j,$$

which are between $\tilde{t}_{-j}$ and $n$: $\tilde{t}_{-j} \le \tilde{z}_{-j} < \cdots < \tilde{z}_{-1} \le n$. Now, a task of finding all $(\alpha,\beta) \in (0,4) \times (0,+\infty)$ such that $(\alpha,\beta) \in \mathcal{C}_k^+$ is equivalent to finding all $(\alpha,\beta) \in (0,4) \times (0,+\infty)$ such that $t_i = \tilde{t}_{-j}$ with $C_2 > 0$ ($C_2 < 0$) for $k$ odd (even), which leads to

$$P_i(\alpha,\beta) + T^\alpha(\vartheta_i(\alpha,\beta)) = n + 1 - P_j(\beta,\alpha) - T^\alpha(\vartheta_j(\beta,\alpha)) \qquad \text{for } k \text{ odd},$$

$$P_i(\alpha,\beta) + T^\alpha(\vartheta_i(\alpha,\beta)) = n + 1 - P_j(\alpha,\beta) - T^\alpha(\vartheta_j(\alpha,\beta)) \qquad \text{for } k \text{ even}.$$

Secondly, in the case of $\alpha > 0$ and $0 < \beta < 4$, we proceed in a similar way. The only difference is that we deal with negative semi-waves instead of positive ones and in (43) and (44), we use $T^\beta$ instead of $T^\alpha$. $\square$

The next theorem is associated with the previous Theorem 26 and contains new sets $\Omega_{i,j}^{+,n}$, which play similar role as sets $\Omega_k^+$ in Theorem 22.

**Theorem 27.** *Let $k,n \in \mathbb{N}$ be such that $k \le n-1$, $n \ge 2$. Moreover, let $i,j \in \mathbb{N}$ be such that $i + j = k + 1$ and let us denote*

$$\Omega_{i,j}^{+,n} := \begin{cases} \{(\alpha,\beta) \in D : \; P_i(\alpha,\beta) + P_j(\beta,\alpha) = n\} & \textit{for } k \textit{ odd}, \\ \{(\alpha,\beta) \in D : \; P_i(\alpha,\beta) + P_j(\alpha,\beta) = n\} & \textit{for } k \textit{ even}. \end{cases}$$

1. *If $k$ is odd then*

$$\mathcal{C}_k^+ = \left\{(\alpha,\beta) \in \Omega_{i,j}^{+,n} : \; \mathcal{W}_i^+(\infty) = \frac{1}{\mathcal{W}_j^-(\infty)}\right\} \cup \left\{(\alpha,\beta) \in \Omega_{i,j}^{+,n+1} : \; \mathcal{W}_i^+(\infty) = \infty = \mathcal{W}_j^-(\infty)\right\}. \tag{45}$$

2. *If $k$ is even then*

$$\mathcal{C}_k^+ = \left\{(\alpha,\beta) \in \Omega_{i,j}^{+,n} : \; \mathcal{W}_i^+(\infty) = \frac{1}{\mathcal{W}_j^+(\infty)}\right\} \cup \left\{(\alpha,\beta) \in \Omega_{i,j}^{+,n+1} : \; \mathcal{W}_i^+(\infty) = \infty = \mathcal{W}_j^+(\infty)\right\}. \tag{46}$$

**Proof.** Let $(\alpha,\beta) \in \mathcal{C}_k^+$ and let $u$ be the corresponding non-trivial solution of (1). Moreover, let us consider that $0 < \alpha < 4$, $\beta > 0$. According to Theorem 26, we have that

$$P_i(\alpha,\beta) + P_j(\beta,\alpha) + T^\alpha(\vartheta_i(\alpha,\beta)) + T^\alpha(\vartheta_j(\beta,\alpha)) = n + 1. \tag{47}$$

There are exactly two possibilities how to satisfy (47):

$$P_i(\alpha,\beta) + P_j(\beta,\alpha) = n+1 \qquad \text{and} \qquad T^\alpha(\vartheta_i(\alpha,\beta)) + T^\alpha(\vartheta_j(\beta,\alpha)) = 0, \tag{48}$$

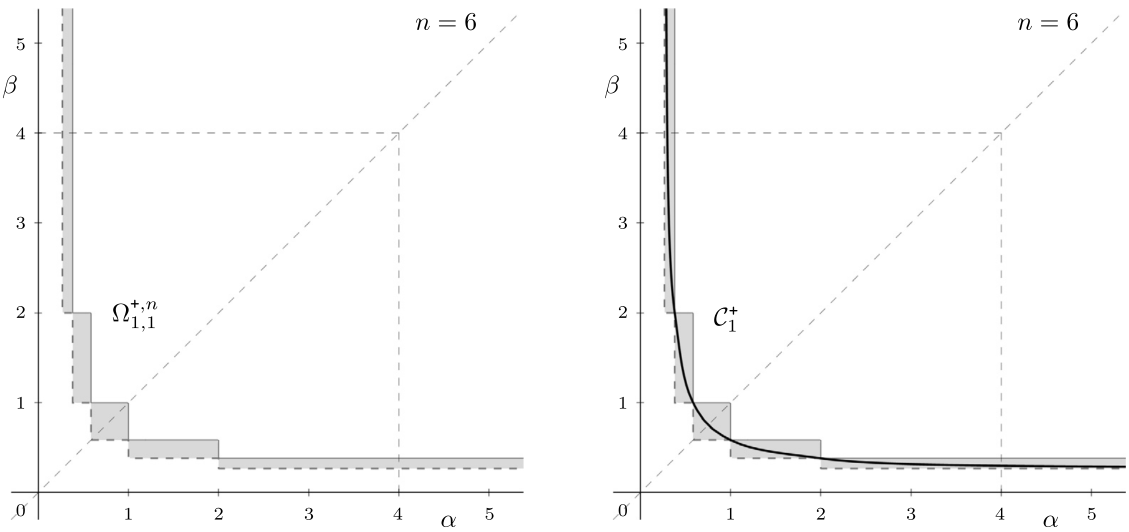


**Fig. 18.** The set $\Omega^{+,n}_{1,1}$ as the grey region for $n = 6$ (left) and the first non-trivial Fučík curve $\mathcal{C}^{+}_{1}$ as the black curve (right) due to Theorems 26 and 27.

or

$$P_i(\alpha,\beta) + P_j(\beta,\alpha) = n \qquad \text{and} \qquad T^\alpha(\vartheta_i(\alpha,\beta)) + T^\alpha(\vartheta_j(\beta,\alpha)) = 1. \tag{49}$$

The second equation in (48) gives us $T^\alpha(\vartheta_i(\alpha,\beta)) = 0$ and $T^\alpha(\vartheta_j(\beta,\alpha)) = 0$, which implies that

$$\vartheta_i(\alpha,\beta) = \mathcal{W}^{+}_{i}(\infty) = \infty \qquad \text{and} \qquad \vartheta_j(\beta,\alpha) = \mathcal{W}^{-}_{j}(\infty) = \infty.$$

Using Lemma 3, we obtain from the second equation in (49) that

$$T^\alpha(\vartheta_i(\alpha,\beta)) = 1 - T^\alpha(\vartheta_j(\beta,\alpha)) = T^\alpha\left(\frac{1}{\vartheta_j(\beta,\alpha)}\right),$$

and thus, according to Remark 18, we get

$$\mathcal{W}^{+}_{i}(\infty) = \vartheta_i(\alpha,\beta) = \frac{1}{\vartheta_j(\beta,\alpha)} = \frac{1}{\mathcal{W}^{-}_{j}(\infty)}. \qquad \square$$

See Fig. 18 for the first non-trivial Fučík curve $\mathcal{C}^{+}_{1}$ for $n = 6$ and the corresponding set $\Omega^{+,n}_{1,1}$ according to Theorem 27. Fig. 19 contains the second non-trivial Fučík curve $\mathcal{C}^{+}_{2}$ for $n = 9$ and also the corresponding set $\Omega^{+,n}_{2,1}$. Let us point out that we have two different sets $\Omega^{+,n}_{i,j}$ available for the third non-trivial Fučík curve $\mathcal{C}^{+}_{3}$, namely $\Omega^{+,n}_{3,1}$ and $\Omega^{+,n}_{2,2}$ (see Figs. 20 and 22). Moreover, see Figs. 20, 22, 23 and 24 to compare all different sets $\Omega^{+,n}_{i,j}$ in the case of $n = 9$.

In the following remark, we reveal the algebraic structure of particular Fučík curves due to Theorem 27 in a similar way as in Remark 23 (compare also with preliminary results for general matrices in [3]).

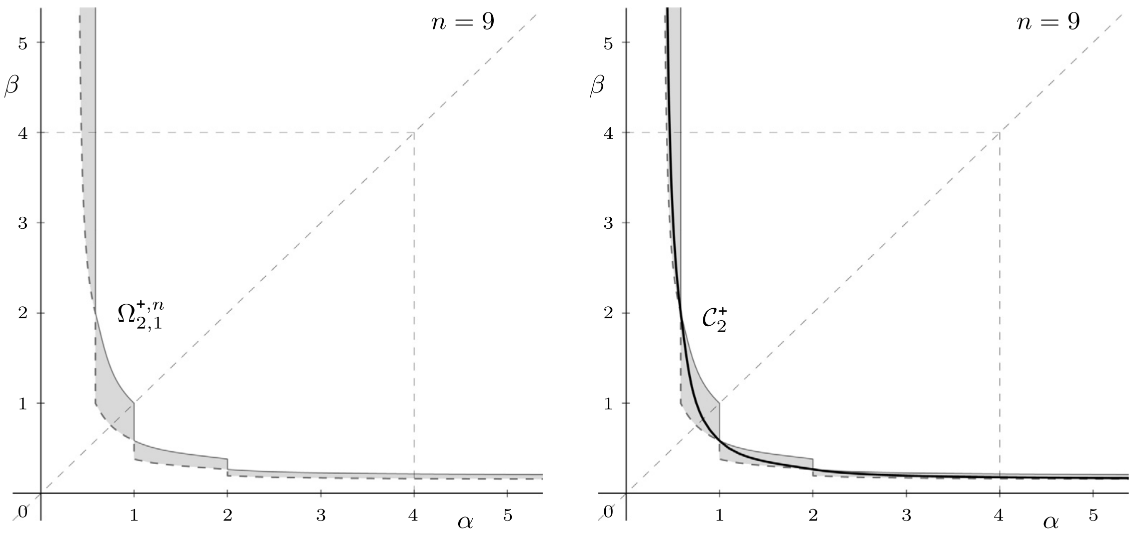


**Fig. 19.** The set $\Omega^{+,n}_{2,1}$ as the grey region for $n = 9$ (left) and the second non-trivial Fučík curve $\mathcal{C}^{+}_{2}$ as the black curve (right) due to Theorems 26 and 27.

**Remark 28.** The condition $\mathcal{W}^{+}_{i}(\infty) = \frac{1}{\mathcal{W}^{-}_{j}(\infty)}$ in (45) reads $\det(\mathbf{A}^{\mathsf{D}} - \mathbf{\Lambda}) = 0$, where

$$\begin{aligned}\mathbf{\Lambda} = \operatorname{diag}\big(&\overbrace{\alpha, \alpha, \ldots\ldots, \alpha}^{p_1(\alpha,\beta)\text{–times}}, \overbrace{\beta, \beta, \ldots\ldots, \beta}^{p_2(\alpha,\beta)\text{–times}}, \ldots, \overbrace{\beta, \beta, \ldots\ldots, \beta}^{p_i(\alpha,\beta)\text{–times}},\\ &\underbrace{\alpha, \alpha, \ldots\ldots, \alpha}_{p_j(\beta,\alpha)\text{–times}}, \ldots, \underbrace{\alpha, \alpha, \ldots\ldots, \alpha}_{p_2(\beta,\alpha)\text{–times}}, \underbrace{\beta, \beta, \ldots\ldots, \beta}_{p_1(\beta,\alpha)\text{–times}}\big)\end{aligned} \tag{50}$$

for $k = i + j - 1$ odd and both $i, j$ even, and

$$\begin{aligned}\mathbf{\Lambda} = \operatorname{diag}\big(&\overbrace{\alpha, \alpha, \ldots\ldots, \alpha}^{p_1(\alpha,\beta)\text{–times}}, \overbrace{\beta, \beta, \ldots\ldots, \beta}^{p_2(\alpha,\beta)\text{–times}}, \ldots, \overbrace{\alpha, \alpha, \ldots\ldots, \alpha}^{p_i(\alpha,\beta)\text{–times}},\\ &\underbrace{\beta, \beta, \ldots\ldots, \beta}_{p_j(\beta,\alpha)\text{–times}}, \ldots, \underbrace{\alpha, \alpha, \ldots\ldots, \alpha}_{p_2(\beta,\alpha)\text{–times}}, \underbrace{\beta, \beta, \ldots\ldots, \beta}_{p_1(\beta,\alpha)\text{–times}}\big)\end{aligned}$$

for $k = i + j - 1$ odd and both $i, j$ odd. Indeed, for $k$ odd and $i, j$ even, the condition $\mathcal{W}^{+}_{i}(\infty) = \frac{1}{\mathcal{W}^{-}_{j}(\infty)}$ can be written as

$$\begin{aligned}&\left(W^{\beta}_{p_i(\alpha,\beta)} \circ \cdots \circ W^{\beta}_{p_2(\alpha,\beta)} \circ W^{\alpha}_{p_1(\alpha,\beta)}\right)(\infty)\\ &\qquad = \frac{1}{\left(W^{\alpha}_{p_j(\beta,\alpha)} \circ \cdots \circ W^{\alpha}_{p_2(\beta,\alpha)} \circ W^{\beta}_{p_1(\beta,\alpha)}\right)(\infty)},\end{aligned}$$

$$\left(W^{\beta}_{p_i(\alpha,\beta)} \circ \cdots \circ W^{\beta}_{p_2(\alpha,\beta)} \circ W^{\alpha}_{p_1(\alpha,\beta)}\right)(\infty)$$

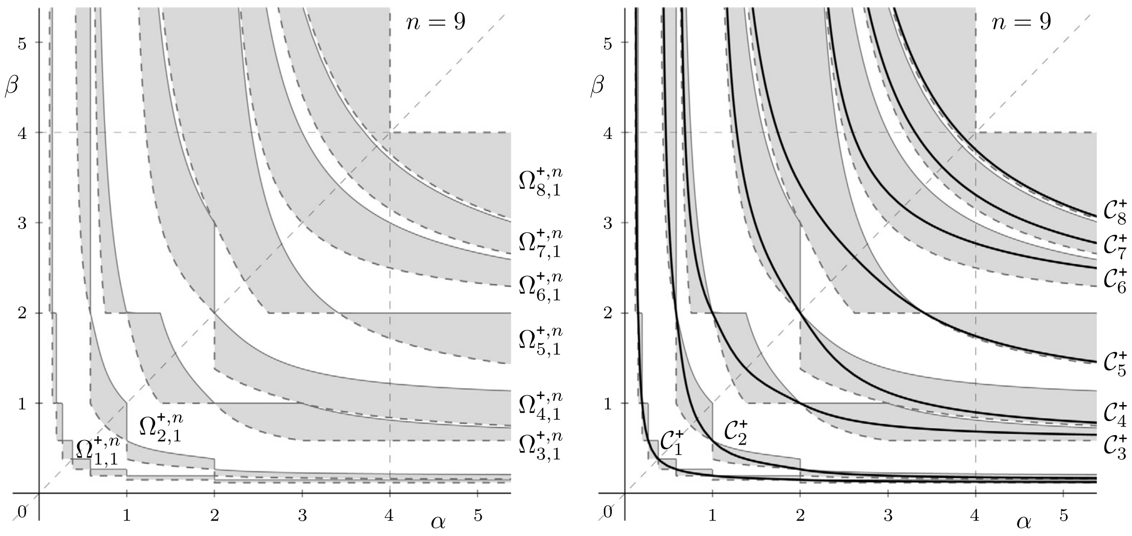


**Fig. 20.** The sets $\Omega^{+,n}_{i,1}$ as grey regions for $n = 9$ (left) and the Fučík curves $\mathcal{C}^+_k$ as black curves (right) due to Theorems 26 and 27.

$$= \left(W^{\alpha}_{-p_j(\beta,\alpha)} \circ \cdots \circ W^{\alpha}_{-p_2(\beta,\alpha)} \circ W^{\beta}_{-p_1(\beta,\alpha)}\right)(0),$$

where we used relation (23) in Lemma 9. Moreover, using (22), we obtain

$$\left(W^{\beta}_{p_1(\beta,\alpha)} \circ W^{\alpha}_{p_2(\beta,\alpha)} \circ \cdots \circ W^{\alpha}_{p_j(\beta,\alpha)} \right.$$
$$\left. \circ W^{\beta}_{p_i(\alpha,\beta)} \circ \cdots \circ W^{\beta}_{p_2(\alpha,\beta)} \circ W^{\alpha}_{p_1(\alpha,\beta)}\right)(\infty) = 0, \tag{51}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \left[W^{\beta}_{p_1(\beta,\alpha)}\right] \cdot \left[W^{\alpha}_{p_2(\beta,\alpha)}\right] \cdots \left[W^{\alpha}_{p_j(\beta,\alpha)}\right]$$
$$\cdot \left[W^{\beta}_{p_i(\alpha,\beta)}\right] \cdots \left[W^{\beta}_{p_2(\alpha,\beta)}\right] \cdot \left[W^{\alpha}_{p_1(\alpha,\beta)}\right] \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0. \tag{52}$$

Using Lemma 12, the condition (52) is exactly $\det(\mathbf{A}^{\mathsf{D}} - \mathbf{\Lambda}) = 0$ with $\mathbf{\Lambda}$ in the form of (50). Similarly, we get that the condition $\mathcal{W}^+_i(\infty) = \frac{1}{\mathcal{W}^+_j(\infty)}$ in (46) reads $\det(\mathbf{A}^{\mathsf{D}} - \mathbf{\Lambda}) = 0$, where

$$\mathbf{\Lambda} = \operatorname{diag}\big(\overbrace{\alpha, \alpha, \ldots\ldots, \alpha}^{p_1(\alpha,\beta)\text{–times}}, \overbrace{\beta, \beta, \ldots\ldots, \beta}^{p_2(\alpha,\beta)\text{–times}}, \ldots, \overbrace{\beta, \beta, \ldots\ldots, \beta}^{p_i(\alpha,\beta)\text{–times}},$$
$$\underbrace{\alpha, \alpha, \ldots\ldots, \alpha}_{p_j(\alpha,\beta)\text{–times}}, \ldots, \underbrace{\beta, \beta, \ldots\ldots, \beta}_{p_2(\alpha,\beta)\text{–times}}, \underbrace{\alpha, \alpha, \ldots\ldots, \alpha}_{p_1(\alpha,\beta)\text{–times}}\big)$$

for $k = i + j - 1$ even, $i$ even and $j$ odd, and

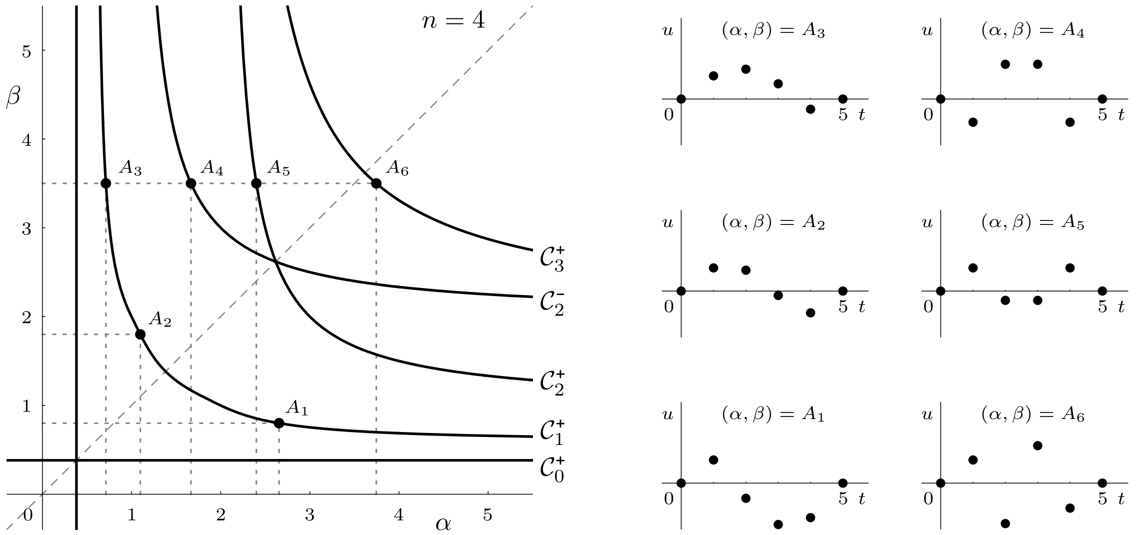


**Fig. 21.** The Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathrm{D}}\right)$ for $n = 4$ (left) and samples of corresponding non-trivial solutions (right).

$$\begin{aligned}\mathbf{\Lambda} = \operatorname{diag}\big(&\overbrace{\alpha, \alpha, \ldots\ldots, \alpha}^{p_1(\alpha,\beta)\text{–times}}, \overbrace{\beta, \beta, \ldots\ldots, \beta}^{p_2(\alpha,\beta)\text{–times}}, \ldots, \overbrace{\alpha, \alpha, \ldots\ldots, \alpha}^{p_i(\alpha,\beta)\text{–times}},\\ &\underbrace{\beta, \beta, \ldots\ldots, \beta}_{p_j(\alpha,\beta)\text{–times}}, \ldots, \underbrace{\beta, \beta, \ldots\ldots, \beta}_{p_2(\alpha,\beta)\text{–times}}, \underbrace{\alpha, \alpha, \ldots\ldots, \alpha}_{p_1(\alpha,\beta)\text{–times}}\big)\end{aligned}$$

for $k = i + j - 1$ even, $i$ odd and $j$ even.

In the following last example we consider $n = 4$ and discuss specific representation for each Fučík curve.

**Example 29.** In this example, let us consider $n = 4$. Thus, we have $4 \times 4$ Dirichlet matrix

$$\mathbf{A}^{\mathrm{D}} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

with the eigenvalues

$$\lambda_0^{\mathrm{D}} = \tfrac{1}{2}(3 - \sqrt{5}), \quad \lambda_1^{\mathrm{D}} = \tfrac{1}{2}(5 - \sqrt{5}), \quad \lambda_2^{\mathrm{D}} = \tfrac{1}{2}(3 + \sqrt{5}), \quad \lambda_3^{\mathrm{D}} = \tfrac{1}{2}(5 + \sqrt{5}).$$

See Fig. 21 for the Fučík spectrum of $\mathbf{A}^{\mathrm{D}}$ (left) and for some corresponding non-trivial solutions (right). Now, we apply Theorem 27 to get the following results. At first, as for the first non-trivial Fučík curve $\mathcal{C}_1^+$, we focus on all pairs $(\alpha, \beta) \in \Omega_{1,1}^{+,n} = \{(\alpha, \beta) \in D : \; p_1(\alpha, \beta) + p_1(\beta, \alpha) = 4\}$, for which the condition

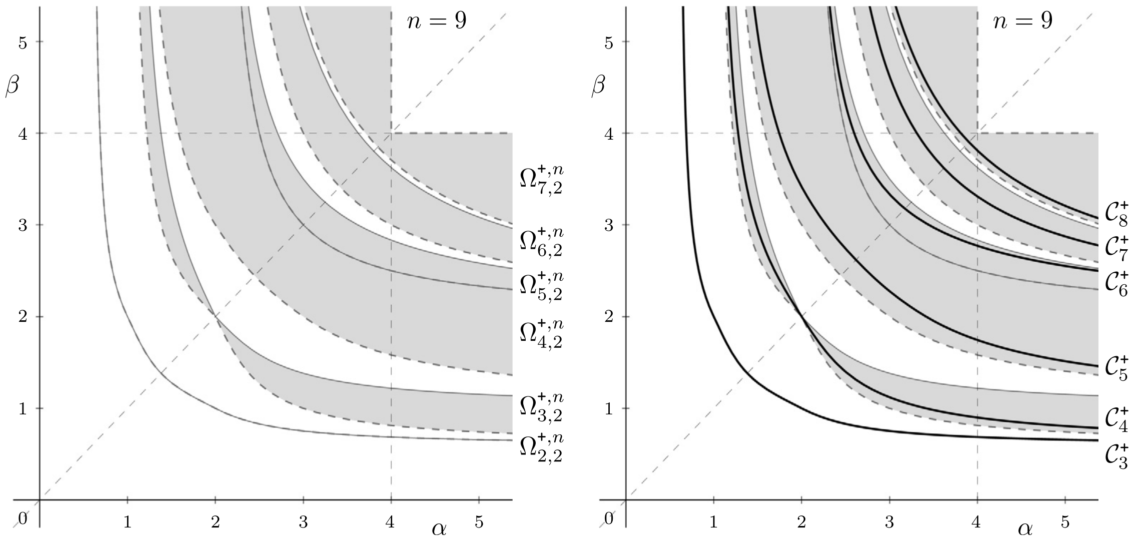


**Fig. 22.** The sets $\Omega^{+;n}_{i,2}$ as grey regions for $n = 9$ (left) and the Fučík curves $\mathcal{C}^+_k$ as black curves (right) due to Theorems 26 and 27.

$$\mathcal{W}^+_1(\infty) = \frac{1}{\mathcal{W}^-_1(\infty)} \tag{53}$$

is satisfied. Since $p_1(\alpha,\beta), p_1(\beta,\alpha) \in \mathbb{N}$, there are exactly three following cases (recall Remark 28).

1. For $p_1(\alpha,\beta) = 1$, $p_1(\beta,\alpha) = 3$, the condition (53) reads

$$\begin{aligned} \det(\mathbf{A}^{\mathsf{D}} - \operatorname{diag}(\alpha,\beta,\beta,\beta)) &= 0, \\ \alpha\beta^3 - 2\beta^3 - 6\alpha\beta^2 + 11\beta^2 + 10\alpha\beta - 4\alpha - 16\beta + 5 &= 0, \\ V^\alpha_1 V^\beta_3 - V^\alpha_0 V^\beta_2 &= 0. \end{aligned}$$

2. For $p_1(\alpha,\beta) = 2$, $p_1(\beta,\alpha) = 2$, the condition (53) has the following form

$$\begin{aligned} \det(\mathbf{A}^{\mathsf{D}} - \operatorname{diag}(\alpha,\alpha,\beta,\beta)) &= 0, \\ \alpha^2\beta^2 - 4\alpha^2\beta - 4\alpha\beta^2 + 3\alpha^2 + 15\alpha\beta + 3\beta^2 - 10\alpha - 10\beta + 5 &= 0, \\ V^\alpha_2 V^\beta_2 - V^\alpha_1 V^\beta_1 &= 0. \end{aligned}$$

3. For $p_1(\alpha,\beta) = 3$, $p_1(\beta,\alpha) = 1$, the condition (53) reads

$$\begin{aligned} \det(\mathbf{A}^{\mathsf{D}} - \operatorname{diag}(\alpha,\alpha,\alpha,\beta)) &= 0, \\ \alpha^3\beta - 2\alpha^3 - 6\alpha^2\beta + 11\alpha^2 + 10\alpha\beta - 16\alpha - 4\beta + 5 &= 0, \\ V^\alpha_3 V^\beta_1 - V^\alpha_2 V^\beta_0 &= 0. \end{aligned}$$

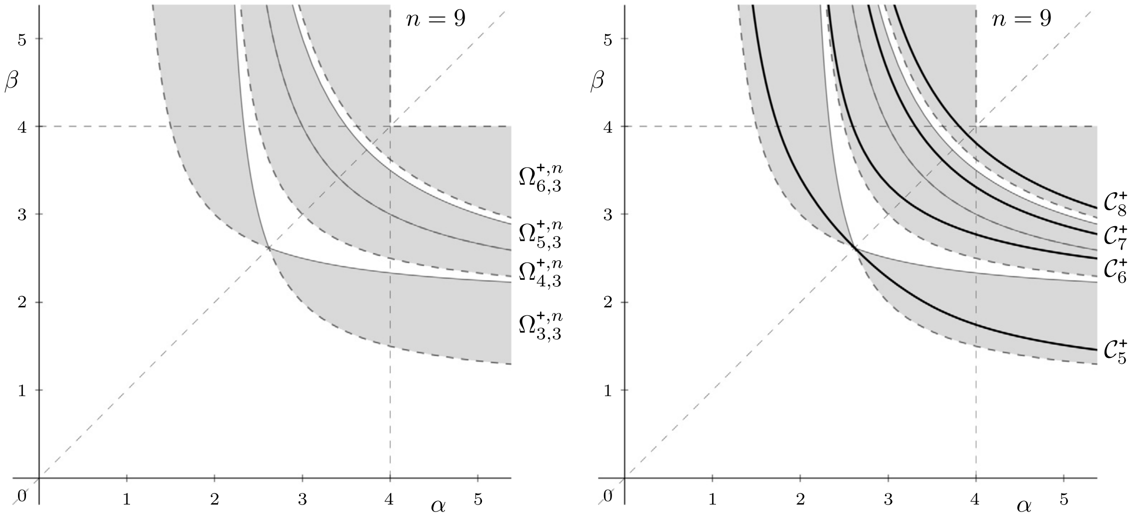


**Fig. 23.** The sets $\Omega^{+,n}_{i,3}$ as grey regions for $n = 9$ (left) and the Fučík curves $\mathcal{C}^+_k$ as black curves (right) due to Theorems 26 and 27.

Secondly, as for the second non-trivial Fučík curve $\mathcal{C}^+_2$, we focus on all pairs $(\alpha, \beta) \in \Omega^{+,n}_{1,2} = \Omega^{+,n}_{2,1} = \{(\alpha, \beta) \in D : \; 2p_1(\alpha, \beta) + p_2(\alpha, \beta) = 4\}$, for which the condition

$$\mathcal{W}^+_1(\infty) = \frac{1}{\mathcal{W}^+_2(\infty)} \tag{54}$$

is satisfied. Since $p_1(\alpha, \beta), p_2(\alpha, \beta) \in \mathbb{N}$, we have that $p_1(\alpha, \beta) = 1$ and $p_2(\alpha, \beta) = 2$ and the condition (54) reads (recall Remark 28)

$$\begin{aligned} \det(\mathbf{A}^{\mathrm{D}} - \operatorname{diag}(\alpha, \beta, \beta, \alpha)) &= 0, \\ \alpha^2\beta^2 - 4\alpha^2\beta - 4\alpha\beta^2 + 3\alpha^2 + 14\alpha\beta + 4\beta^2 - 8\alpha - 12\beta + 5 &= 0, \\ V^\alpha_1 V^\alpha_1 V^\beta_2 - 2V^\alpha_1 V^\beta_1 + 1 &= 0. \end{aligned} \tag{55}$$

Moreover, using $V^\beta_2 = V^\beta_1 V^\beta_1 - 1$, the equality (55) can be simplified as

$$\begin{aligned} (V^\alpha_1 V^\beta_1 - V^\alpha_1 - 1)(V^\alpha_1 V^\beta_1 + V^\alpha_1 - 1) &= 0, \\ (\alpha\beta - 2\beta - \alpha + 1)(\alpha\beta - 2\beta - 3\alpha + 5) &= 0. \end{aligned}$$

Finally, as for the last non-trivial Fučík curve $\mathcal{C}^+_3$, we focus on all pairs $(\alpha, \beta)$ from sets

$$\begin{aligned} \Omega^{+,n}_{1,3} &= \{(\alpha, \beta) \in D : \; p_1(\alpha, \beta) + p_1(\beta, \alpha) + p_2(\beta, \alpha) + p_3(\beta, \alpha) = 4\}, \\ \Omega^{+,n}_{2,2} &= \{(\alpha, \beta) \in D : \; p_1(\alpha, \beta) + p_2(\alpha, \beta) + p_1(\beta, \alpha) + p_2(\beta, \alpha) = 4\}, \\ \Omega^{+,n}_{3,1} &= \{(\alpha, \beta) \in D : \; p_1(\alpha, \beta) + p_2(\alpha, \beta) + p_3(\alpha, \beta) + p_1(\beta, \alpha) = 4\}, \end{aligned}$$

for which one of the following conditions is satisfied, in particular,

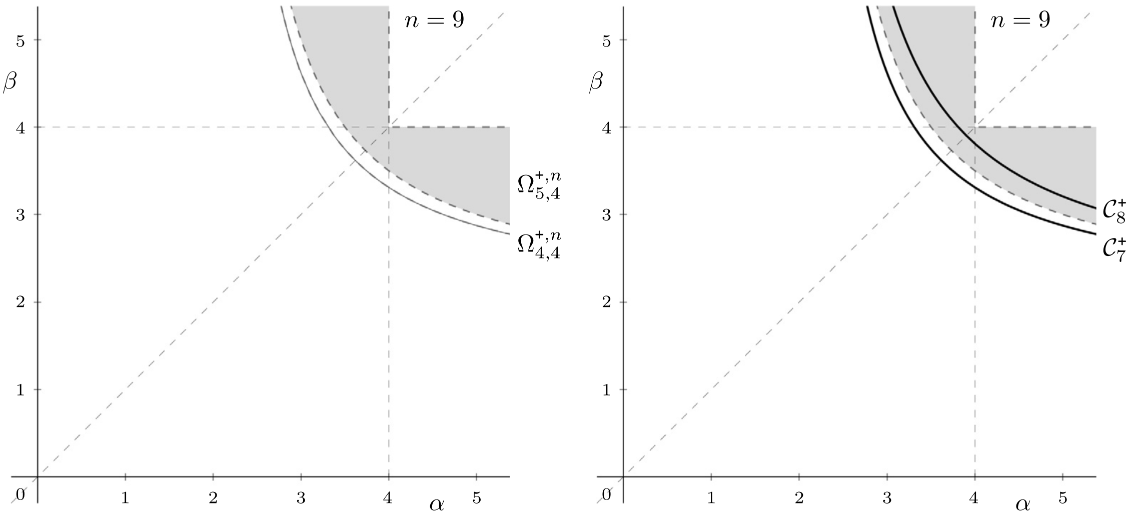


**Fig. 24.** The sets $\Omega^{+,n}_{i,4}$ as grey regions for $n = 9$ (left) and the Fučík curves $\mathcal{C}^{+}_{k}$ as black curves (right) due to Theorems 26 and 27.

$$\mathcal{W}^{+}_{1}(\infty) = \frac{1}{\mathcal{W}^{-}_{3}(\infty)}, \qquad \mathcal{W}^{+}_{2}(\infty) = \frac{1}{\mathcal{W}^{-}_{2}(\infty)}, \qquad \mathcal{W}^{+}_{3}(\infty) = \frac{1}{\mathcal{W}^{-}_{1}(\infty)}. \tag{56}$$

For $j = 1, 2, 3$, $p_j(\alpha, \beta)$ and $p_j(\beta, \alpha)$ are positive integers, which implies that

$$p_1(\alpha, \beta) = p_2(\alpha, \beta) = p_3(\alpha, \beta) = p_1(\beta, \alpha) = p_2(\beta, \alpha) = p_3(\beta, \alpha) = 1,$$

and thus, each condition in (56) can be simplified to the following form (recall Remark 28)

$$\begin{aligned} \det(\mathbf{A}^{\mathtt{D}} - \operatorname{diag}(\alpha, \beta, \alpha, \beta)) &= 0, \\ \alpha^2\beta^2 - 4\alpha^2\beta - 4\alpha\beta^2 + 4\alpha^2 + 13\alpha\beta + 4\beta^2 - 10\alpha - 10\beta + 5 &= 0, \\ V^{\alpha}_{1} V^{\alpha}_{1} V^{\beta}_{1} V^{\beta}_{1} - 3V^{\alpha}_{1} V^{\beta}_{1} + 1 &= 0. \end{aligned}$$

In the final corollary, we reveal the algebraic structure of the first non-trivial Fučík curve $\mathcal{C}^{\pm}_{1}$ in detail.

**Corollary 30.** *For the first non-trivial Fučík curve, $n \in \mathbb{N}$, $n \geq 3$, we have that*

$$\mathcal{C}^{+}_{1} = \mathcal{C}^{-}_{1} = \mathcal{C}^{\mathtt{P}}_{1} \cup \mathcal{C}^{\mathtt{C}}_{1},$$

*where*

$$\begin{aligned} \mathcal{C}^{\mathtt{P}}_{1} &:= \bigcup_{i=1}^{n-2} \Big\{ (\xi_{n-i}, \xi_{i+1}) \Big\}, \qquad \xi_k := 4\sin^2 \frac{\pi}{2k}, \quad k = 2, \dots, n, \\ \mathcal{C}^{\mathtt{C}}_{1} &:= \bigcup_{i=1}^{n-1} \Big\{ (\alpha, \beta) \in \Omega_i : \; W^{\alpha}_{n-i}(\infty) \cdot W^{\beta}_{i}(\infty) = 1 \Big\}, \end{aligned}$$

*and*

$$\begin{aligned}
\Omega_1 &:= (\xi_n, \xi_{n-1}) \times (\xi_2, +\infty)\,, \\
\Omega_i &:= (\xi_{n-i+1}, \xi_{n-i}) \times (\xi_{i+1}, \xi_i)\,, \quad i = 2, \dots, n-2 \quad \textit{for } n \geq 4, \\
\Omega_{n-1} &:= (\xi_2, +\infty) \times (\xi_n, \xi_{n-1})\,.
\end{aligned}$$

**Proof.** According to Theorem 27, we have the following characterization of the first non-trivial Fučík curve

$$\mathcal{C}_1^+ = \left\{(\alpha,\beta) \in \Omega_{1,1}^{+,n} :\ \mathcal{W}_1^+(\infty) = \frac{1}{\mathcal{W}_1^-(\infty)}\right\} \cup \\ \left\{(\alpha,\beta) \in \Omega_{1,1}^{+,n+1} :\ \mathcal{W}_1^+(\infty) = \infty = \mathcal{W}_1^-(\infty)\right\}.$$

Firstly, we detect all points $(\alpha,\beta) \in \Omega_{1,1}^{+,n+1}$ such that $\mathcal{W}_1^+(\infty) = \infty = \mathcal{W}_1^-(\infty)$. Thus, we look for all $\alpha, \beta > 0$ such that

$$p_1(\alpha,\beta) + p_1(\beta,\alpha) = n+1, \qquad W^{\alpha}_{p_1(\alpha,\beta)}(\infty) = \infty, \qquad W^{\beta}_{p_1(\beta,\alpha)}(\infty) = \infty, \tag{57}$$

where

$$p_1(\alpha,\beta) = \begin{cases} \left\lfloor \frac{\pi}{\omega_\alpha} \right\rfloor & \text{for } 0 < \alpha < 4,\ \beta > 0, \\ \left\lfloor T^{\beta}(2-\alpha) + 1 \right\rfloor & \text{for } \alpha \geq 4,\ 0 < \beta < 4, \end{cases} \tag{58}$$

$$p_1(\beta,\alpha) = \begin{cases} \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor & \text{for } \alpha > 0,\ 0 < \beta < 4, \\ \left\lfloor T^{\alpha}(2-\beta) + 1 \right\rfloor & \text{for } 0 < \alpha < 4,\ \beta \geq 4. \end{cases} \tag{59}$$

Now, if we denote $i := p_1(\beta,\alpha) - 1$ then (57) reads

$$p_1(\alpha,\beta) = n - i, \qquad W^{\alpha}_{n-i}(\infty) = \infty, \tag{60}$$

$$p_1(\beta,\alpha) = i + 1, \qquad W^{\beta}_{i+1}(\infty) = \infty. \tag{61}$$

Both conditions in (60) imply that $i \leq n-2$ and $\frac{\pi}{\omega_\alpha} = n - i$, i.e. $\alpha = \xi_{n-i}$. On the other hand, conditions in (61) imply that $i \geq 1$ and $\frac{\pi}{\omega_\beta} = i + 1$, which means that $\beta = \xi_{i+1}$.

Secondly, we determine all points $(\alpha,\beta) \in \Omega_{1,1}^{+,n}$ such that $\mathcal{W}_1^+(\infty) = \frac{1}{\mathcal{W}_1^-(\infty)}$. Thus, we look for all $\alpha, \beta > 0$ such that

$$p_1(\alpha,\beta) + p_1(\beta,\alpha) = n, \qquad W^{\alpha}_{p_1(\alpha,\beta)}(\infty) = \frac{1}{W^{\beta}_{p_1(\beta,\alpha)}(\infty)}, \tag{62}$$

where $p_1(\alpha,\beta)$ and $p_1(\beta,\alpha)$ are given by (58) and (59). If we denote $i := p_1(\beta,\alpha)$ then $1 \le i \le n-1$, $p_1(\alpha,\beta) = n-i$ and the second equation in (62) reads

$$W^{\alpha}_{n-i}(\infty) \cdot W^{\beta}_{i}(\infty) = 1.$$

The condition $p_1(\beta,\alpha) = i$ implies that (recall the basic properties of $\omega_\beta$ and $T^\alpha$ and note that for $i \ge 2$, we have that $\frac{\pi}{\omega_\beta} = i$ if and only if $\beta = \xi_i$)

$$\begin{aligned} \xi_2 < \beta \qquad & \text{for } i = 1, \\ \xi_{i+1} < \beta \le \xi_i \qquad & \text{for } i \ge 2. \end{aligned}$$

Finally, the condition $p_1(\alpha,\beta) = n-i$ implies that

$$\begin{aligned} \xi_2 < \alpha \qquad & \text{for } i = n-1, \\ \xi_{n-i+1} < \alpha \le \xi_{n-i} \qquad & \text{for } i \le n-2, \end{aligned}$$

which finishes the proof. □

## 5. Conclusion

In this paper, we introduced a new approach how to investigate the Fučík spectrum for the discrete Dirichlet operator of the second order, which allowed us to reveal its algebraic structure. We started with the initial value problems (9) and (29) and we discussed properties of their solutions. A suitable continuous extension of the discrete solution was used to localize all its generalized zeros. We defined recurrently two sequences of functions $(p_i)$ and $(\vartheta_i)$ (recall Definition 17) in order to localize all generalized zeros and to obtain several descriptions of particular Fučík curves. Thus, we introduced various analytic implicit formulas for Fučík curves and we also identified sets, where these curves are localized. Let us point out that our description of the Fučík spectrum has the form of necessary and sufficient conditions.

Approach presented in this paper can be directly applied for other discrete operators of the second order as well (e.g. with Neumann or mixed boundary conditions) and provides a new way how to deal with the discrete Fučík spectrum problems.

## Acknowledgement

The authors were supported by the Grant Agency of the Czech Republic, grant no. 13-00863S.

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Bull. Sci. math. 171 (2021) 103014

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Bulletin des Sciences Mathématiques

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# Localization of Fučík curves for the second order discrete Dirichlet operator

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A R T I C L E I N F O

*Article history:*
Received 25 January 2021
Available online 16 June 2021

*MSC:*
39A12
39A70
34B15

*Keywords:*
Asymmetric nonlinearities
Fučík spectrum
Difference operators
Chebyshev polynomials of the second kind
Möbius transformation

A B S T R A C T

In this paper, we deal with the second order difference equation with asymmetric nonlinearities on the integer lattice and we investigate the distribution of zeros for continuous extensions of positive semi-waves. The distance between two consecutive zeros of two different positive semi-waves depends not only on the parameters of the problem but also on the position of one of these zeros with respect to the integer lattice. We provide an explicit formula for this distance, which allows us to obtain a new simple implicit description of all non-trivial Fučík curves for the discrete Dirichlet operator. Moreover, for fixed parameters of the problem, we show that this distance is bounded and attains its global extrema that are explicitly described in terms of Chebyshev polynomials of the second kind. Finally, for each non-trivial Fučík curve, we provide suitable bounds by two curves with a simple description similar to the description of the first non-trivial Fučík curve.



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https://doi.org/10.1016/j.bulsci.2021.103014
0007-4497/

## 1. Introduction

In 1976, two papers [10] by Fučík and [5] by Dancer were published concerning the solvability of the following Dirichlet problem

$$\begin{cases} v''(x) + g(v(x)) = f(x), \quad x \in (0,1), \\ v(0) = v(1) = 0, \end{cases} \tag{1}$$

where $g$ is a jumping nonlinearity, i.e., $\lim\limits_{s\to-\infty} \frac{g(s)}{s} =: a \neq b := \lim\limits_{s\to+\infty} \frac{g(s)}{s}$. Both authors independently recognized that the solvability of the problem (1) depends strongly on the fact if there exists a non-trivial solution $v$ of the following problem

$$\begin{cases} v''(x) + av^+(x) - bv^-(x) = 0, \quad x \in (0,1), \\ v(0) = v(1) = 0, \end{cases} \tag{2}$$

where $v^+$ and $v^-$ are the positive and negative parts of $v$, respectively, i.e. $v^\pm(x) := \max\{\pm v(x), 0\}$. The following set

$$\Sigma^{\mathsf{c}} := \left\{(a,b) \in \mathbb{R}^2 : \text{ the problem (2) has a non-trivial solution } v\right\}$$

is usually called as the Fučík spectrum for (2) and can be expressed analytically in the following way (see [10,11]). The Fučík spectrum $\Sigma^{\mathsf{c}}$ consists of two lines $\mathsf{C}_0^\pm : \left(a - \pi^2\right) \cdot \left(b - \pi^2\right) = 0$ and countably many curves $\mathsf{C}_l^\pm$ (see Fig. 1, left) given by $(j \in \mathbb{N})$

$$\mathsf{C}_{2j-1}^\pm : \frac{j\pi}{\sqrt{a}} + \frac{j\pi}{\sqrt{b}} = 1, \qquad \mathsf{C}_{2j}^+ : \frac{(j+1)\pi}{\sqrt{a}} + \frac{j\pi}{\sqrt{b}} = 1, \qquad \mathsf{C}_{2j}^- : \frac{j\pi}{\sqrt{a}} + \frac{(j+1)\pi}{\sqrt{b}} = 1. \tag{3}$$

Let us note that for a pair $(a,b) \in \mathsf{C}_l^\pm$, the corresponding non-trivial solution $v$ of (2) has exactly $l$ zeros in $(0,1)$ and consists of positive and negative semi-waves of lengths $\frac{\pi}{\sqrt{a}}$ and $\frac{\pi}{\sqrt{b}}$, respectively (see Fig. 1, right).

In 1987, Lazer and McKenna introduced a new nonlinear model of a suspension bridge using the *asymmetric* nonlinearity $g(v) = kv^+$ to describe supporting cable stays as one-sided springs which do not exert restoring force if they are compressed. They studied periodic solutions of such asymmetric systems and showed in [17] that a sufficiently large asymmetry in the system leads to large oscillations which cannot be predicted by the linear theory. In [8], authors consider the following normalized symmetric model of the vertical motion of a suspension bridge

$$\begin{cases} v_{tt}(x,t) + v_{xxxx}(x,t) + kv^+(x,t) = f(x,t) \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\ v\left(\pm\frac{\pi}{2}, t\right) = v_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \quad t \in \mathbb{R}, \\ v(x,t) = v(-x,t) = v(x,-t) = v(x,t+T), \end{cases} \tag{4}$$

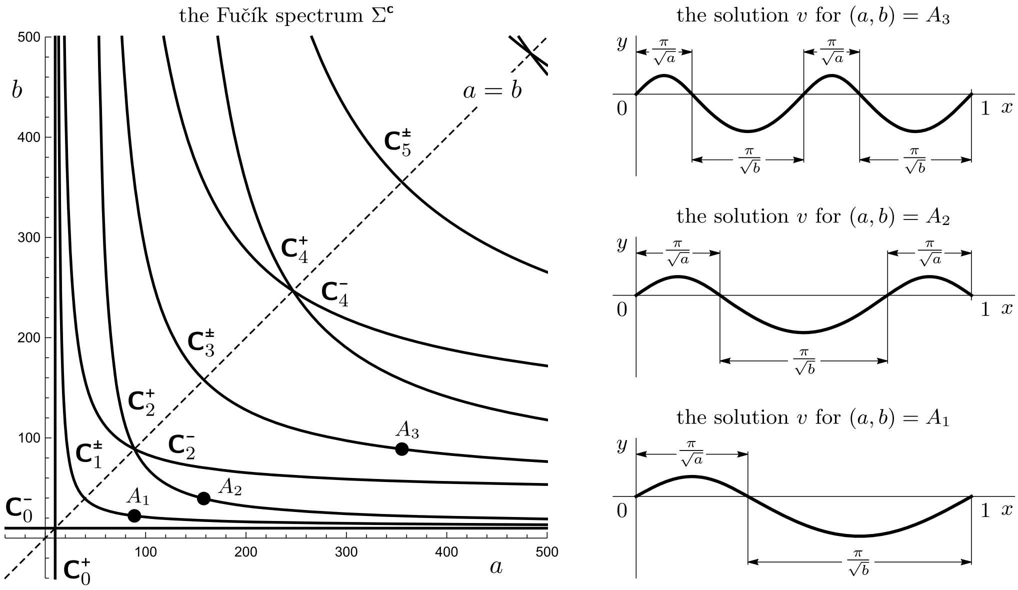


**Fig. 1.** The Fučík spectrum $\Sigma^{\mathbf{c}}$ (left) for the continuous problem (2) given by countable many Fučík curves $\mathbf{C}_l^{\pm}$, $l \in \mathbb{N} \cup \{0\}$, and corresponding non-trivial solutions $v$ (right) for three different pairs $(a, b)$ as points $A_1 \in \mathbf{C}_1^{+}$, $A_2 \in \mathbf{C}_2^{+}$ and $A_3 \in \mathbf{C}_3^{+}$, where $A_1 = \left(9\pi^2, \frac{9}{4}\pi^2\right)$, $A_2 = \left(16\pi^2, 4\pi^2\right)$ and $A_3 = \left(36\pi^2, 9\pi^2\right)$.

and investigate its set of solutions $v$ that are $T$-periodic in the second variable. For special right hand sides $f$ and $T > \pi$, they show that it has very rich set of non-stationary solutions with blow up points in the sense that for bounded values of the parameter $k$ there are non-stationary solutions of (4) with the amplitude approaching infinity. Let us point out that the blow up points are determined by the Fučík spectrum of the beam operator $v \mapsto -(v_{tt} + v_{xxxx})$ with the boundary conditions given in (4). However, the knowledge of the Fučík spectrum of this operator seems to be a hard problem. For other one dimensional models of suspension bridges, we recommend the reader the book [12] by Gazzola with a focus on Subchapter 2.8 concerning models with asymmetric nonlinearities. Finally, let us note that asymmetric nonlinearities also surprisingly appear in the study of competing systems of species with large interactions in biology (see [4,6,22]) and the Fučík spectrum of the Dirichlet Laplacian (the Laplace operator $u \mapsto -\Delta u$ with zero Dirichlet boundary conditions) is needed (see [6] for details).

Nowadays, there are a number of papers in which authors study the structure of the Fučík spectrum for particular linear differential operators, let us mention here only some of them: [1,2,7,9,14,23,24] for the Dirichlet Laplacian on bounded domains, [3,13,15,16, 26,27] for the ordinary differential operators with various boundary conditions (Dirichlet, Neumann, Robin, Navier, periodic, multipoint, integral type).

In [22] and [25], authors consider a finite dimensional nonlinear matrix-vector equation

$$\mathbf{A}\mathbf{u} = g(\mathbf{u}), \tag{5}$$

where $\mathbf{A}$ is an $n \times n$ matrix and $g : \mathbb{R}^n \to \mathbb{R}^n$ is mildly nonlinear, i.e. $g(\mathbf{u}) = a\mathbf{u}^+ - b\mathbf{u}^- + h(\mathbf{u})$, where $h : \mathbb{R}^n \to \mathbb{R}^n$ is such that $\lim\limits_{\|\mathbf{u}\| \to +\infty} \frac{h(\mathbf{u})}{\|\mathbf{u}\|} = \mathbf{0}$. Equations of this type (5) can represent numerical approximations of continuous boundary value problems describing nonlinear oscillations in asymmetric systems such as suspension bridges (see [25] and [18]). The Fučík spectrum of the matrix $\mathbf{A}$ is defined as the set of all pairs $(a, b) \in \mathbb{R}^2$ such that the problem $\mathbf{A}\mathbf{u} = a\mathbf{u}^+ - b\mathbf{u}^-$ has a non-trivial solution $\mathbf{u}$ and plays an important role in questions of the solvability of the discrete equation (5). More precisely, in [22], the solvability of (5) is provided in the so-called *nonresonance* case when the point $(a, b)$ is not in the Fučík spectrum of $\mathbf{A}$ and can be connected by a continuous curve to a point $(\lambda, \lambda)$ on the diagonal $a = b$ such that this curve belongs to the complement of the Fučík spectrum. In [25], authors investigate the Fučík spectrum of the following tridiagonal persymmetric matrix $(\delta \geq 0)$

$$\mathbf{A}_\delta := \begin{bmatrix} 2+\delta & -(1+\delta) & & & \\ -1 & 2+\delta & -(1+\delta) & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2+\delta & -(1+\delta) \\ & & & -1 & 2+\delta \end{bmatrix} \tag{6}$$

which represents a discrete approximation of the differential operator $u \mapsto -(u'' + \delta u')$ with zero Dirichlet boundary conditions. Moreover, the solvability of (5) is investigated in both the *resonance* and *nonresonance* case, i.e. when the point $(a, b)$ is, or is not in the Fučík spectrum of $\mathbf{A}_\delta$. Finally, at the end of the paper [25], authors leave the reader with two interesting problems and one of them is to determine a complete description of the Fučík spectrum of the $n \times n$ matrix $\mathbf{A}_\delta$ for $n \geq 3$. In the special case of $\delta = 0$, the Fučík spectrum of $\mathbf{A}_0$ has been also studied in [20–22,28] and let us note that its known description for $n \geq 3$ is rather more complicated in comparison to the simple description of Fučík curves $\mathsf{C}_l^\pm$ given in (3) for the continuous problem (2).

In this paper, we continue in studying the Fučík spectrum of the matrix $\mathbf{A}_0$ given in (6) for $\delta = 0$ and thus, we deal with the following discrete Dirichlet problem

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, \quad k \in \mathbb{T}, \\ u(0) = u(n+1) = 0, \end{cases} \tag{7}$$

where $\mathbb{T} := \{1, \ldots, n\}$, $n \in \mathbb{N}$, and $u : \widehat{\mathbb{T}} \to \mathbb{R}$, $\widehat{\mathbb{T}} := \mathbb{T} \cup \{0, n+1\}$. Moreover, $\alpha, \beta \in \mathbb{R}$, $\Delta^2$ denotes the second order forward difference operator, i.e.

$$\Delta^2 u(k-1) := u(k-1) - 2u(k) + u(k+1),$$

$u^\pm : \widehat{\mathbb{T}} \to \mathbb{R}$ are positive and negative parts of $u$, i.e. $u^\pm(k) := \max\{\pm u(k), 0\}$. The aim of this paper is to investigate the Fučík spectrum for the problem (7) as the set

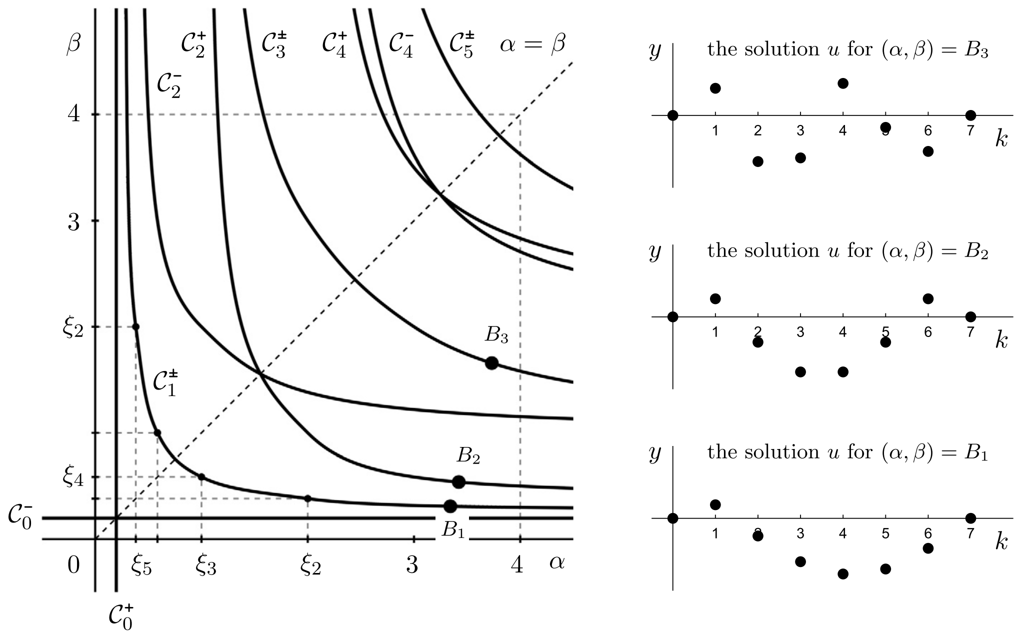


**Fig. 2.** The Fučík spectrum $\Sigma$ (left) for the discrete problem (7) given by twelve Fučík curves $\mathcal{C}_0^\pm$, $\mathcal{C}_1^\pm$, $\mathcal{C}_2^\pm$, $\mathcal{C}_3^\pm$, $\mathcal{C}_4^\pm$, $\mathcal{C}_5^\pm$ in the case of $n = 6$ (note that $\mathcal{C}_1^+ = \mathcal{C}_1^-$, $\mathcal{C}_3^+ = \mathcal{C}_3^-$ and $\mathcal{C}_5^+ = \mathcal{C}_5^-$) and corresponding non-trivial solutions $u$ (right) for three different pairs $(\alpha, \beta)$ as points $B_1 \in \mathcal{C}_1^+$, $B_2 \in \mathcal{C}_2^+$ and $B_3 \in \mathcal{C}_3^+$, where $B_1 \doteq (3.342, 0.309)$, $B_2 \doteq (3.421, 0.538)$ and $B_3 \doteq (3.732, 1.657)$.

$$\Sigma := \left\{(\alpha, \beta) \in \mathbb{R}^2 : \text{ the problem (7) has a non-trivial solution } u\right\}.$$

Let us note that the set $\Sigma$ is exactly the Fučík spectrum of the matrix $\mathbf{A}_0$.

Let us briefly recall some known results concerning the set $\Sigma$ (for a more detailed overview see the first section in [20]). The Fučík spectrum consists of a finite number of algebraic curves (see Fig. 2)

$$\Sigma = \bigcup_{l=0}^{n-1} \left(\mathcal{C}_l^+ \cup \mathcal{C}_l^-\right),$$

where

$$\mathcal{C}_l^\pm := \Big\{(\alpha, \beta) \in \mathbb{R}^2 : \quad \text{the problem (7) has a non-trivial solution } u \\ \text{with exactly } l \text{ generalized zeros on } \mathbb{T} \text{ and } u(1) \gtrless 0\Big\}.$$

Let us note that $j \in \mathbb{T}$ is a *generalized zero* of the solution $u$ of (7) if $u(j) = 0$ or $u(j)u(j-1) < 0$. Fučík curves $\mathcal{C}_0^\pm$ are trivial ones (lines $\alpha = \lambda_0$ and $\beta = \lambda_0$, where $\lambda_0 := 4\sin^2 \frac{\pi}{2(n+1)}$), each non-trivial Fučík curve $\mathcal{C}_l^\pm$, $l \in \{1, \ldots, n-1\}$ is located in the domain $D := ((0, 4) \times (0, +\infty)) \cup ((0, +\infty) \times (0, 4))$. For $\alpha = \beta = \lambda$, the problem (7) is a linear one and thus, it is straightforward to verify that it has a non-trivial solution $u$ if and only if $\lambda = \lambda_j := 4\sin^2 \frac{(j+1)\pi}{2(n+1)}$, $j = 0, \ldots, n-1$. Moreover, the corresponding

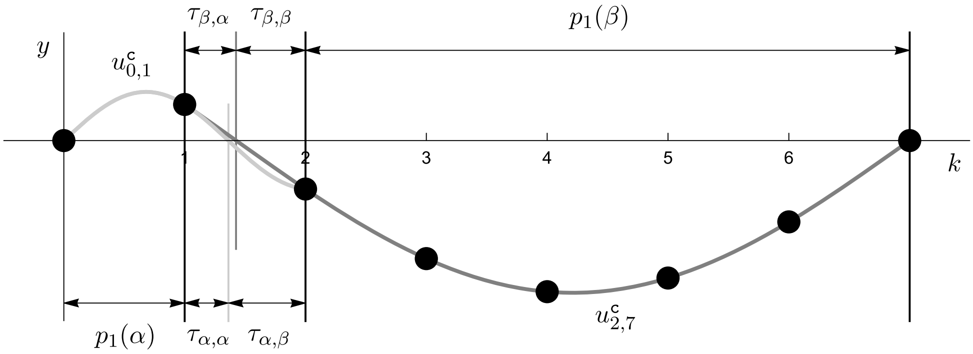


**Fig. 3.** The geometry of the discrete solution $u$ of (7) for $(\alpha,\beta) = B_1 \in \mathcal{C}_1^{\pm}$, where $B_1 \doteq (3.342, 0.309)$. The solution $u$ has one generalized zero at $j = 2$ and two continuous extensions $u^{\mathsf{c}}_{0,1}$ and $u^{\mathsf{c}}_{2,7}$.

non-trivial solution is $u_j(k) = \sin(\omega_{\lambda_j} k)/\sin\omega_{\lambda_j}$, where $\omega_{\lambda_j} := \arccos\frac{2-\lambda_j}{2}$, and thus, the point $(\lambda_j, \lambda_j)$ on the diagonal $\alpha = \beta$ belongs to both Fučík curves $\mathcal{C}_j^{+}$ and $\mathcal{C}_j^{-}$.

The qualitative properties of the first non-trivial Fučík curve $\mathcal{C}_1^{\pm}$ were studied in [20,28]. In [28], a conjecture is stated that $\mathcal{C}_1^{\pm}$ has no elementary parametrization and possible ways to prove it are also discussed. On the other hand, in [20], it is shown that the first non-trivial Fučík curve $\mathcal{C}_1^{\pm}$ has an elementary parametrization for $n \le 7$. The reason is that it is possible to provide the implicit description of $\mathcal{C}_1^{\pm}$ in terms of Chebyshev polynomials of the second kind. More precisely, due to Corollary 30 in [20], the first non-trivial Fučík curve $\mathcal{C}_1^{\pm}$ consists of the following $(n-1)$ algebraic curves in prescribed rectangles

$$
\begin{aligned}
V^{\alpha}_{n-1}\cdot(2-\beta) - V^{\alpha}_{n-2} &= 0 && \text{for } (\alpha,\beta) \in (\xi_n, \xi_{n-1}) \times (\xi_2, +\infty)\,,\\
V^{\alpha}_{n-i}\cdot V^{\beta}_{i} - V^{\alpha}_{n-i-1}\cdot V^{\beta}_{i-1} &= 0 && \text{for } (\alpha,\beta) \in (\xi_{n-i+1}, \xi_{n-i}) \times (\xi_{i+1}, \xi_i)\,,\\
& && i = 2,\dots,n-2,\\
(2-\alpha)\cdot V^{\beta}_{n-1} - V^{\beta}_{n-2} &= 0 && \text{for } (\alpha,\beta) \in (\xi_2, +\infty) \times (\xi_n, \xi_{n-1})\,,
\end{aligned}
$$

where $V^{\alpha}_k$ and $V^{\beta}_k$ are defined by the Chebyshev polynomial $U_k = U_k(x)$ of the second kind of degree $k$

$$V^{\lambda}_k := U_k\left(\tfrac{2-\lambda}{2}\right), \quad k \in \mathbb{Z},\ \lambda \in \mathbb{R}, \tag{8}$$

and the values $\xi_k$ for $k = 2,\dots,n$ are given by the formula

$$\xi_k := 4\sin^2\frac{\pi}{2k}, \quad k \in \mathbb{N}. \tag{9}$$

Moreover, the first non-trivial Fučík curve $\mathcal{C}_1^{\pm}$ contains also $(n-2)$ points $(\xi_{n-i}, \xi_{i+1})$, $i = 1,\dots,n-2$ (see Fig. 2 for the Fučík curve $\mathcal{C}_1^{\pm}$ in the case of $n = 6$, which consists of four points and five algebraic curves).

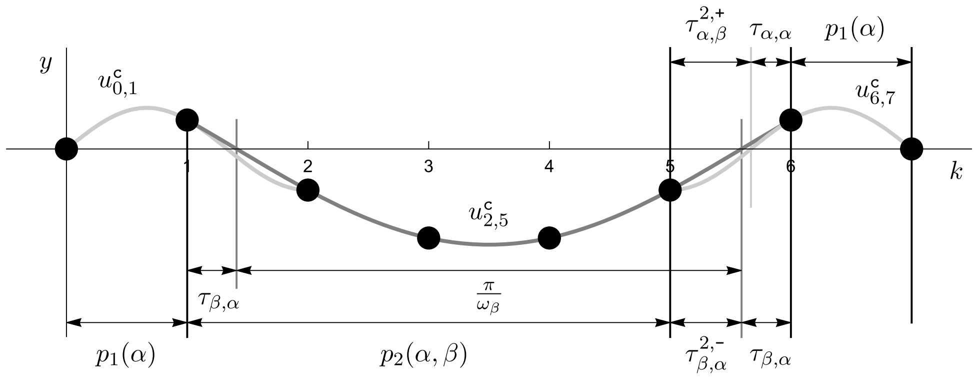


**Fig. 4.** The geometry of the discrete solution $u$ of (7) for $(\alpha,\beta) = B_2 \in \mathcal{C}_2^+$, where $B_2 \doteq (3.421, 0.538)$. The solution $u$ has two generalized zeros (2 and 6) and three continuous extensions $u^{\mathsf{c}}_{0,1}$, $u^{\mathsf{c}}_{2,5}$ and $u^{\mathsf{c}}_{6,7}$.

Now, let us recall the discrete anchoring procedure introduced in [28] which is also called the matching-extension method in [21] and can be used to obtain an implicit description of all Fučík curves $\mathcal{C}_l^\pm$. This technique consists of successive anchoring positive and negative continuous semi-waves which are defined as continuous extensions of the discrete solution $u$ of (7) on intervals determined by generalized zeros of $u$. See Figs. 2 and 3 for a non-trivial discrete solution $u$ of (7) for $(\alpha,\beta) = B_1 \in \mathcal{C}_1^+$. This discrete solution $u$ has one generalized zero on $\mathbb{T}$ at $j = 2$ and thus, we have one positive continuous semi-wave $u^{\mathsf{c}}_{0,1}$ on the interval $[0,2]$ and one negative continuous semi-wave $u^{\mathsf{c}}_{2,7}$ on the interval $[1,7]$. These two continuous semi-waves are anchored on the interval $[1,2]$ such that $u^{\mathsf{c}}_{0,1}(1) = u^{\mathsf{c}}_{2,7}(1)$ and $u^{\mathsf{c}}_{0,1}(2) = u^{\mathsf{c}}_{2,7}(2)$. Now, for simplicity, let us consider that $0 < \alpha, \beta < 4$. By Theorem 26 in [20], the problem (7) has a non-trivial solution $u$ with $u(1) > 0$ and exactly one generalized zero on $\mathbb{T}$ if and only if

$$p_1(\alpha) + p_1(\beta) + \tau_{\alpha,\alpha} + \tau_{\alpha,\beta} = n + 1, \tag{10}$$

where we have denoted

$$p_1(\alpha) := \left\lfloor \tfrac{\pi}{\omega_\alpha} \right\rfloor, \quad p_1(\beta) = \left\lfloor \tfrac{\pi}{\omega_\beta} \right\rfloor, \quad \tau_{\alpha,\beta} := T^\alpha\left(\frac{V^\beta_{p_1(\beta)}}{V^\beta_{p_1(\beta)-1}}\right), \quad \tau_{\alpha,\alpha} = T^\alpha\left(\frac{V^\alpha_{p_1(\alpha)}}{V^\alpha_{p_1(\alpha)-1}}\right), \tag{11}$$

and $\lfloor\cdot\rfloor$ is the floor function, $\omega_\alpha := \arccos\frac{2-\alpha}{2}$ and the function $T^\alpha : \mathbb{R}^* \to \mathbb{R}$ with the domain $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$ (the one-point compactification of $\mathbb{R}$) is defined as

$$T^\alpha(\infty) := 0, \qquad T^\alpha(q) := \frac{1}{\omega_\alpha} \operatorname{arccot} \frac{\cos\omega_\alpha - q}{\sin\omega_\alpha} \quad \text{for } q \in \mathbb{R}. \tag{12}$$

Let us point out that the function arccotangent in (12) is strictly decreasing on $\mathbb{R}$ with the range $(0,\pi)$. Thus, a pair $(\alpha,\beta)$ belongs to $\mathcal{C}_1^+$ if and only if (10) holds. Moreover, the equation (10) can be equivalently replaced by

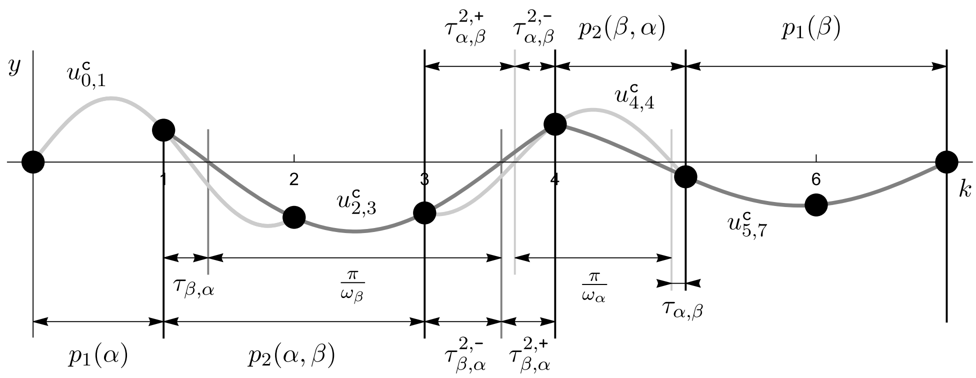


**Fig. 5.** The geometry of the discrete solution $u$ of (7) for $(\alpha,\beta) = B_3 \in \mathcal{C}_3^{\pm}$, where $B_3 \doteq (3.732, 1.657)$. The solution has three generalized zeros (2, 4 and 5) and four continuous extensions $u^{c}_{0,1}$, $u^{c}_{2,3}$, $u^{c}_{4,4}$ and $u^{c}_{5,7}$.

$$p_1(\alpha) + p_1(\beta) + \tau_{\beta,\beta} + \tau_{\beta,\alpha} = n + 1. \tag{13}$$

Let us point out that if $(\alpha,\beta) \in \mathcal{C}_1^{+}$ such that $\alpha \neq \beta$ and $\beta \neq \xi_k$ for all $k \in \{2,\dots,n-1\}$ then zeros of the positive and the negative continuous semi-waves do not coincide (see Fig. 3). Indeed, $(p_1(\alpha) + \tau_{\alpha,\alpha})$ and $(p_1(\alpha) + \tau_{\beta,\alpha})$ are zeros of the positive and the negative semi-waves, respectively, and we have that $\tau_{\alpha,\alpha} = \tau_{\beta,\alpha}$ if and only if $\alpha = \beta$ or $\beta \in \{\xi_2,\dots,\xi_{n-1}\}$.

Now, using Theorem 26 in [20], the second non-trivial Fučík curve $\mathcal{C}_2^{+}$ can be implicitly described as (see Figs. 2 and 4)

$$2p_1(\alpha) + p_2(\alpha,\beta) + \tau_{\alpha,\alpha} + \tau^{2,+}_{\alpha,\beta} = n + 1, \tag{14}$$

where we have denoted

$$\begin{aligned} p_2(\alpha,\beta) &:= \left\lfloor \tau_{\beta,\alpha} + \tfrac{\pi}{\omega_\beta} \right\rfloor, \\ \tau^{2,+}_{\alpha,\beta} &:= T^{\alpha}\left( \frac{V^{\alpha}_{p_1(\alpha)} V^{\beta}_{p_2(\alpha,\beta)} - V^{\alpha}_{p_1(\alpha)-1} V^{\beta}_{p_2(\alpha,\beta)-1}}{V^{\alpha}_{p_1(\alpha)} V^{\beta}_{p_2(\alpha,\beta)-1} - V^{\alpha}_{p_1(\alpha)-1} V^{\beta}_{p_2(\alpha,\beta)-2}} \right). \end{aligned} \tag{15}$$

As in the previous case, the equation (14) can be equivalently replaced by

$$2p_1(\alpha) + p_2(\alpha,\beta) + \tau^{2,-}_{\alpha,\beta} + \tau_{\beta,\alpha} = n + 1, \tag{16}$$

where we have denoted

$$\tau^{2,-}_{\alpha,\beta} := T^{\alpha}\left( \frac{V^{\beta}_{p_1(\beta)} V^{\alpha}_{p_2(\beta,\alpha)} - V^{\beta}_{p_1(\beta)-1} V^{\alpha}_{p_2(\beta,\alpha)-1}}{V^{\beta}_{p_1(\beta)} V^{\alpha}_{p_2(\beta,\alpha)-1} - V^{\beta}_{p_1(\beta)-1} V^{\alpha}_{p_2(\beta,\alpha)-2}} \right).$$

Finally, the third non-trivial Fučík curve $\mathcal{C}_3^{+}$ can be implicitly described as (see Figs. 2 and 5)

$$p_1(\alpha) + p_1(\beta) + p_2(\alpha,\beta) + p_2(\beta,\alpha) + \tau^{2,+}_{\alpha,\beta} + \tau^{2,-}_{\alpha,\beta} = n+1, \tag{17}$$

or as

$$p_1(\alpha) + p_1(\beta) + p_2(\alpha,\beta) + p_2(\beta,\alpha) + \tau^{2,-}_{\beta,\alpha} + \tau^{2,+}_{\beta,\alpha} = n+1. \tag{18}$$

Now, if we would like to describe higher Fučík curves then we have to use functions with higher level of nesting depth and also a higher number of different Chebyshev polynomials. Let us only note that to obtain an implicit description of the Fučík curve $\mathcal{C}^+_l$ for $l \geq 4$, we need to use nested functions $p_3(\alpha,\beta) := \left\lfloor \tau^{2,+}_{\alpha,\beta} + \frac{\pi}{\omega_\alpha} \right\rfloor$ and

$$\tau^{3,+}_{\alpha,\beta} := T^\alpha \left( \frac{V^\alpha_{p_1} V^\beta_{p_2} V^\alpha_{p_3} - V^\alpha_{p_1-1} V^\beta_{p_2-1} V^\alpha_{p_3} - V^\alpha_{p_1} V^\beta_{p_2-1} V^\alpha_{p_3-1} + V^\alpha_{p_1-1} V^\beta_{p_2-2} V^\alpha_{p_3-1}}{V^\alpha_{p_1} V^\beta_{p_2} V^\alpha_{p_3-1} - V^\alpha_{p_1-1} V^\beta_{p_2-1} V^\alpha_{p_3-1} - V^\alpha_{p_1} V^\beta_{p_2-1} V^\alpha_{p_3-2} + V^\alpha_{p_1-1} V^\beta_{p_2-2} V^\alpha_{p_3-2}} \right), \tag{19}$$

where $p_1 = p_1(\alpha)$, $p_2 = p_2(\alpha,\beta)$ and $p_3 = p_3(\alpha,\beta)$. If we compare the definitions of $\tau_{\alpha,\beta}$, $\tau^{2,+}_{\alpha,\beta}$ and $\tau^{3,+}_{\alpha,\beta}$ in (11), (15) and (19), respectively, we have to conclude that the complexity of the known implicit description of Fučík curves $\mathcal{C}^\pm_l$ substantially increases with increasing numbers of generalized zeros $l$ of the solution $u$. As far as we know, it is not possible to lower the level of used nested functions. Thus, in this paper, for each non-trivial Fučík curve $\mathcal{C}^\pm_l$, we provide new bounds with the same description complexity as the implicit description (10) for the first non-trivial Fučík curve $\mathcal{C}^\pm_1$. In the following section, we introduce these new bounds and present the main results of this paper.

## 2. Main results

In this section, we introduce two main results of this paper concerning the discrete problem (7), namely Theorems 3 and 5. Proofs of both these theorems are provided in the following sections.

One of the main goals of this paper is to provide new suitable bounds for each non-trivial Fučík curve $\mathcal{C}^\pm_l$ such that all these bounds will have the same simplicity of description as used in (10) for the first non-trivial Fučík curve $\mathcal{C}^\pm_1$. Let us recall that the Fučík spectrum $\Sigma$ is symmetric to the diagonal $\alpha = \beta$ and each of its non-trivial Fučík curve $\mathcal{C}^\pm_l$ is in the domain $D = ((0,4) \times (0,+\infty)) \cup ((0,+\infty) \times (0,4))$ (see Fig. 6). Thus, it is enough to construct bounds for Fučík curves only in the following half-strip

$$\mathcal{D} := (0,4) \times (0,+\infty).$$

Now, let us define the basic map $\kappa_\beta : (0,+\infty) \to \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, as

$$\kappa_\beta := \begin{cases} \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor - 1 & \text{for } 0 < \beta < 4, \\ 0 & \text{for } \beta \geq 4, \end{cases} \tag{20}$$

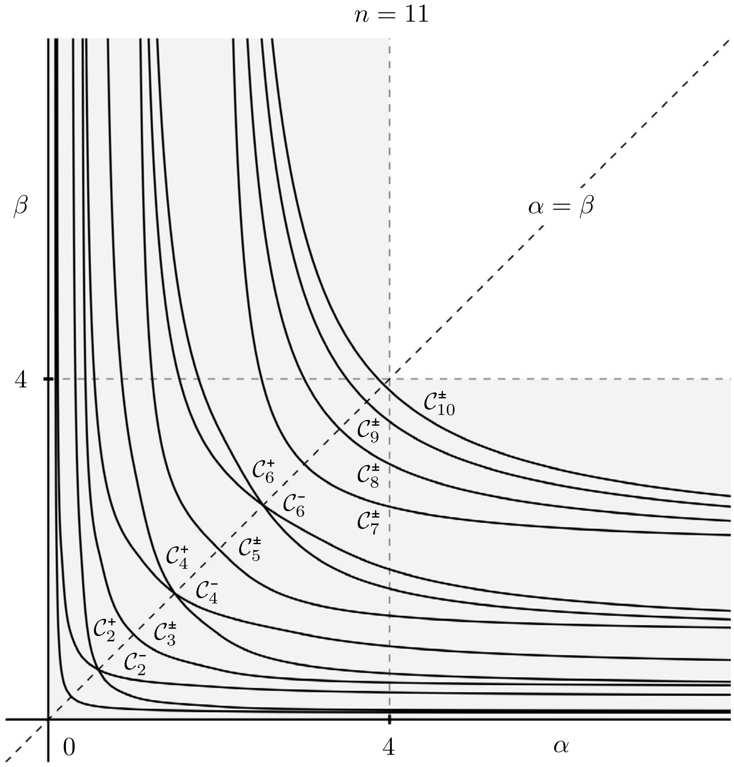


**Fig. 6.** All non-trivial Fučík curves $\mathcal{C}_l^{\pm}$ (black curves) for $l = 1, \ldots, n-1$ are contained in the domain $D = ((0,4) \times (0,+\infty)) \cup ((0,+\infty) \times (0,4))$ (grey region). Let us note that for $n = 11$, we have $\mathcal{C}_l^{+} = \mathcal{C}_l^{-}$ for $l = 1, 3, 5, 7, 8, 9, 10$.

where $\omega_\beta := \arccos \frac{2-\beta}{2}$ for $0 < \beta < 4$. Using $\kappa_\beta$, we decompose the half-strip $\mathcal{D}$ into rectangles by $\kappa_\beta = k$, $k \in \mathbb{N}_0$, i.e. we have (see Fig. 8)

$$\mathcal{D} = \big((0,4) \times (\xi_2, +\infty]\big) \cup \big((0,4) \times (\xi_3, \xi_2]\big) \cup \cdots \cup \big((0,4) \times (\xi_{k+2}, \xi_{k+1}]\big) \cup \ldots,$$

where $\xi_k$ is defined in (9). On each rectangle given by $\kappa_\beta = k$, we use Chebyshev polynomials of two degrees $V^\beta_{\kappa_\beta}$ and $V^\beta_{\kappa_\beta+1}$ to introduce three basic elements $\eta_{\alpha,\beta}$, $\tau_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ in the following definition. Let us note that for fixed $\beta \in (0,4)$, the value $\frac{\pi}{\omega_\beta}$ used in (20) represents the distance between zeros of the continuous extension of a negative semi-wave (see Figs. 4 and 5 for continuous extensions $u^{\mathfrak{c}}_{2,5}$ and $u^{\mathfrak{c}}_{2,3}$, respectively).

**Definition 1.** For $0 < \alpha < 4$ and $\beta > 0$, let us define

$$\eta_{\alpha,\beta} := T^\alpha\left(\frac{V^\beta_{\kappa_\beta+1} - 1}{V^\beta_{\kappa_\beta}}\right), \qquad \tau_{\alpha,\beta} := T^\alpha\left(\frac{V^\beta_{\kappa_\beta+1}}{V^\beta_{\kappa_\beta}}\right), \qquad \mu_{\alpha,\beta} := T^\alpha\left(\frac{V^\beta_{\kappa_\beta+1}}{V^\beta_{\kappa_\beta} + 1}\right),$$

where the function $T^\alpha$ is given by (12) and $V^\beta_k$ is given in (8) by Chebyshev polynomials of the second type of degree $k$.

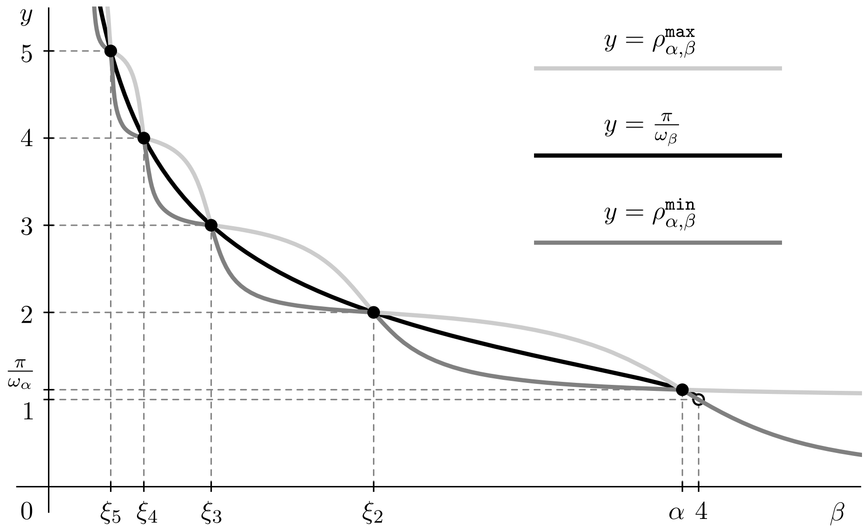


**Fig. 7.** Graphs of functions $\beta \mapsto \rho^{\mathtt{min}}_{\alpha,\beta}$ and $\beta \mapsto \rho^{\mathtt{max}}_{\alpha,\beta}$ for fixed $\alpha = 3.9$ and the graph of the function $\beta \mapsto \frac{\pi}{\omega_\beta}$.

Let us recall that using $\tau_{\alpha,\beta}$, we can formulate an implicit description of the first non-trivial Fučík curve $\mathcal{C}^{\pm}_1$ as in (10) or (13). Now, using $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ in the following definition, let us introduce $\rho^{\mathtt{min}}_{\alpha,\beta}$ and $\rho^{\mathtt{max}}_{\alpha,\beta}$ that are given on the half-strip $\mathcal{D}$. See also Fig. 7 and note that $\rho^{\mathtt{min}}_{\alpha,\beta} \leq \frac{\pi}{\omega_\beta} \leq \rho^{\mathtt{max}}_{\alpha,\beta}$ for $0 < \beta < 4$.

**Definition 2.** For $0 < \alpha < 4$ and $\beta > 0$, let us define

$$\rho^{\mathtt{min}}_{\alpha,\beta} := \begin{cases} 2\mu_{\alpha,\beta} + \kappa_\beta & \text{for } \alpha \leq \beta, \\ 2\eta_{\alpha,\beta} + \kappa_\beta + 1 & \text{for } \alpha > \beta, \end{cases}$$

$$\rho^{\mathtt{max}}_{\alpha,\beta} := \begin{cases} 2\eta_{\alpha,\beta} + \kappa_\beta + 1 & \text{for } \alpha \leq \beta, \\ 2\mu_{\alpha,\beta} + \kappa_\beta & \text{for } \alpha > \beta. \end{cases}$$

In the following theorem, we use $\rho^{\mathtt{min}}_{\alpha,\beta}$ and $\rho^{\mathtt{max}}_{\alpha,\beta}$ to construct sets $\Upsilon^{\pm}_l$ as bounds for Fučík curves $\mathcal{C}^{\pm}_l$ such that (see Figs. 9 and 8)

$$(\mathcal{C}^{\pm}_l \cap \mathcal{D}) \subset \Upsilon^{\pm}_l.$$

**Theorem 3.** *In the domain $\mathcal{D} = (0,4) \times (0,+\infty)$, we have the following bounds for Fučík curves $\mathcal{C}^{\pm}_l$, $l = 1, \ldots, n-1$,*

$$\begin{aligned} \left(\mathcal{C}^{\pm}_{2j-1} \cap \mathcal{D}\right) &\subset \Upsilon_{j,j} &&=: \Upsilon^{\pm}_{2j-1}, \\ \left(\mathcal{C}^{+}_{2j} \cap \mathcal{D}\right) &\subset \Upsilon_{j+1,j} &&=: \Upsilon^{+}_{2j}, \\ \left(\mathcal{C}^{-}_{2j} \cap \mathcal{D}\right) &\subset \Upsilon_{j,j+1} &&=: \Upsilon^{-}_{2j}, \end{aligned}$$

*$j \in \mathbb{N}$, where for $k, s \in \mathbb{N}$, sets $\Upsilon_{k,s}$ are given by*

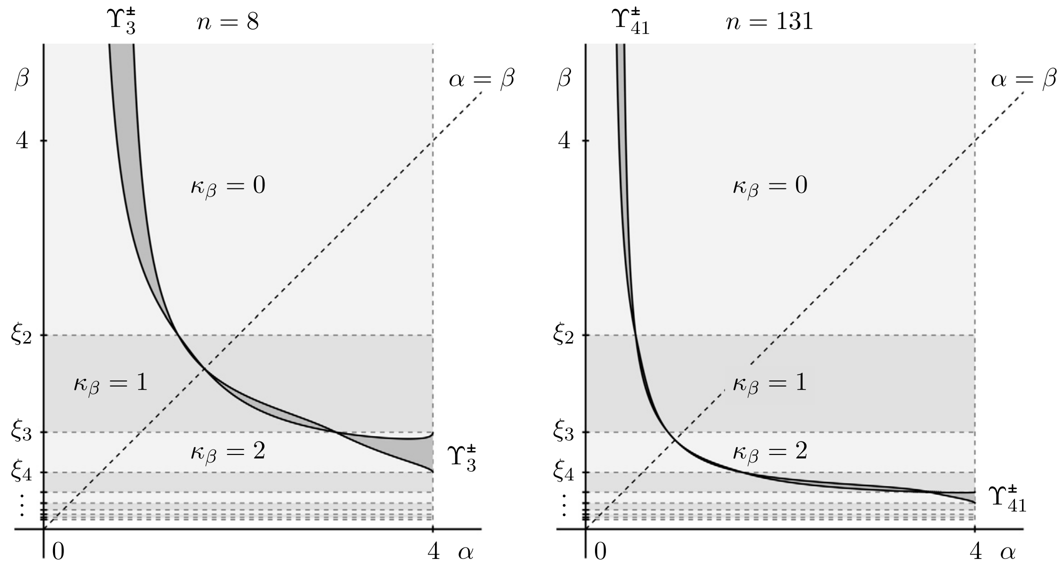


**Fig. 8.** The decomposition of the half-strip $\mathcal{D}$ into rectangles by $\kappa_\beta = k$, $k \in \mathbb{N}_0$, and the set $\Upsilon_3^{\pm} \subset \mathcal{D}$ as a bound for the third non-trivial Fučík curve $\mathcal{C}_3^{\pm}$ for $n = 8$ (left) and the set $\Upsilon_{41}^{\pm} \subset \mathcal{D}$ as a bound for the forty-first non-trivial Fučík curve $\mathcal{C}_{41}^{\pm}$ for $n = 131$ (right).

$$\Upsilon_{k,s} := \left\{(\alpha,\beta) \in \mathcal{D} :\ \rho_{\alpha,\beta}^{\mathtt{min}} \ \leq\ \tfrac{1}{s}\left(n+1-k\tfrac{\pi}{\omega_\alpha}\right) \ \leq\ \rho_{\alpha,\beta}^{\mathtt{max}}\right\}.$$

**Remark 4.** Due to Theorem 3, the part of the Fučík curve $\mathcal{C}_{2j-1}^{\pm}$ that belongs to the half-strip $\mathcal{D}$ is in the set $\Upsilon_{2j-1}^{\pm}$ with the boundary determined by two curves

$$s\left(\kappa_\beta + 2\mu_{\alpha,\beta}\right) + k\tfrac{\pi}{\omega_\alpha} = n+1, \qquad s\left(\kappa_\beta + 2\eta_{\alpha,\beta} + 1\right) + k\tfrac{\pi}{\omega_\alpha} = n+1, \tag{21}$$

where $k = s = j$. And similarly, parts of Fučík curves $\mathcal{C}_{2j}^{+} \cap \mathcal{D}$ and $\mathcal{C}_{2j}^{-} \cap \mathcal{D}$ are in sets $\Upsilon_{2j}^{+}$ and $\Upsilon_{2j}^{-}$ with boundaries given by curves in (21) for $k = j+1$, $s = j$ and $k = j$, $s = j+1$, respectively.

For $0 < \alpha, \beta < 4$, the equation (10), which describes the first non-trivial Fučík curve $\mathcal{C}_1^{\pm}$, can be written in the following form

$$\kappa_\beta + \tau_{\alpha,\beta} + 1 + \tfrac{\pi}{\omega_\alpha} = n+1, \tag{22}$$

since $\left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor = \kappa_\beta + 1$ and $\left\lfloor \frac{\pi}{\omega_\alpha} \right\rfloor + \tau_{\alpha,\alpha} = \frac{\pi}{\omega_\alpha}$ (see Lemma 16). Let us note that the equation (22) has the same structure as equations in (21) which describe the boundary of the set $\Upsilon_l^{\pm}$ containing the particular Fučík curve $\mathcal{C}_l^{\pm}$ ($\tau_{\alpha,\beta}$ is used in (22) instead of $\mu_{\alpha,\beta}$ or $\eta_{\alpha,\beta}$ in (21)). On the other hand, the structure of equations in (21) is much simpler than the known precise description of higher non-trivial Fučík curves $\mathcal{C}_l^{\pm}$ for $l \geq 2$. For example, compare (21) for $k = 3$ and $s = 2$ to the description of the fourth non-trivial Fučík curve $\mathcal{C}_4^{+}$ which has the following form

$$2\left\lfloor \tfrac{\pi}{\omega_\alpha} \right\rfloor + 2\left\lfloor \tau_{\beta,\alpha} + \tfrac{\pi}{\omega_\beta} \right\rfloor + \left\lfloor \tau_{\alpha,\beta}^{2,+} + \tfrac{\pi}{\omega_\alpha} \right\rfloor + \tau_{\alpha,\beta}^{2,+} + \tau_{\alpha,\beta}^{3,+} = n+1, \tag{23}$$

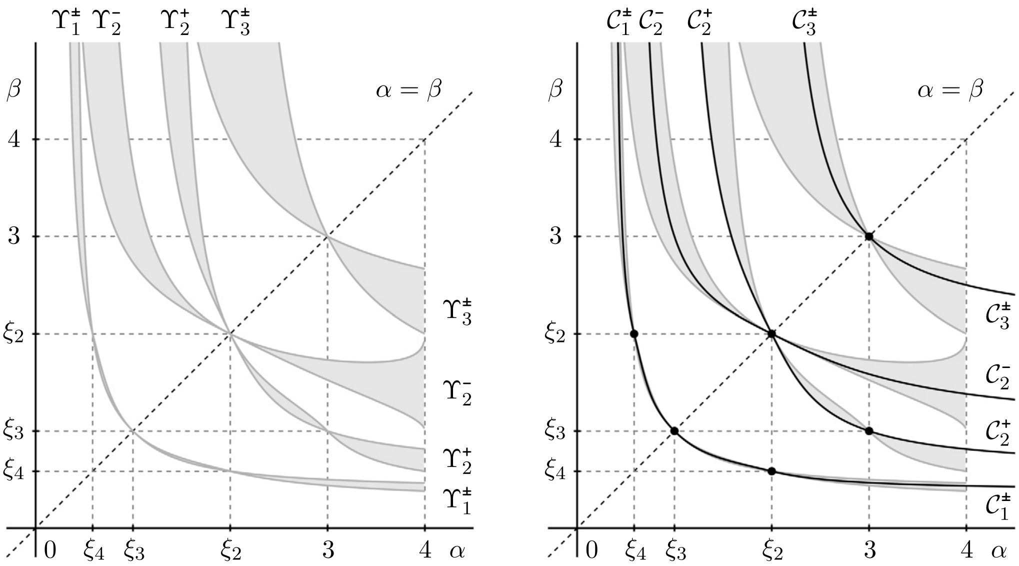


**Fig. 9.** Sets $\Upsilon_l^\pm$ in $\mathcal{D}$ (grey regions) as bounds for Fučík curves $\mathcal{C}_l^\pm$ (black curves, right) for $n = 5$.

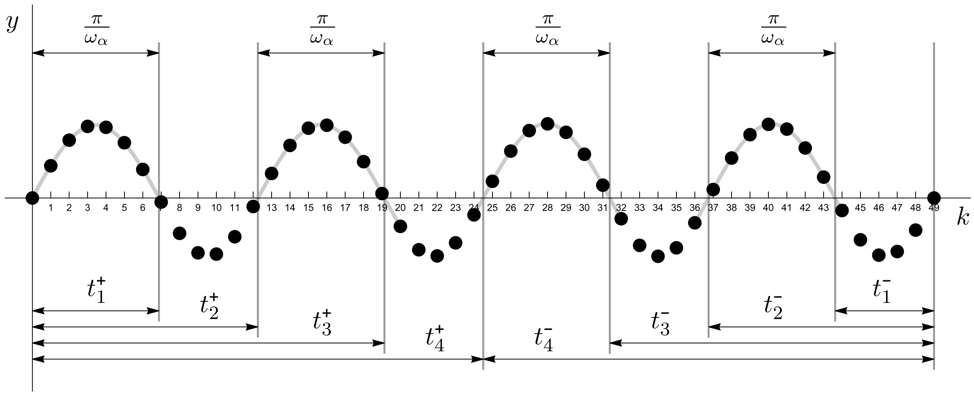


**Fig. 10.** A non-trivial solution of the problem (7) with 7 generalized zeros on $\mathbb{T}$ for $(\alpha, \beta) \in \mathcal{C}_7^+$ ($n = 48$, $\alpha \doteq 0.205$, $\beta \doteq 0.332$).

where $\tau_{\alpha,\beta}^{2,+}$ and $\tau_{\alpha,\beta}^{3,+}$ are defined in (15) and (19), respectively.

The implicit description of all non-trivial Fučík curves $\mathcal{C}_l^\pm$ is provided in the next Theorem 5. Let us note that $t_j^+$ and $t_j^-$ determine zeros of positive semi-waves (as continuous extensions) and $\rho_{\alpha,\beta}$ (introduced in Definition 19 in Section 5) measures the distance between every two consecutive zeros of two different positive semi-waves. See Fig. 10 and observe that $t_1^+ = \frac{\pi}{\omega_\alpha}$ and ($\lceil\cdot\rceil$ denotes the ceil function)

$$t_2^+ = t_1^+ + \rho_{\alpha,\beta}\left(\lceil t_1^+\rceil - t_1^+\right), \qquad t_3^+ = t_2^+ + \tfrac{\pi}{\omega_\alpha}, \qquad t_4^+ = t_3^+ + \rho_{\alpha,\beta}\left(\lceil t_3^+\rceil - t_3^+\right).$$

See also Figs. 29, 30 and 31 at the end of Section 5 for other examples of non-trivial solutions of the problem (7).

**Theorem 5.** *In the domain* $\mathcal{D} = (0,4) \times (0,+\infty)$*, we have the following description of Fučík curves* $\mathcal{C}_l^{\pm}$*,* $l = 1,\dots,n-1$*,*

$$\begin{aligned}\mathcal{C}_{2j-1}^{\pm} \cap \mathcal{D} &= \left\{(\alpha,\beta) \in \mathcal{D} :\ t_j^{+}(\alpha,\beta) + t_j^{-}(\alpha,\beta) = n+1\right\},\\ \mathcal{C}_{2j}^{+} \cap \mathcal{D} &= \left\{(\alpha,\beta) \in \mathcal{D} :\ t_{j+1}^{+}(\alpha,\beta) + t_j^{+}(\alpha,\beta) = n+1\right\},\\ \mathcal{C}_{2j}^{-} \cap \mathcal{D} &= \left\{(\alpha,\beta) \in \mathcal{D} :\ t_{j+1}^{-}(\alpha,\beta) + t_j^{-}(\alpha,\beta) = n+1\right\},\end{aligned}$$

*where*

$$t_1^{+} := \frac{\pi}{\omega_\alpha}, \qquad t_j^{+} := \begin{cases} t_{j-1}^{+} + \rho_{\alpha,\beta}\left(\lceil t_{j-1}^{+}\rceil - t_{j-1}^{+}\right) & \text{for } j \text{ even},\\ t_{j-1}^{+} + \frac{\pi}{\omega_\alpha} & \text{for } j \text{ odd}, \end{cases} \tag{24}$$

$$t_1^{-} := \rho_{\alpha,\beta}(0), \qquad t_j^{-} := \begin{cases} t_{j-1}^{-} + \frac{\pi}{\omega_\alpha} & \text{for } j \text{ even},\\ t_{j-1}^{-} + \rho_{\alpha,\beta}\left(\lceil t_{j-1}^{-}\rceil - t_{j-1}^{-}\right) & \text{for } j \text{ odd}. \end{cases} \tag{25}$$

Finally, let us point out that the value $\rho_{\alpha,\beta}(s)$ of the distance function $\rho_{\alpha,\beta}$ is bounded by $\rho_{\alpha,\beta}^{\mathtt{min}}$ and $\rho_{\alpha,\beta}^{\mathtt{max}}$ used in Theorem 3 to construct sets $\Upsilon_l^{\pm}$ as bounds for Fučík curves $\mathcal{C}_l^{\pm}$.

## 3. Connections between the Fučík spectra for discrete and continuous problems

In this section, we show some consequences of obtained results in Theorems 3 and 5 in order to reveal the link between the Fučík spectrum $\Sigma$ for the discrete problem (7) and the Fučík spectrum $\Sigma^{\mathtt{c}}$ for the continuous problem (2). For this purpose, let us consider the following discrete Dirichlet problem

$$\begin{cases} \Delta_h^2 v(k-h) + av^+(k) - bv^-(k) = 0, & k \in \mathbb{T}_h,\\ u(0) = u(1) = 0, \end{cases} \tag{26}$$

where $a, b \in \mathbb{R}$, $h := \frac{1}{n+1}$, $n \in \mathbb{N}$, $\mathbb{T}_h := \{ih :\ i = 1,\dots,n\}$ and

$$\Delta_h^2 v(k-h) = \frac{v(k+h) - 2v(k) + v(k-h)}{h^2}.$$

Thus, the problem (26) is the rescaled version of the original problem (7) and it can be also viewed as the result of a discretization of the continuous Dirichlet problem (2). The Fučík spectrum for the rescaled discrete problem (26) consists of finite number of Fučík curves $\mathcal{C}_{h,l}^{\pm}$, $l = 0,\dots,n-1$, such that

$$\mathcal{C}_{h,l}^{\pm} = \left\{(a,b) \in \mathbb{R}^2 :\ \left(ah^2, bh^2\right) \in \mathcal{C}_l^{\pm}\right\},$$

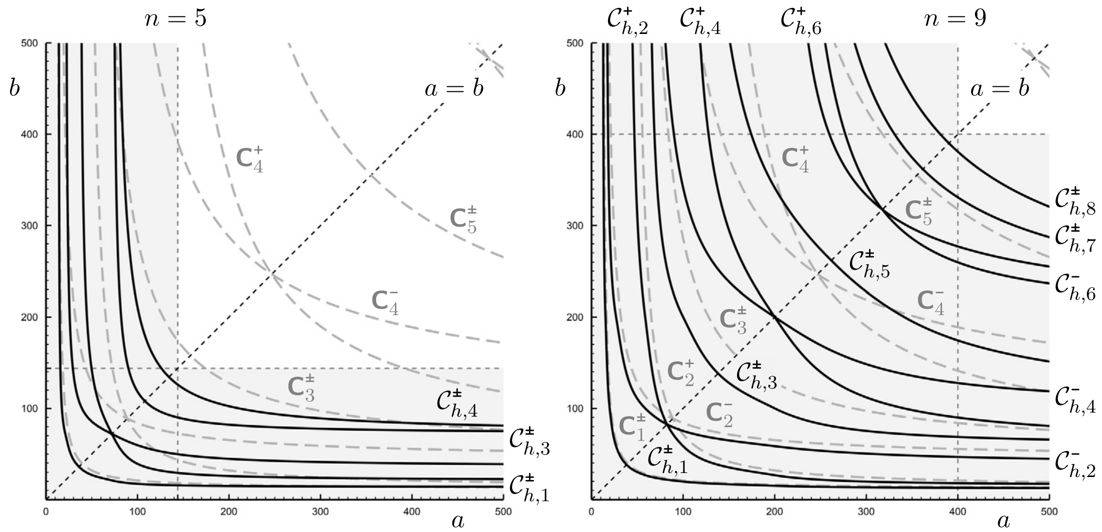


**Fig. 11.** Non-trivial Fučík curves $\mathbf{C}_l^\pm$ (grey dashed curves) for the continuous problem (2) and non-trivial Fučík curves $\mathcal{C}_{h,l}^\pm$ (black curves) for the rescaled discrete problem (26): five curves $\mathcal{C}_{h,1}^\pm$, $\mathcal{C}_{h,2}^+$, $\mathcal{C}_{h,2}^-$, $\mathcal{C}_{h,3}^\pm$ and $\mathcal{C}_{h,4}^\pm$ for $n = 5$ in the domain $D_h = ((0, 144) \times (0, +\infty)) \cup ((0, +\infty) \times (0, 144))$ (grey region, left) and eleven curves $\mathcal{C}_{h,1}^\pm$, $\mathcal{C}_{h,2}^+$, $\mathcal{C}_{h,2}^-$, $\ldots$, $\mathcal{C}_{h,8}^\pm$ for $n = 9$ in the domain $D_h = ((0, 400) \times (0, +\infty)) \cup ((0, +\infty) \times (0, 400))$ (grey region, right).

where non-trivial Fučík curves $\mathcal{C}_l^\pm$ are described implicitly in Theorem 5. Since each non-trivial Fučík curve $\mathcal{C}_l^\pm$ is located in the domain $D = ((0, 4) \times (0, +\infty)) \cup ((0, +\infty) \times (0, 4))$ then each non-trivial Fučík curve $\mathcal{C}_{h,l}^\pm$ is contained in the domain

$$D_h := \left(\left(0, 4h^{-2}\right) \times (0, +\infty)\right) \cup \left((0, +\infty) \times \left(0, 4h^{-2}\right)\right).$$

See Fig. 11 for the domain $D_h$ containing all non-trivial Fučík curves $\mathcal{C}_{h,l}^\pm$ for the rescaled problem (26) and notice their correspondence to Fučík curves $\mathbf{C}_l^\pm$ for the continuous problem (2). Moreover, according to Theorem 3, we have for $l = 1, \ldots, n-1$ that

$$\left(\mathcal{C}_{h,l}^\pm \cap \mathcal{D}_h\right) \subset \Upsilon_{h,l}^\pm := \left\{(a, b) \in \mathbb{R}^2 : \ \left(ah^2, bh^2\right) \in \Upsilon_l^\pm\right\},$$

where $\mathcal{D}_h := \left(0, 4h^{-2}\right) \times (0, +\infty)$. See Figs. 12 and 13 for sets $\Upsilon_{h,l}^\pm$ and check their correspondence to Fučík curves $\mathbf{C}_l^\pm$ for the continuous problem (2).

This paper is organized in the following way. Firstly, we recall some basic facts and results concerning mainly the semi-linear initial value problem in Section 4. At the end of this section, in Theorem 13, we obtain some basic bounds for each non-trivial Fučík curve $\mathcal{C}_l^\pm$ using $\kappa_\beta$. The next Section 5 is devoted to the investigation of the distance $\rho_{\alpha,\beta}$ of two consecutive zeros of two different positive semi-waves as continuous extensions. We explore the properties of $\eta_{\alpha,\beta}$, $\tau_{\alpha,\beta}$, $\mu_{\alpha,\beta}$ and $\rho_{\alpha,\beta}$ in detail. This careful analysis leads to the proof of Theorem 5, which is available at the end of this section. The next Section 6 is devoted to the construction of improved bounds $\Upsilon_l^\pm$ for non-trivial Fučík curves $\mathcal{C}_l^\pm$. In Theorem 31, we prove that $\rho_{\alpha,\beta}$ is a differentiable function which attains its global

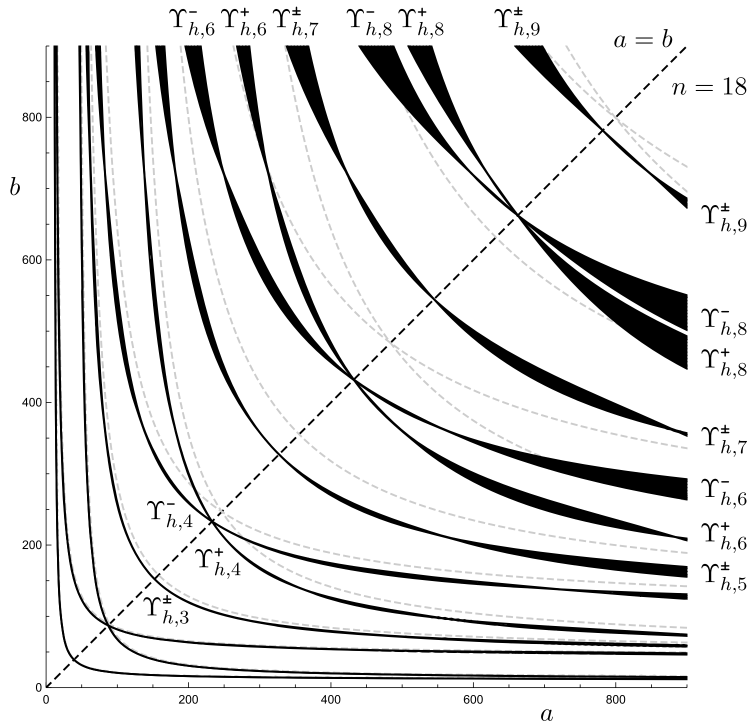


**Fig. 12.** Sets $\Upsilon^\pm_{h,l}$ (black regions), $l = 1, \dots, 9$, as bounds for Fučík curves $\mathcal{C}^\pm_{h,l}$ for the discrete rescaled problem (26) ($n = 18$) and Fučík curves $\mathbf{C}^\pm_l$ (grey dashed curves) for the continuous problem (2).

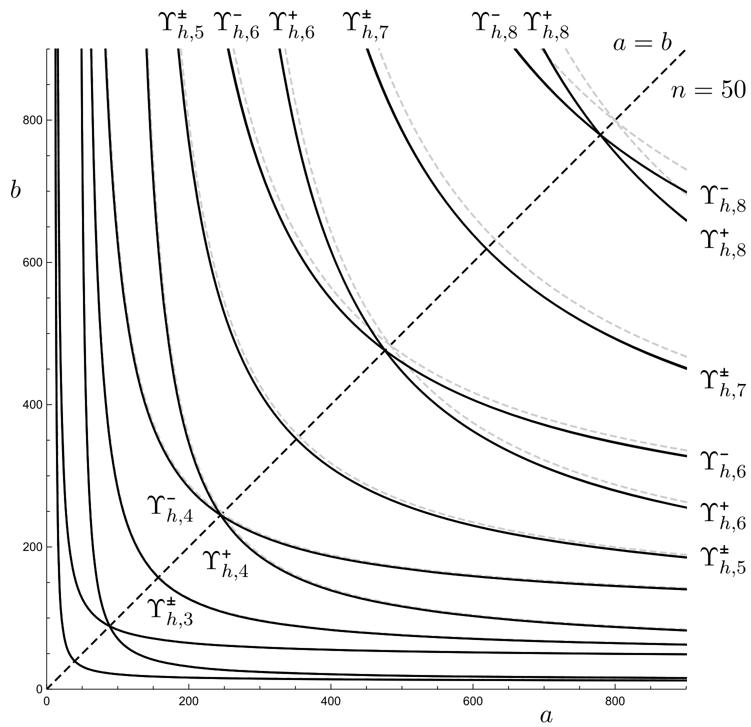


**Fig. 13.** Sets $\Upsilon^\pm_{h,l}$ (black thin regions), $l = 1, \dots, 8$, as bounds for Fučík curves $\mathcal{C}^\pm_{h,l}$ for the discrete rescaled problem (26) ($n = 50$) and Fučík curves $\mathbf{C}^\pm_l$ (grey dashed curves) for the continuous problem (2).

extrema at points $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$. Finally, the proof of the main Theorem 3 is available at the end of Section 6 and let us note that it is based on both Theorems 5 and 31.

## 4. Preliminaries and basic bounds for Fučík curves

In the first part of this section, we recall some preliminaries used in [20], and we also prove some basic properties of $V^{\beta}_{\kappa_\beta}$ and $V^{\beta}_{\kappa_\beta+1}$ defined by Chebyshev polynomials of the second kind. Let us note that we follow the notation used in [20]. In the second part of this section, we deal with the sequence of functions $p_i$ introduced in [20] that are used to describe implicitly a non-trivial Fučík curve $\mathcal{C}^{\pm}_l$ (recall (10), (14) and (17), where $p_1$ and $p_2$ are used). Using $\kappa_\beta$, we provide a new description of functions $p_i$ in Lemma 14. Moreover, due to this description, we obtain some basic bounds for each non-trivial Fučík curve $\mathcal{C}^{\pm}_l$ in Theorem 13.

For $0 < \alpha < 4$, the function $T^\alpha$ defined in (12) is strictly increasing on $\mathbb{R}$ (see Fig. 14), maps $\mathbb{R}^*$ onto $\left[0, \frac{\pi}{\omega_\alpha}\right)$ and

$$T^\alpha(0) = 1, \qquad T^\alpha(-1) = \tfrac{1}{2}, \qquad T^\alpha(1) = \tfrac{1}{2} + \tfrac{\pi}{2\omega_\alpha}, \qquad T^\alpha\left(\tfrac{2-\alpha}{2}\right) = \tfrac{\pi}{2\omega_\alpha}.$$

Moreover, we have the following useful formula (see Lemma 3 in [20])

$$T^\alpha(q) + T^\alpha\left(\tfrac{1}{q}\right) = 1 \qquad \text{for } q \le 0 \text{ or } q = \infty. \tag{27}$$

Let us denote the inverse function of $T^\alpha$ by $Q^\alpha : \left[0, \frac{\pi}{\omega_\alpha}\right) \to \mathbb{R}^*$

$$Q^\alpha(0) = \infty, \qquad Q^\alpha(t) = -\frac{\sin(\omega_\alpha(1-t))}{\sin(\omega_\alpha t)} \qquad \text{for } 0 < t < \tfrac{\pi}{\omega_\alpha}, \tag{28}$$

where $\omega_\alpha = \arccos \frac{2-\alpha}{2}$. Let us point out that $1 < \frac{\pi}{\omega_\alpha}$ and that $Q^\alpha$ is a strictly increasing function on $\left(0, \frac{\pi}{\omega_\alpha}\right)$. Moreover, using (28), we obtain that

$$Q^\alpha(t) = \frac{1}{Q^\alpha(1-t)} \qquad \text{for } 0 \le t \le 1. \tag{29}$$

Let us consider the following semi-linear initial value problem

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, \quad k \in \mathbb{Z}, \\ u(0) = 0, \quad u(1) = C_1, \end{cases} \tag{30}$$

where $C_1 \in \mathbb{R}$, $C_1 \neq 0$ and $(\alpha, \beta) \in D := ((0,4) \times (0,+\infty)) \cup ((0,+\infty) \times (0,4))$. For $0 < \alpha = \beta < 4$, the problem (30) is a linear one and it has a unique solution $u$ of the form

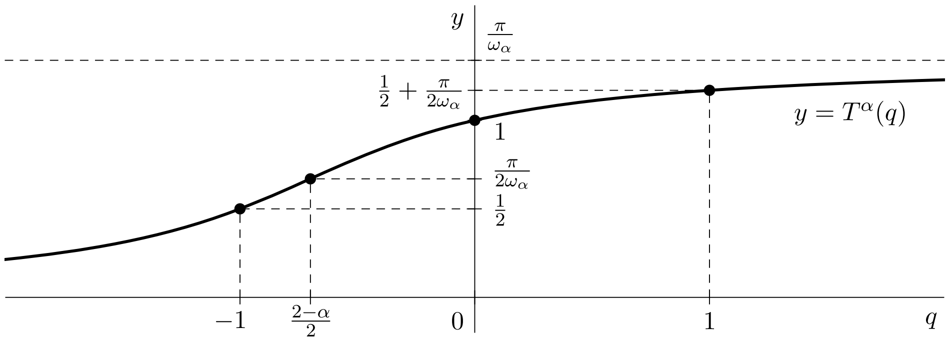


**Fig. 14.** The graph of the function $T^\alpha = T^\alpha(q)$ for fixed $\alpha = 3.4$.

$$u(k) = C_1 \frac{\sin(\omega_\beta k)}{\sin \omega_\beta} = C_1 V_{k-1}^\beta, \tag{31}$$

where $V_{k-1}^\beta$ is given in (8) by the Chebyshev polynomial of the second kind. For $(\alpha, \beta) \in D$, the problem (30) has a unique solution $u$ which consists of infinitely many positive and negative semi-waves (as continuous extensions). Moreover, for $0 < \alpha < 4$, $\beta > 0$ and $C_1 > 0$, we have due to Lemma 19 and Remark 20 in [20] that all non-negative zero points of all positive semi-waves form a sequence $(t_j)_{j=0}^{+\infty}$ such that

$$t_0 = 0, \qquad t_j = \begin{cases} \sum_{i=1}^{j} p_i(\alpha,\beta) + T^\alpha(\vartheta_j(\alpha,\beta)) & \text{for } C_1 > 0, \\ \sum_{i=1}^{j} p_i(\beta,\alpha) + T^\alpha(\vartheta_j(\beta,\alpha)) & \text{for } C_1 < 0, \end{cases} \qquad j \in \mathbb{N},$$

where functions $p_i$ and $\vartheta_i$ are given recurrently for $i \in \mathbb{N}$ in the following way (see Definition 17 in [20])

$$\vartheta_0(\alpha,\beta) := \infty,$$

$$p_{2i-1}(\alpha,\beta) := \begin{cases} \left\lfloor T^\alpha(\vartheta_{2i-2}(\alpha,\beta)) + \frac{\pi}{\omega_\alpha} \right\rfloor & \text{for } \alpha < 4, \\ \left\lfloor T^\beta(\vartheta_{2i-2}(\alpha,\beta)) + T^\beta(2-\alpha) + 1 \right\rfloor & \text{for } \alpha \geq 4, \end{cases} \tag{32}$$

$$p_{2i}(\alpha,\beta) := \begin{cases} \left\lfloor T^\beta(\vartheta_{2i-1}(\alpha,\beta)) + \frac{\pi}{\omega_\beta} \right\rfloor & \text{for } \beta < 4, \\ \left\lfloor T^\alpha(\vartheta_{2i-1}(\alpha,\beta)) + T^\alpha(2-\beta) + 1 \right\rfloor & \text{for } \beta \geq 4, \end{cases} \tag{33}$$

$$\vartheta_{2i-1}(\alpha,\beta) := W^\alpha_{p_{2i-1}(\alpha,\beta)}(\vartheta_{2i-2}(\alpha,\beta)), \tag{34}$$

$$\vartheta_{2i}(\alpha,\beta) := W^\beta_{p_{2i}(\alpha,\beta)}(\vartheta_{2i-1}(\alpha,\beta)). \tag{35}$$

Finally, to complete the definition of $\vartheta_i$ in (34) and (35), let us recall the function $W_k^\lambda : \mathbb{R}^* \to \mathbb{R}^*$ for $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$ as (see Definition 5 in [20])

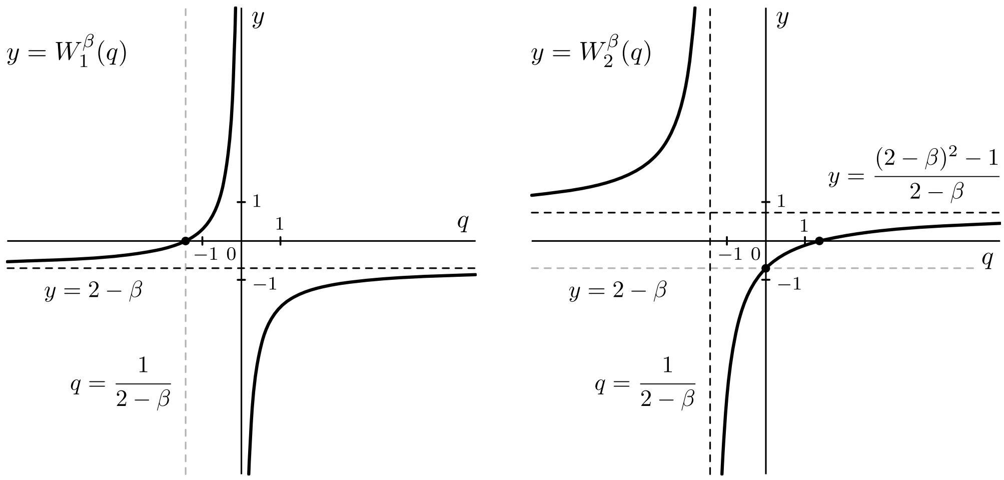


**Fig. 15.** Graphs of functions $W_1^\beta = W_1^\beta(q)$ (left) and $W_2^\beta = W_2^\beta(q)$ (right) for fixed $\beta = 2.7$.

$$W_k^\lambda(q) := \begin{cases} \dfrac{q \cdot V_k^\lambda - V_{k-1}^\lambda}{q \cdot V_{k-1}^\lambda - V_{k-2}^\lambda} & \text{for } q \in \mathbb{R}, \\ \dfrac{V_k^\lambda}{V_{k-1}^\lambda} & \text{for } q = \infty. \end{cases} \tag{36}$$

Let us note that the function $W_k^\lambda$ is the restriction of a complex Möbius transformation on $\mathbb{R}^*$ (see Fig. 15). Now, let us recall some useful properties of $W_k^\lambda$ due to Lemma 9 in [20]

$$W_l^\lambda(W_k^\lambda(q)) = W_{k+l}^\lambda(q), \qquad W_{-k}^\lambda(W_k^\lambda(q)) = q, \qquad W_{-k}^\lambda(q) = \frac{1}{W_k^\lambda\left(\frac{1}{q}\right)}, \tag{37}$$

where $k, l \in \mathbb{Z}$ and $q \in \mathbb{R}^*$. Moreover, due to Remark 10 in [20], we have for $\lambda \in \mathbb{R}$ and $k, l \in \mathbb{Z}$ that

$$q_{k+l} = W_l^\lambda(q_k), \qquad q_k := \frac{u(k)}{u(k-1)}, \tag{38}$$

where $u$ is a non-trivial solution of the linear equation $\Delta^2 u(k-1) + \lambda u(k) = 0$.

In (36), the coefficients $V_k^\lambda$ are defined in (8) using Chebyshev polynomial of the second kind and thus, $V_k^\lambda$ satisfies the three terms recurrence formula

$$V_{k-1}^\lambda - (2-\lambda)V_k^\lambda + V_{k+1}^\lambda = 0. \tag{39}$$

Moreover, by Lemma 4 in [20], we also have

$$\left(V_k^\lambda\right)^2 - V_{k+1}^\lambda V_{k-1}^\lambda = 1. \tag{40}$$

Let us introduce the next identity for Chebyshev polynomials of the second kind.

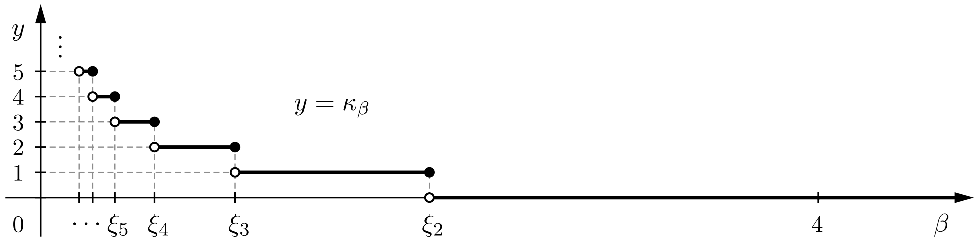


**Fig. 16.** The graph of the piecewise constant function $\beta \mapsto \kappa_\beta$.

**Lemma 6.** *For $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, the following equality holds*

$$\left(V_{k+1}^{\lambda} - V_k^{\lambda}\right)^2 = 1 - \lambda \cdot V_{k+1}^{\lambda} V_k^{\lambda}. \tag{41}$$

**Proof.** Using (39) and (40), we obtain

$$\begin{aligned}
\left(V_{k+1}^{\lambda} - V_k^{\lambda}\right)^2 &= \left(V_{k+1}^{\lambda}\right)^2 - 2V_{k+1}^{\lambda}V_k^{\lambda} + \left(V_k^{\lambda}\right)^2 + \lambda V_{k+1}^{\lambda}V_k^{\lambda} - \lambda V_{k+1}^{\lambda}V_k^{\lambda} \\
&= \left(V_{k+1}^{\lambda}\right)^2 - V_{k+1}^{\lambda}(2-\lambda)V_k^{\lambda} + \left(V_k^{\lambda}\right)^2 - \lambda V_{k+1}^{\lambda}V_k^{\lambda} \\
&= \left(V_{k+1}^{\lambda}\right)^2 - V_{k+1}^{\lambda}\left(V_{k-1}^{\lambda} + V_{k+1}^{\lambda}\right) + 1 + V_{k+1}^{\lambda}V_{k-1}^{\lambda} - \lambda V_{k+1}^{\lambda}V_k^{\lambda} \\
&= 1 - \lambda \cdot V_{k+1}^{\lambda}V_k^{\lambda}. \quad \square
\end{aligned}$$

Now, let us take into account $\kappa_\beta$ defined in (20) for $\beta > 0$ (see Fig. 16). The function $\beta \mapsto \kappa_\beta$ is a piecewise constant and decreasing function, which has a jump discontinuity at $\xi_k$ for $k \in \mathbb{N}$, $k \geq 2$, defined in (9). Let us note that for $\beta = \xi_k$, $k \in \mathbb{N}$, $k \geq 2$, we have $\omega_\beta = \frac{\pi}{k}$, $\kappa_\beta = k-1$ and $W_k^\beta$ is the identity function on $\mathbb{R}^*$ (see Lemma 11 in [20] for $\lambda = \beta$ and $j = 1$). Thus, we have

$$W_{\kappa_\beta+1}^{\beta}(q) = q, \quad q \in \mathbb{R}^*, \qquad \text{for } \beta = \xi_k,\ k \in \mathbb{N},\ k \geq 2. \tag{42}$$

Let us investigate some basic properties of $V_{\kappa_\beta}^{\beta}$ and $V_{\kappa_\beta+1}^{\beta}$ (see Figs. 18 and 19).

**Lemma 7.** *For $\beta > 0$, we have $0 \leq V_{\kappa_\beta}^{\beta} \leq 1$ and $V_{\kappa_\beta+1}^{\beta} < 0$. Moreover, $V_{\kappa_\beta}^{\beta}$ and $V_{\kappa_\beta+1}^{\beta}$ have the following properties:*

1. *$V_{\kappa_\beta}^{\beta} = 1$ if and only if $\beta > \xi_2 = 2$.*
2. *$V_{\kappa_\beta}^{\beta} = 0$ if and only if $\beta = \xi_k$ for some $k \in \mathbb{N}$, $k \geq 2$.*
3. *$V_{\kappa_\beta+1}^{\beta} = -1$ for $\beta = \xi_k$, $k \in \mathbb{N}$, $k \geq 2$.*
4. *If $V_{\kappa_\beta}^{\beta} + V_{\kappa_\beta+1}^{\beta} = -1$ for $0 < \beta \neq 4$ then $\beta = \xi_k$ for some $k \in \mathbb{N}$, $k \geq 2$.*

**Proof.** At first, let us assume that $\beta > \xi_2 = 2$. In this case, we have $\kappa_\beta = 0$ and thus $V_{\kappa_\beta}^{\beta} = V_0^{\beta} \equiv 1$ and $V_{\kappa_\beta+1}^{\beta} = V_1^{\beta} = 2 - \beta < 0$.

At second, let us assume that $\xi_{k+2} < \beta \leq \xi_{k+1} \leq \xi_2$ for fixed $k \in \mathbb{N}$. Then $\kappa_\beta = k$ and it suffices to show that (see Fig. 17)

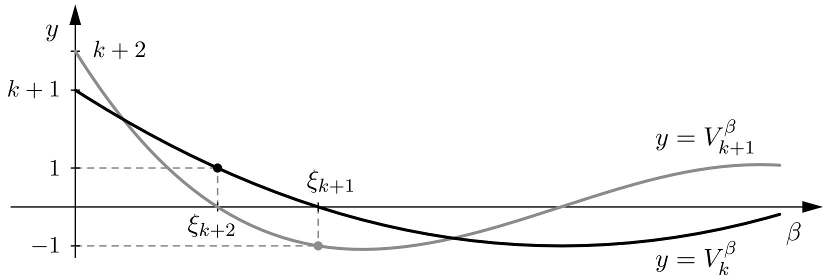


**Fig. 17.** Graphs of functions $\beta \mapsto V_k^\beta$ (black curve) and $\beta \mapsto V_{k+1}^\beta$ (grey curve).

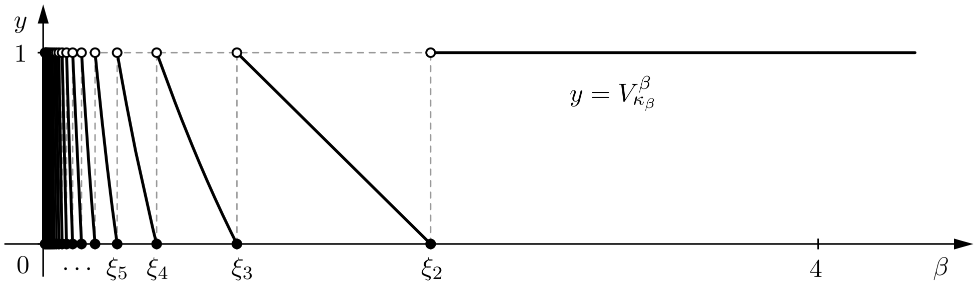


**Fig. 18.** The graph of the function $\beta \mapsto V_{\kappa_\beta}^\beta$.

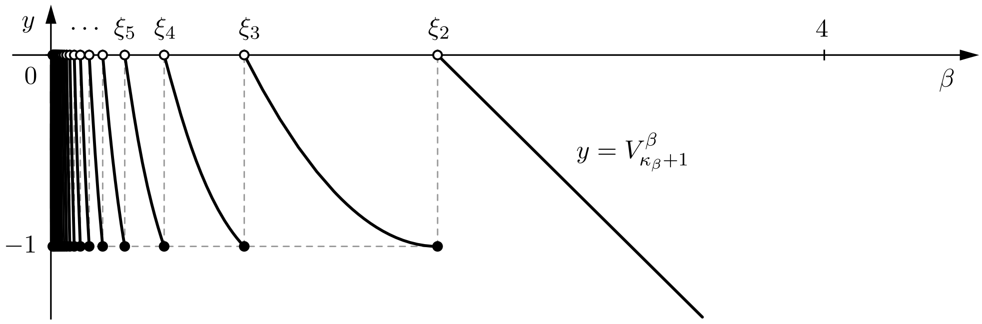


**Fig. 19.** The graph of the function $\beta \mapsto V_{\kappa_\beta+1}^\beta$.

1. $V_k^\beta = 0$ and $V_{k+1}^\beta = -1$ for $\beta = \xi_{k+1}$,
2. $0 < V_k^\beta < 1$ and $V_{k+1}^\beta < 0$ for $\xi_{k+2} < \beta < \xi_{k+1}$.

Now, we have that $V_k^\beta = 0$ if and only if $\beta = 4\sin^2 \frac{m\pi}{2(k+1)}$, $m \in \{1, \dots, k\}$. Thus, the first zero of $V_k^\beta$ is $\beta = \xi_{k+1}$. Similarly, $\beta = \xi_{k+2} < \xi_{k+1}$ is the first zero of $V_{k+1}^\beta$. Moreover, we have that $V_k^\beta > 0$ for $0 < \beta < \xi_{k+1}$ since for $\beta = 0$, we have $V_k^\beta = k + 1 > 0$. Using (41) for $\lambda = \beta = \xi_{k+2}$, we obtain

$$\left(V_{k+1}^\beta - V_k^\beta\right)^2 + \xi_{k+2} \cdot V_{k+1}^\beta V_k^\beta = 1,$$

which simplifies to $\left(V_k^\beta\right)^2 = 1$ since $V_{k+1}^\beta = 0$ for $\beta = \xi_{k+2}$. Thus, we conclude that $V_k^\beta = 1$ for $\beta = \xi_{k+2} < \xi_{k+1}$ since $V_k^\beta$ is positive for $0 < \beta < \xi_{k+1}$. The Chebyshev polynomial of the second kind monotonically oscillates between its extrema and the first extreme of $\beta \mapsto V_k^\beta$ does not belong to the interval $(0, \xi_{k+1})$. Thus, for $\xi_{k+2} < \beta < \xi_{k+1}$, we have that $0 < V_k^\beta < 1$. Since Chebyshev polynomials of the second kind are orthogonal with weight function $\omega(x) = \sqrt{1-x^2}$, using Corollary 3.3.3 on page 93 in [19], we have that two consecutive polynomials strictly interlace, i.e. between two consecutive zeros of $V_{k+1}^\beta$ is exactly one zero of $V_k^\beta$. Since $V_{k+1}^\beta = 0$ for $\beta = \xi_{k+2}$ and $V_k^\beta = 0$ for $\beta = \xi_{k+1}$, we have that $V_{k+1}^\beta < 0$ for $\xi_{k+2} < \beta < \xi_{k+1}$. Finally, using (41) for $\lambda = \beta = \xi_{k+1}$, we obtain $\left(V_{k+1}^\beta\right)^2 = 1$ and thus, we conclude that $V_{k+1}^\beta = -1$.

Now, it remains to justify the last statement. Thus, let us assume that $V_{\kappa_\beta}^\beta + V_{\kappa_\beta+1}^\beta = -1$ for $0 < \beta \neq 4$. Using (41) for $\lambda = \beta$ and $k = \kappa_\beta$, we obtain

$$\begin{aligned}
\left(V_{\kappa_\beta+1}^\beta - V_{\kappa_\beta}^\beta\right)^2 + \beta \cdot V_{\kappa_\beta+1}^\beta V_{\kappa_\beta}^\beta &= 1,\\
\left(2V_{\kappa_\beta}^\beta + 1\right)^2 - \beta \cdot \left(V_{\kappa_\beta}^\beta + 1\right) V_{\kappa_\beta}^\beta &= 1,\\
4\left(V_{\kappa_\beta}^\beta\right)^2 + 4V_{\kappa_\beta}^\beta - \beta \cdot \left(V_{\kappa_\beta}^\beta\right)^2 - \beta \cdot V_{\kappa_\beta}^\beta &= 0,\\
(4-\beta)(V_{\kappa_\beta}^\beta + 1)V_{\kappa_\beta}^\beta &= 0,\\
(\beta - 4)V_{\kappa_\beta+1}^\beta V_{\kappa_\beta}^\beta &= 0,
\end{aligned}$$

which implies that $V_{\kappa_\beta}^\beta = 0$ and thus, $\beta = \xi_k$ for some $k \in \mathbb{N}$, $k \geq 2$. $\square$

In the second part of this section, we simplify the definition of functions $p_i = p_i(\alpha, \beta)$ given by (32) and (33) within the following four lemmas. As a consequence of this simplification, we also obtain the basic bounds for each non-trivial Fučík curve $\mathcal{C}_l^\pm$.

**Lemma 8.** *For $0 < \beta < 4$, we have*

$$\tfrac{\pi}{\omega_\beta} - \left\lfloor \tfrac{\pi}{\omega_\beta} \right\rfloor = T^\beta\left(W_{\kappa_\beta+1}^\beta(\infty)\right). \tag{43}$$

**Proof.** We have that $\frac{\pi}{\omega_\beta} - \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor = \frac{\pi}{\omega_\beta} - 1 - \kappa_\beta$ and thus, using (31), we get

$$\begin{aligned}
Q^\beta\left(\tfrac{\pi}{\omega_\beta} - 1 - \kappa_\beta\right) &= -\frac{\sin\left(\omega_\beta\left(1 - \frac{\pi}{\omega_\beta} + 1 + \kappa_\beta\right)\right)}{\sin\left(\omega_\beta\left(\frac{\pi}{\omega_\beta} - 1 - \kappa_\beta\right)\right)} = \frac{\sin\left(\omega_\beta(\kappa_\beta + 2)\right)}{\sin\left(\omega_\beta(\kappa_\beta+1)\right)}\\
&= \frac{V_{\kappa_\beta+1}^\beta}{V_{\kappa_\beta}^\beta} = W_{\kappa_\beta+1}^\beta(\infty). \quad \square
\end{aligned}$$

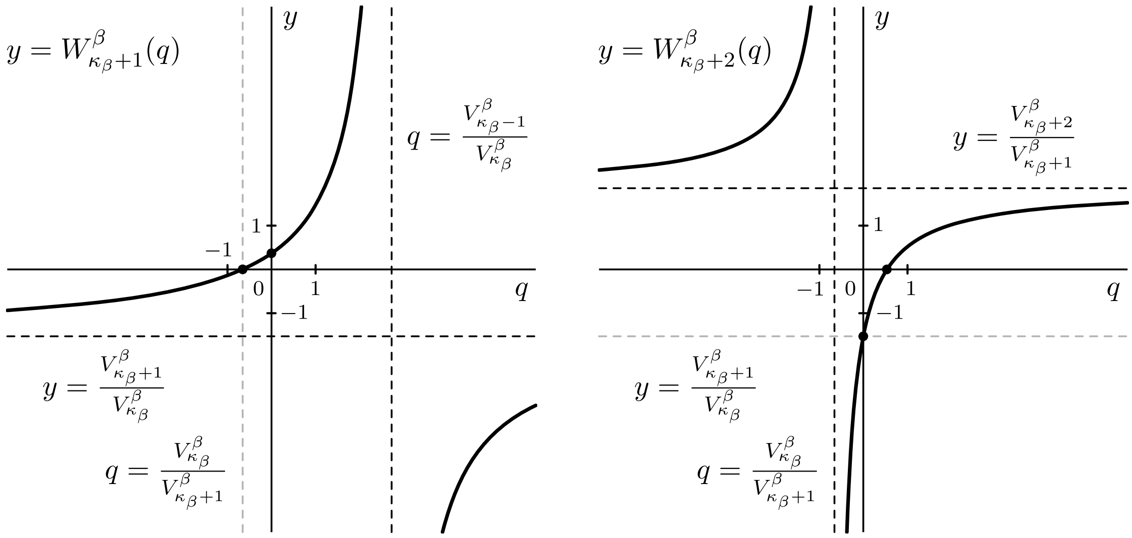


**Fig. 20.** Graphs of functions $W^\beta_{\kappa_\beta+1} = W^\beta_{\kappa_\beta+1}(q)$ (left) and $W^\beta_{\kappa_\beta+2} = W^\beta_{\kappa_\beta+2}(q)$ (right) for fixed $\beta = 0.8$ (i.e. $\kappa_\beta = 2$).

**Lemma 9.** *Let $\beta > 0$.*

1. *If $\beta = \xi_k$, $k \in \mathbb{N}$, $k \geq 2$, then $W^\beta_{\kappa_\beta+1}(q) = \infty$ if and only if $q = \infty$.*
2. *If $\beta > 2$ then $W^\beta_{\kappa_\beta+1}(q) = \infty$ if and only if $q = 0$.*
3. *If $\beta < 2$ and $\beta \neq \xi_k$, $k \in \mathbb{N}$, $k \geq 2$, then $W^\beta_{\kappa_\beta+1}(q)$ is finite for $q \leq 0$ and for $q = \infty$.*

**Proof.** Firstly, in the case of $\beta = \xi_k$, $k \in \mathbb{N}$, $k \geq 2$, we have $k = \kappa_\beta + 1$ and $W^\beta_{\kappa_\beta+1}$ is the identity function (recall (42)). Secondly, for $\beta > 2$, we have $\kappa_\beta = 0$ and $W^\beta_{\kappa_\beta+1}(q) = W^\beta_1(q) = 2 - \beta - 1/q$ (see Fig. 15). Thirdly, let us assume that $\beta \neq \xi_k$, $k \in \mathbb{N}$, $k \geq 2$, and that $0 < \beta < 2$. Then we have (see Fig. 20)

$$W^\beta_{\kappa_\beta+1}(q) = \begin{cases} \dfrac{q \cdot V^\beta_{\kappa_\beta+1} - V^\beta_{\kappa_\beta}}{q \cdot V^\beta_{\kappa_\beta} - V^\beta_{\kappa_\beta-1}} & \text{for } q \in \mathbb{R}, \\ \dfrac{V^\beta_{\kappa_\beta+1}}{V^\beta_{\kappa_\beta}} & \text{for } q = \infty. \end{cases} \tag{44}$$

Using Lemma 7, we obtain that $W^\beta_{\kappa_\beta+1}(\infty)$ is negative and that $W^\beta_{\kappa_\beta+1}(q)$ is finite for $q \leq 0$. Indeed, using (40) for $\lambda = \beta$ and $k = \kappa_\beta$, we have

$$q \cdot V^\beta_{\kappa_\beta} - V^\beta_{\kappa_\beta-1} = \frac{q \cdot V^\beta_{\kappa_\beta} V^\beta_{\kappa_\beta+1} + 1 - \left(V^\beta_{\kappa_\beta}\right)^2}{V^\beta_{\kappa_\beta+1}} < 0. \quad \square$$

The following lemma is based on Lemmas 14 and 16 in [20] and it allows us to determine the length of the interval $[i-1, j+1]$ for a positive or negative semi-wave

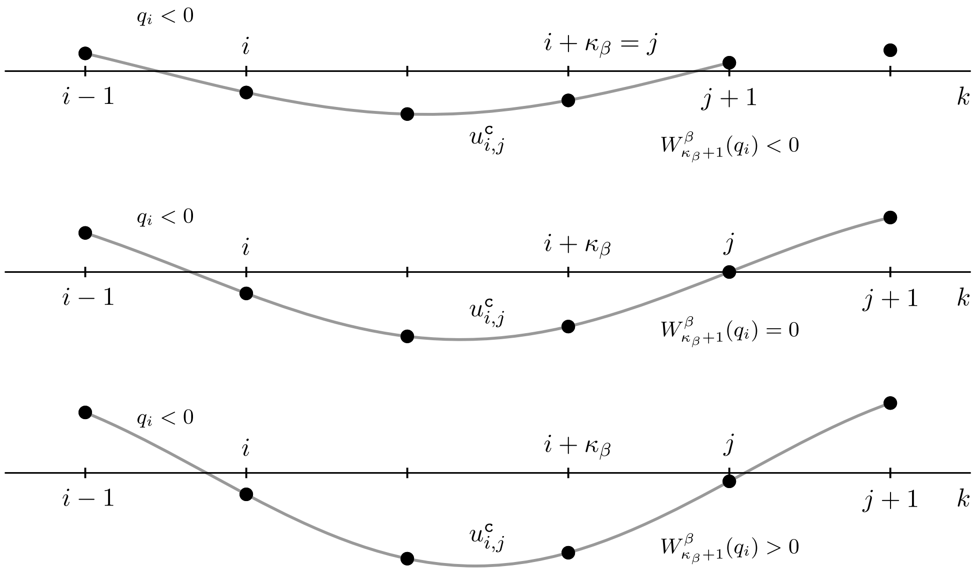


**Fig. 21.** The length of the interval $[i-1, j+1]$ for a negative semi-wave $u^{\mathsf{c}}_{i,j}$ of the solution $u$ of (30) for fixed $(\alpha, \beta) \in D$ according to the sign of $W^\beta_{\kappa_\beta+1}(q_i)$: $j = i + \kappa_\beta + 1$ and $u(j) < 0$ (bottom), $j = i + \kappa_\beta + 1$ and $u(j) = 0$ (middle) and $j = i + \kappa_\beta$ and $u(j) < 0$ (top).

of the solution $u$ according to the ratio $q_i$ of the values $u(i)$ and $u(i-1)$ (see Fig. 21). Let us note that conditions in (45) or (46) mean that the solution $u$ has a positive or negative semi-wave on the interval $[i-1, j+1]$.

**Lemma 10.** *Let $(\alpha, \beta) \in D$ and $u$ be the solution of the initial value problem (30). Moreover, let $i, j \in \mathbb{Z}$ be such that $i \le j$ and*

$$u(i-1) < 0, \quad u(k) \ge 0 \quad \textit{for } k = i, \dots, j, \quad u(j+1) < 0, \tag{45}$$

*or*

$$u(i-1) > 0, \quad u(k) \le 0 \quad \textit{for } k = i, \dots, j, \quad u(j+1) > 0. \tag{46}$$

*Then we have*

$$j = \begin{cases} i + \kappa_\lambda & \textit{for} \quad W^\lambda_{\kappa_\lambda+1}(q_i) < 0, \\ i + \kappa_\lambda + 1 & \textit{for} \quad W^\lambda_{\kappa_\lambda+1}(q_i) \ge 0, \end{cases} \tag{47}$$

*where we denoted $q_i := \frac{u(i)}{u(i-1)} \le 0$ and $\lambda = \alpha$ if (45) holds or $\lambda = \beta$ if (46) holds. Moreover, we have $u(k) \ne 0$ for $k \in \mathbb{Z}$ such that $i < k < j$, and $u(j) = 0$ if and only if $W^\lambda_{\kappa_\lambda+1}(q_i) = 0$.*

**Proof.** Let us assume that conditions in (46) hold, which means that we have a negative semi-wave $u^{\mathsf{c}}_{i,j}$ of $u$ defined on the interval $[i-1, j+1]$ (see Fig. 21). Moreover, let us assume that the value $W^{\beta}_{\kappa_\beta+1}(q_i)$ is finite and we split the proof according to the value of $\beta$.

At first, let us consider $0 < \beta < 4$. Using Lemma 14 in [20], we have that

$$j = i + \left\lfloor T^{\beta}(q_i) + \tfrac{\pi}{\omega_\beta} \right\rfloor - 1, \tag{48}$$

where $q_i = \frac{u(i)}{u(i-1)} \le 0$. If we denote $s = 1 - T^{\beta}(q_i)$ then (48) reads

$$j = i + \left\lfloor \tfrac{\pi}{\omega_\beta} - s \right\rfloor \tag{49}$$

and $s \in [0, 1)$ since $0 < T^{\beta}(q_i) \le 1$. Now, let us consider that

$$s > \tfrac{\pi}{\omega_\beta} - \left\lfloor \tfrac{\pi}{\omega_\beta} \right\rfloor \ge 0, \tag{50}$$

which implies that $\left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor - 1 \le \frac{\pi}{\omega_\beta} - 1 < \frac{\pi}{\omega_\beta} - s < \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor$ and that $\left\lfloor \frac{\pi}{\omega_\beta} - s \right\rfloor = \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor - 1 = \kappa_\beta$. Thus, we obtain using (49) that

$$j = i + \kappa_\beta. \tag{51}$$

Moreover, using (43), (27) and (37), the strict inequality in (50) reads

$$\begin{aligned}
1 - T^{\beta}(q_i) &> T^{\beta}\left(W^{\beta}_{\kappa_\beta+1}(\infty)\right), \\
T^{\beta}(q_i) &< T^{\beta}\left(W^{\beta}_{-(\kappa_\beta+1)}(0)\right), \\
W^{\beta}_{\kappa_\beta+1}(q_i) &< 0,
\end{aligned}$$

which justifies (47) if we take into account (51). Now, let us consider that

$$0 \le s \le \tfrac{\pi}{\omega_\beta} - \left\lfloor \tfrac{\pi}{\omega_\beta} \right\rfloor, \tag{52}$$

which implies $\left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor \le \frac{\pi}{\omega_\beta} - s \le \frac{\pi}{\omega_\beta} < \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor + 1$ and that $\left\lfloor \frac{\pi}{\omega_\beta} - s \right\rfloor = \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor = \kappa_\beta + 1$. Thus, we obtain using (49) that

$$j = i + \kappa_\beta + 1. \tag{53}$$

And similarly as in the previous case, using (43), (27) and (37), the second inequality in (52) reads $W^{\beta}_{\kappa_\beta+1}(q_i) \ge 0$, which justifies (47) if we take into account (53).

At second, let us consider $\beta \ge 4$. Then we have $0 < \alpha < 4$ and using Lemma 16 in [20], we obtain that

$$j = i + \lfloor T^{\alpha}(q_i) + T^{\alpha}(2 - \beta) \rfloor, \tag{54}$$

where $q_i = \frac{u(i)}{u(i-1)} \leq 0$. Since $0 < T^\alpha(q_i) \leq 1$ and $0 < T^\alpha(2-\beta) < \frac{1}{2}$, we have $0 < T^\alpha(q_i) + T^\alpha(2-\beta) < \frac{3}{2}$ and thus, (54) reads

$$j = i \qquad \text{for} \quad T^\alpha(q_i) + T^\alpha(2-\beta) < 1, \tag{55}$$

$$j = i+1 \qquad \text{for} \quad T^\alpha(q_i) + T^\alpha(2-\beta) \geq 1. \tag{56}$$

The inequality in (55) reads $T^\alpha(q_i) < T^\alpha\left(\frac{1}{2-\beta}\right)$ or $q_i < W^\beta_{-1}(0)$, which justifies (47) since $\kappa_\beta = 0$ for $\beta \geq 4$. And similarly, (56) can be identified with the second case in (47).

Finally, for $\beta > 0$ and $k \in \mathbb{Z}$ such that $i < k < j$, we have that $u(k) < 0$. In contrary, if we assume that $u(k) = 0$ for some $k$ strictly between $i$ and $j$ then $u(k-1)u(k+1) < 0$, which contradicts (46). Moreover, we have $u(j) = 0$ if and only if $W^\beta_{\kappa_\beta+1}(q_i) = 0$. Indeed, using (38), we have

$$W^\beta_{\kappa_\beta+1}(q_i) = q_{i+\kappa_\beta+1} = \begin{cases} q_{j+1} = \frac{u(j+1)}{u(j)} & \text{for } W^\beta_{\kappa_\beta+1}(q_i) < 0, \\ q_j = \frac{u(j)}{u(j-1)} & \text{for } W^\beta_{\kappa_\beta+1}(q_i) \geq 0. \end{cases}$$

Thus, the proof is complete in the case of a negative semi-wave such that the value $W^\beta_{\kappa_\beta+1}(q_i)$ is finite. Now, let us clarify that the case $W^\beta_{\kappa_\beta+1}(q_i) = \infty$ cannot occur. If we assume that $W^\beta_{\kappa_\beta+1}(q_i) = \infty$ then we have $q_{i+\kappa_\beta+1} = \infty$, $u(i+\kappa_\beta) = 0$ and $j = i + \kappa_\beta$. Taking into account that $q_i$ is finite, we obtain using Lemma 9 that $q_i = 0$ and $\beta > 2$. Thus, we have that $\kappa_\beta = 0$, $i = j$ and that $u(i-1) > u(i) = 0 = u(j) < u(j+1)$, which is a contradiction.

In the case of a positive semi-wave on $[i-1, j+1]$, i.e. if conditions in (45) hold, we prove statements in an analogous way. □

**Remark 11.** Let $u$ be the solution of (30) for $(\alpha, \beta) \in D$ such that $u(i-1) = 0$ and on the interval $[i-2, j+1]$, we have a negative semi-wave (cf. (46) in Lemma 10)

$$u(i-2) > 0, \quad u(i-1) = 0, \quad u(k) \leq 0 \quad \text{for } k = i, \ldots, j, \quad u(j+1) > 0.$$

Then we have $q_i = \frac{u(i)}{u(i-1)} = \infty$, $j = i + \kappa_\beta$ and $W^\beta_{\kappa_\beta+1}(q_i)$ is negative or infinity. Moreover, using Lemma 9, we conclude that $W^\beta_{\kappa_\beta+1}(q_i) = \infty$ if and only if $\beta = \xi_k$, $k \in \mathbb{N}$, $k \geq 2$.

We provide a new expression for the values of $p_i(\alpha, \beta)$ (defined in (32) and (33)) using the Heaviside unit step function $H$ defined as

$$H(q) := \begin{cases} 1 & \text{for } q \geq 0, \\ 0 & \text{for } q < 0 \text{ or } q = \infty. \end{cases}$$

**Lemma 12.** *For* $(\alpha,\beta)\in D$ *and* $k\in\mathbb{N}$, *we have*

$$\begin{aligned} p_1(\alpha,\beta) &= \kappa_\alpha + 1, \\ p_{2k}(\alpha,\beta) &= \kappa_\beta + 1 + H\big(W^{\beta}_{\kappa_\beta+1}(\vartheta_{2k-1}(\alpha,\beta))\big), && (57) \\ p_{2k+1}(\alpha,\beta) &= \kappa_\alpha + 1 + H\big(W^{\alpha}_{\kappa_\alpha+1}(\vartheta_{2k}(\alpha,\beta))\big). && (58) \end{aligned}$$

**Proof.** Firstly, the statement (47) in Lemma 10 can be equivalently written in the following form

$$j = i + \kappa_\lambda + H\big(W^{\lambda}_{\kappa_\lambda+1}(q_i)\big). \tag{59}$$

Now, let $u$ be the solution of the initial value problem (30) with $C_1 > 0$. Then $u$ has a positive semi-wave on the interval $[-1, p_1+1]$, where $p_1$ is defined in (32) as

$$p_1(\alpha,\beta) = \begin{cases} \left\lfloor T^{\alpha}(\infty) + \frac{\pi}{\omega_\alpha} \right\rfloor & \text{for } \alpha < 4, \\ \left\lfloor T^{\beta}(\infty) + T^{\beta}(2-\alpha) + 1 \right\rfloor & \text{for } \alpha \ge 4. \end{cases}$$

Since $T^{\alpha}(\infty) = T^{\beta}(\infty) = 0$, we have that $p_1(\alpha,\beta) = \kappa_\alpha + 1$. Indeed, for $\alpha \ge 4$, we have $\kappa_\alpha = 0$ and $0 < T^{\beta}(2-\alpha) < \frac{1}{2}$. Moreover, using (34), we have $\vartheta_1(\alpha,\beta) = W^{\alpha}_{p_1(\alpha,\beta)}(\infty)$ which is exactly the value of $\frac{u(p_1+1)}{u(p_1)}$. The solution $u$ has a negative semi-wave on the interval $[p_1, p_1+p_2+1]$, where $p_2$ is given by (33) in the following way

$$p_2(\alpha,\beta) = \begin{cases} \left\lfloor T^{\beta}(\vartheta_1(\alpha,\beta)) + \frac{\pi}{\omega_\beta} \right\rfloor & \text{for } \beta < 4, \\ \left\lfloor T^{\alpha}(\vartheta_1(\alpha,\beta)) + T^{\alpha}(2-\beta) + 1 \right\rfloor & \text{for } \beta \ge 4. \end{cases}$$

Thus, using (48), (54) and (59) for $\lambda = \beta$, we get

$$p_2(\alpha,\beta) = j - i + 1 = \kappa_\beta + H\big(W^{\beta}_{\kappa_\beta+1}(\vartheta_1(\alpha,\beta))\big) + 1,$$

which corresponds to (57). Moreover, using (35), we have that $\vartheta_2(\alpha,\beta) = W^{\beta}_{p_2(\alpha,\beta)}(\vartheta_1(\alpha,\beta))$ which is equal to $\frac{u(p_1+p_2+1)}{u(p_1+p_2)}$. And similarly, the solution $u$ has a positive semi-wave on $[p_1+p_2, p_1+p_2+p_3+1]$, where $p_3$ is defined in (32) as

$$p_3(\alpha,\beta) = \begin{cases} \left\lfloor T^{\alpha}(\vartheta_2(\alpha,\beta)) + \frac{\pi}{\omega_\alpha} \right\rfloor & \text{for } \alpha < 4, \\ \left\lfloor T^{\beta}(\vartheta_2(\alpha,\beta)) + T^{\beta}(2-\alpha) + 1 \right\rfloor & \text{for } \alpha \ge 4. \end{cases}$$

Thus, using (59) for $\lambda = \alpha$, we obtain

$$p_3(\alpha,\beta) = j - i + 1 = \kappa_\alpha + H\big(W^{\alpha}_{\kappa_\alpha+1}(\vartheta_2(\alpha,\beta))\big) + 1,$$

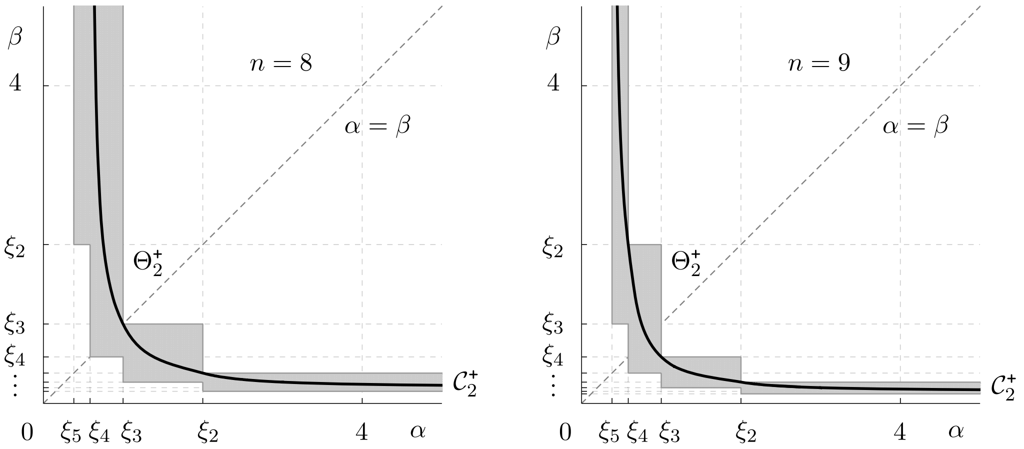


**Fig. 22.** The set $\Theta_2^+$ (grey region) as the basic bound for the second non-trivial Fučík curve $\mathcal{C}_2^+ \subset \Theta_2^+$ (black curve) for $n = 8$ (left) and for $n = 9$ (right).

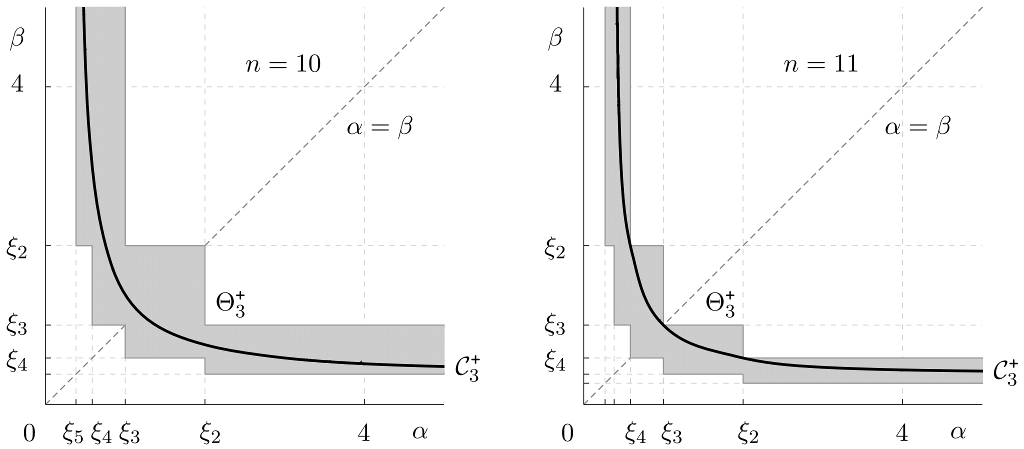


**Fig. 23.** The set $\Theta_3^+$ (grey region) as the basic bound for the third non-trivial Fučík curve $\mathcal{C}_3^+ \subset \Theta_3^+$ (black curve) for $n = 10$ (left) and for $n = 11$ (right).

which corresponds to (58). To conclude, we have justified (57) and (58) for $k = 1$ (i.e. for $p_2$ and $p_3$, respectively). In the case of $k \geq 2$, the proof of (57) and (58) concerning $p_{2k}$ and $p_{2k+1}$ can be done in an analogous way. □

At the end of this section, using Lemma 12, we obtain some basic bounds for each Fučík curve $\mathcal{C}_l^\pm \subset \Theta_l^\pm$ (see Figs. 22 and 23).

**Theorem 13.** *In the domain $D$, we have the following bounds for Fučík curves $\mathcal{C}_l^\pm \subset \Theta_l^\pm$, $l = 1, \dots, n-1$, where*

$$
\begin{aligned}
\Theta_{2j-1}^\pm &:= \{(\alpha,\beta) \in D : \; 0 \leq n+1 - j(\kappa_\alpha + 1) - j(\kappa_\beta + 1) \leq 2j - 1\}, \\
\Theta_{2j}^+ &:= \{(\alpha,\beta) \in D : \; 0 \leq n+1 - (j+1)(\kappa_\alpha + 1) - j(\kappa_\beta + 1) \leq 2j\}, \\
\Theta_{2j}^- &:= \{(\alpha,\beta) \in D : \; 0 \leq n+1 - j(\kappa_\alpha + 1) - (j+1)(\kappa_\beta + 1) \leq 2j\}.
\end{aligned}
$$

**Proof.** First of all, it is enough to focus only on Fučík curves $\mathcal{C}_l^+$ since we have

$$\mathcal{C}_l^- = \{(\alpha,\beta) \in D : \ (\beta,\alpha) \in \mathcal{C}_l^+\}.$$

By Theorem 22 in [20], we have that $\mathcal{C}_l^+ \subset \Omega_l^+$, where $\Omega_l^+$ is the set of all pairs $(\alpha,\beta) \in D$ such that

$$\sum_{i=1}^{l+1} p_i(\alpha,\beta) = n+1.$$

Thus, using Lemma 12, we obtain for $l = 2j-1$ that

$$j(\kappa_\alpha + 1) + j(\kappa_\beta + 1) \le n+1 \le j(\kappa_\alpha + 1) + j(\kappa_\beta + 1) + 2j - 1$$

and for $l = 2j$ that

$$(j+1)(\kappa_\alpha + 1) + j(\kappa_\beta + 1) \le n+1 \le (j+1)(\kappa_\alpha + 1) + j(\kappa_\beta + 1) + 2j. \quad \square$$

## 5. Implicit description of Fučík curves

In this section, we investigate the distance between zeros of two consecutive continuous positive semi-waves of the solution $u$ of the initial value problem (30) for $0 < \alpha < 4$ and $\beta > 0$. Thus, let $i, j \in \mathbb{Z}$ be such that $i < j$ and that (46) holds, i.e. $i$ is the generalized zero of $u$ and the next generalized zero of $u$ is $j$ or $(j+1)$ if $u(j) = 0$ or $u(j) < 0$, respectively. Moreover, we have two consecutive continuous positive semi-waves $u_1^{\mathrm{c}}$ and

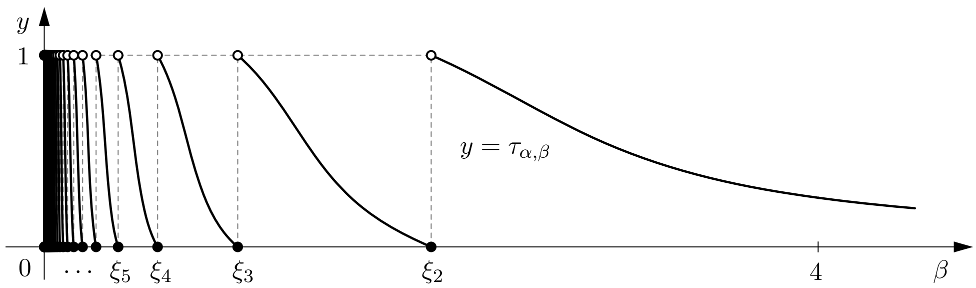


**Fig. 24.** The graph of the function $\beta \mapsto \tau_{\alpha,\beta}$ for fixed $\alpha = 2.9$.

$u_2^{\mathrm{c}}$ of $u$ with zeros $t_1 \in (i-1, i]$ and $t_2 \in [j, j+1)$, respectively. In the following Lemma 14, we show how to reconstruct the zero $t_2$ according to values of $t_1$, $\alpha$ and $\beta$. For this reconstruction, we use $\tau_{\alpha,\beta} = T^\alpha\big(V^\beta_{\kappa_\beta+1}/V^\beta_{\kappa_\beta}\big)$ introduced in Definition 1 (see Fig. 24) to distinguish between two disjoint cases (see Fig. 25)

$$j = i + \kappa_\beta \qquad \text{and} \qquad j = i + \kappa_\beta + 1.$$

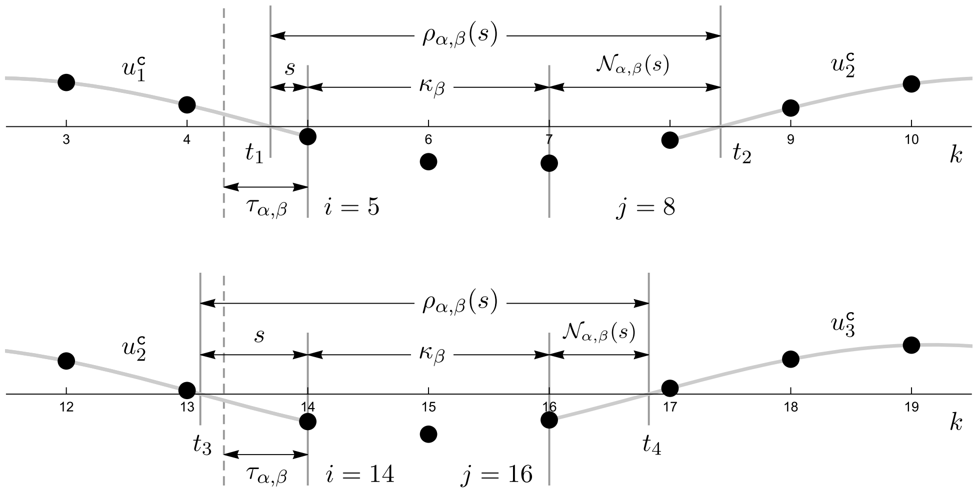


**Fig. 25.** Two details of the solution $u$ of the initial value problem (30) for fixed $\alpha = 0.432$ and $\beta = 0.671$ (i.e. $\kappa_\beta = 2$). On top, we have two continuous positive semi-waves $u_1^{\mathsf{c}}$ and $u_2^{\mathsf{c}}$ with zeros $t_1$ and $t_2$ such that $j = i + \kappa_\beta + 1$. Bottom, we have two continuous positive semi-waves $u_2^{\mathsf{c}}$ and $u_3^{\mathsf{c}}$ with zeros $t_3$ and $t_4$ such that $j = i + \kappa_\beta$.

Let us note that $0 \le \tau_{\alpha,\beta} < 1$ since $V^\beta_{\kappa_\beta+1}/V^\beta_{\kappa_\beta}$ is negative or equal to $\infty$ according to Lemma 7.

**Lemma 14.** *Let $u$ be the solution of the initial value problem (30) for $0 < \alpha < 4$ and $\beta > 0$ and let $u_1^{\mathsf{c}}$ and $u_2^{\mathsf{c}}$ be two consecutive continuous positive semi-waves of $u$. Moreover, let $t_1$ be the second zero of $u_1^{\mathsf{c}}$ and let $t_2$ be the first zero of $u_2^{\mathsf{c}}$. If we denote $s = \lceil t_1 \rceil - t_1$ then we have*

$$t_2 = \begin{cases} t_1 + s + \kappa_\beta + T^\alpha\left(W^\beta_{\kappa_\beta+1}\left(Q^\alpha(1-s)\right)\right) & \text{for } s > \tau_{\alpha,\beta}, \\ t_1 + s + \kappa_\beta + 1 + T^\alpha\left(W^\beta_{\kappa_\beta+2}\left(Q^\alpha(1-s)\right)\right) & \text{for } s \le \tau_{\alpha,\beta}. \end{cases} \tag{60}$$

**Proof.** We have $t_1 \in (i-1, i]$ and $t_2 \in [j, j+1)$, where $i, j \in \mathbb{Z}$ are such that $i < j$ and that (46) holds. Moreover, we have

$$q_i = Q^\alpha\left(1-s\right), \qquad q_{j+1} = W^\beta_{j-i+1}(q_i), \tag{61}$$

where we denoted $q_k := \frac{u(k)}{u(k-1)}$ for $k = i, \ldots, j+1$. Now, using Lemma 10 and (47) for $\lambda = \beta$, we obtain

$$j = \begin{cases} i + \kappa_\beta & \text{for } W^\beta_{\kappa_\beta+1}(q_i) < 0, \\ i + \kappa_\beta + 1 & \text{for } W^\beta_{\kappa_\beta+1}(q_i) \ge 0. \end{cases} \tag{62}$$

Since $t_2 - j = t_2 - \lfloor t_2 \rfloor = T^\alpha\left(q_{j+1}\right)$, we get using the second equality in (61) that $t_2 = j + T^\alpha\left(W^\beta_{j-i+1}(q_i)\right)$, which implies using (62) that

$$t_2 = \begin{cases} i + \kappa_\beta + T^\alpha \left( W^\beta_{\kappa_\beta+1}(q_i) \right) & \text{for } W^\beta_{\kappa_\beta+1}(q_i) < 0, \\ i + \kappa_\beta + 1 + T^\alpha \left( W^\beta_{\kappa_\beta+2}(q_i) \right) & \text{for } W^\beta_{\kappa_\beta+1}(q_i) \geq 0. \end{cases} \tag{63}$$

Using the first equality in (61), the inequality $W^\beta_{\kappa_\beta+1}(q_i) < 0$ in (63) reads $Q^\alpha(1-s) < V^\beta_{\kappa_\beta}/V^\beta_{\kappa_\beta+1}$ which can be equivalently written as $Q^\alpha(s) > V^\beta_{\kappa_\beta+1}/V^\beta_{\kappa_\beta}$ or as $s > \tau_{\alpha,\beta}$ due to (29). Similarly, we obtain that the second inequality $W^\beta_{\kappa_\beta+1}(q_i) \geq 0$ in (63) reads $s \leq \tau_{\alpha,\beta}$. To conclude, (63) can be also written in the following way

$$t_2 = \begin{cases} i + \kappa_\beta + T^\alpha \left( W^\beta_{\kappa_\beta+1}(q_i) \right) & \text{for } s > \tau_{\alpha,\beta}, \\ i + \kappa_\beta + 1 + T^\alpha \left( W^\beta_{\kappa_\beta+2}(q_i) \right) & \text{for } s \leq \tau_{\alpha,\beta}, \end{cases}$$

which is exactly (60) if we take into account the first equality in (61) and that $i = \lceil t_1 \rceil = t_1 + s$. $\square$

**Remark 15.** Let us note that for $s = \tau_{\alpha,\beta}$, we have using (60) that $t_2 = t_1 + s + \kappa_\beta + 1$, since $Q^\alpha(1 - \tau_{\alpha,\beta}) = 1/Q^\alpha(\tau_{\alpha,\beta}) = V^\beta_{\kappa_\beta}/V^\beta_{\kappa_\beta+1}$ and thus

$$T^\alpha \left( W^\beta_{\kappa_\beta+2} \left( Q^\alpha(1 - \tau_{\alpha,\beta}) \right) \right) = T^\alpha(\infty) = 0.$$

See also Fig. 20 and note that for $q = V^\beta_{\kappa_\beta}/V^\beta_{\kappa_\beta+1}$, we have $W^\beta_{\kappa_\beta+1}(q) = 0$ and $W^\beta_{\kappa_\beta+2}(q) = W^\beta_1 \left( W^\beta_{\kappa_\beta+1}(q) \right) = \infty$.

In the following lemma, we provide some basic properties of $\tau_{\alpha,\beta}$ (see Fig. 24).

**Lemma 16.** *For $0 < \alpha < 4$ and $\beta > 0$, we have that $0 \leq \tau_{\alpha,\beta} < 1$. Moreover, if we denote*

$$\xi_k := 4 \sin^2 \frac{\pi}{2k}, \qquad \zeta_k := 4 \sin^2 \frac{\pi}{2k-1} \qquad \textit{for } k \in \mathbb{N},\ k \geq 2,$$

*then we have*

1. $\tau_{\alpha,\beta} = 0$ *if and only if* $\beta = \xi_k$ *for some* $k \in \mathbb{N}$, $k \geq 2$,
2. $\tau_{\alpha,\beta} = \frac{1}{2}$ *if and only if* $\beta = \zeta_k$ *for some* $k \in \mathbb{N}$, $k \geq 2$,
3. $\tau_{\beta,\beta} = \frac{\pi}{\omega_\beta} - \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor$ *for* $0 < \beta < 4$.

**Proof.** First of all, for $0 < \alpha < 4$ and $\beta \geq 4$, we have that $\tau_{\alpha,\beta} = T^\alpha \left( \frac{V^\beta_1}{V^\beta_0} \right) = T^\alpha(2 - \beta)$ and thus, $0 < \tau_{\alpha,\beta} < \frac{1}{2}$ since $(2 - \beta) < -1$. For the rest of the proof, let us restrict to the case $0 < \beta < 4$. We claim that

$$\tau_{\alpha,\beta} = T^\alpha \left( Q^\beta \left( \frac{\pi}{\omega_\beta} - \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor \right) \right) \qquad \text{for } 0 < \alpha, \beta < 4. \tag{64}$$

Indeed, using Lemma 8, we get

$$Q^\beta\left(\tfrac{\pi}{\omega_\beta}-\left\lfloor\tfrac{\pi}{\omega_\beta}\right\rfloor\right)=W^\beta_{\kappa_\beta+1}(\infty)=\frac{V^\beta_{\kappa_\beta+1}}{V^\beta_{\kappa_\beta}},$$

which justifies (64) according to the definition of $\tau_{\alpha,\beta}$. Now, since $0<\frac{\pi}{\omega_\beta}-\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor<1$ or $\frac{\pi}{\omega_\beta}-\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor=0$, we have that

$$Q^\beta\left(\tfrac{\pi}{\omega_\beta}-\left\lfloor\tfrac{\pi}{\omega_\beta}\right\rfloor\right)<0 \qquad \text{or} \qquad Q^\beta\left(\tfrac{\pi}{\omega_\beta}-\left\lfloor\tfrac{\pi}{\omega_\beta}\right\rfloor\right)=\infty,$$

respectively. Thus, using (64), we get that $0<\tau_{\alpha,\beta}<1$ for $\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor<\frac{\pi}{\omega_\beta}$ and that $\tau_{\alpha,\beta}=0$ for $\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor=\frac{\pi}{\omega_\beta}$. Moreover, $\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor=\frac{\pi}{\omega_\beta}$ if and only if $\frac{\pi}{\omega_\beta}=k$, $k\in\mathbb{N}$, $k\geq 2$ (recall that $1<\frac{\pi}{\omega_\beta}$), and let us note that $\frac{\pi}{\omega_\beta}=k$ reads $\beta=2-2\cos\frac{\pi}{k}$ or $\beta=\xi_k$. Finally, using (64), we conclude that $\tau_{\alpha,\beta}=\frac{1}{2}$ can be equivalently written as $Q^\beta\left(\frac{\pi}{\omega_\beta}-\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor\right)=-1$ or $\frac{\pi}{\omega_\beta}=\left\lfloor\frac{\pi}{\omega_\beta}\right\rfloor+\frac{1}{2}$ or $\frac{\pi}{\omega_\beta}=k+\frac{1}{2}$, $k\in\mathbb{N}$, $k\geq 2$, and let us note that $\frac{\pi}{\omega_\beta}=k+\frac{1}{2}$ reads $\beta=2-2\cos\frac{2\pi}{2k+1}$ or $\beta=\zeta_k$. □

Let us introduce the function $\mathcal{N}_{\alpha,\beta}:[0,1+\tau_{\alpha,\beta}]\to\mathbb{R}$, which we use to measure the distance $t_2-t_1$ between zeros $t_1$ and $t_2$ of two consecutive continuous positive semi-waves of $u$ (see Figs. 25 and 26).

**Definition 17.** For $0<\alpha<4$ and $\beta>0$, let us define

$$\mathrm{Dom}(\mathcal{N}_{\alpha,\beta}):=[0,1+\tau_{\alpha,\beta}], \qquad \mathcal{N}_{\alpha,\beta}(s):=\begin{cases}\bar{\bar{M}}_{\alpha,\beta}(s)+1 & \text{for } s\in[0,\tau_{\alpha,\beta}],\\ \bar{M}_{\alpha,\beta}(s) & \text{for } s\in(\tau_{\alpha,\beta},1),\\ \bar{\bar{M}}_{\alpha,\beta}(s-1) & \text{for } s\in[1,1+\tau_{\alpha,\beta}],\end{cases}$$

where

$$\begin{aligned}\bar{M}_{\alpha,\beta}(s) &:= T^\alpha\left(W^\beta_{\kappa_\beta+1}\left(Q^\alpha(1-s)\right)\right), \quad s\in[\tau_{\alpha,\beta},1],\\ \bar{\bar{M}}_{\alpha,\beta}(s) &:= T^\alpha\left(W^\beta_{\kappa_\beta+2}\left(Q^\alpha(1-s)\right)\right), \quad s\in[0,\tau_{\alpha,\beta}].\end{aligned}$$

**Remark 18.** Let $u$ be the solution of the initial value problem (30) for $0<\alpha<4$ and $\beta>0$ and let $t_1$ and $t_2$ be two zeros of positive semi-waves of $u$ as in Lemma 14 (see Fig. 25). Then we have

$$t_2=t_1+s+\kappa_\beta+\mathcal{N}_{\alpha,\beta}(s), \quad 0\leq s\leq 1+\tau_{\alpha,\beta}. \tag{65}$$

Indeed, for $0\leq s<1$, we have that $s=\lceil t_1\rceil-t_1$ and we obtain using (60) that

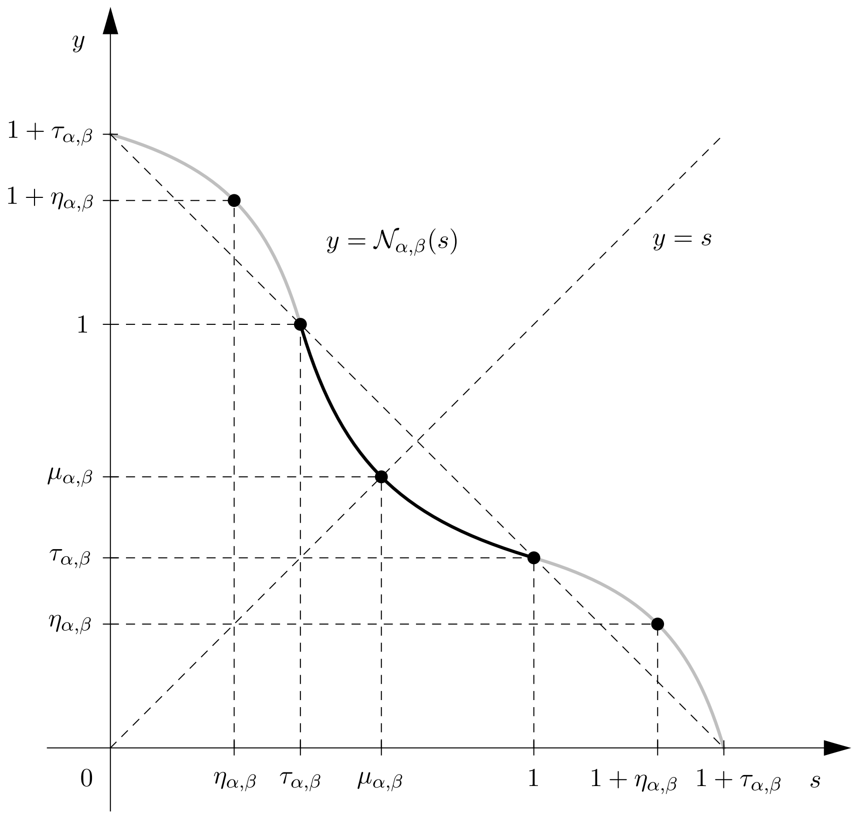


**Fig. 26.** The graph of the function $\mathcal{N}_{\alpha,\beta} = \mathcal{N}_{\alpha,\beta}(s)$ for fixed $\alpha = 1.2$ and $\beta = 3.2$.

$$t_2 = \begin{cases} t_1 + s + \kappa_\beta + 1 + \bar{\bar{M}}_{\alpha,\beta}(s) & \text{for } 0 \le s \le \tau_{\alpha,\beta}, \\ t_1 + s + \kappa_\beta + \bar{M}_{\alpha,\beta}(s) & \text{for } \tau_{\alpha,\beta} < s < 1. \end{cases}$$

Moreover, for $1 \le s \le 1 + \tau_{\alpha,\beta}$, we have $0 \le s - 1 \le \tau_{\alpha,\beta}$ and thus,

$$t_2 = t_1 + s - 1 + \kappa_\beta + 1 + \bar{\bar{M}}_{\alpha,\beta}(s-1) = t_1 + s + \kappa_\beta + \mathcal{N}_{\alpha,\beta}(s).$$

Let us introduce the function $\rho_{\alpha,\beta}$ according to (65), which measures the distance between zeros of two consecutive continuous positive semi-waves.

**Definition 19.** Let $0 < \alpha < 4$ and $\beta > 0$. Let us define

$$\rho_{\alpha,\beta}(s) := s + \kappa_\beta + \mathcal{N}_{\alpha,\beta}(s), \qquad 0 \le s \le 1 + \tau_{\alpha,\beta}.$$

Now, using (65), we have for zeros $t_1$ and $t_2$ in Lemma 14 that

$$t_2 = t_1 + \rho_{\alpha,\beta}(\lceil t_1 \rceil - t_1). \tag{66}$$

In the following three lemmas, let us investigate some basic properties of $\mathcal{N}_{\alpha,\beta}$.

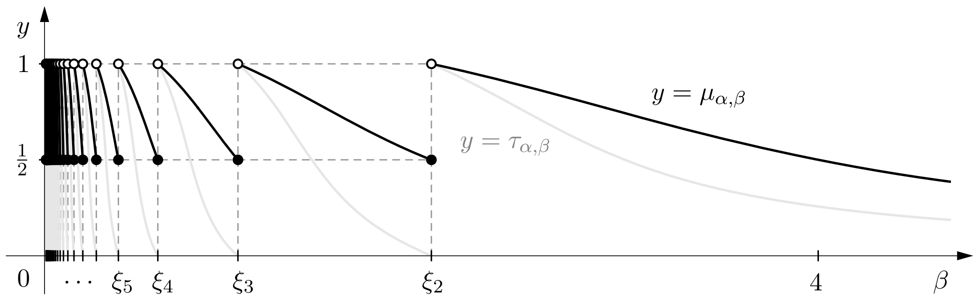


**Fig. 27.** The graph of the function $\beta \mapsto \mu_{\alpha,\beta}$ for fixed $\alpha = 2.9$.

**Lemma 20.** *The function $\mathcal{N}_{\alpha,\beta}$ is a continuous involution, i.e.*

$$\forall s \in [0, 1 + \tau_{\alpha,\beta}] : \; \mathcal{N}_{\alpha,\beta}(\mathcal{N}_{\alpha,\beta}(s)) = s.$$

*Moreover, we have $\mathcal{N}_{\alpha,\beta}(0) = 1 + \tau_{\alpha,\beta}$ and $\mathcal{N}_{\alpha,\beta}(\tau_{\alpha,\beta}) = 1$.*

**Proof.** At first, $\bar{M}_{\alpha,\beta}$ is the continuous strictly decreasing function on $[\tau_{\alpha,\beta}, 1]$, which maps this interval onto itself. Moreover, $\bar{M}_{\alpha,\beta}$ is an involution. Indeed, for $\tau_{\alpha,\beta} \le s \le 1$, we have

$$\bar{M}_{\alpha,\beta}(s) = T^{\alpha}\left(W^{\beta}_{\kappa_\beta+1}\left(1/Q^{\alpha}(s)\right)\right) = T^{\alpha}\left(1/W^{\beta}_{-\kappa_\beta-1}\left(Q^{\alpha}(s)\right)\right)$$

and thus, we obtain

$$\bar{M}_{\alpha,\beta}(\bar{M}_{\alpha,\beta}(s)) = T^{\alpha}\left(1/W^{\beta}_{-\kappa_\beta-1}\left(Q^{\alpha}\left(T^{\alpha}\left(W^{\beta}_{\kappa_\beta+1}\left(1/Q^{\alpha}(s)\right)\right)\right)\right)\right) = s.$$

At second, $\bar{\bar{M}}_{\alpha,\beta}$ is the continuous strictly decreasing function on $[0, \tau_{\alpha,\beta}]$, which maps this interval onto itself. Moreover, $\bar{\bar{M}}_{\alpha,\beta}$ is an involution, which can be justified similarly as in the case of $\bar{M}_{\alpha,\beta}$.

Finally, $\mathcal{N}_{\alpha,\beta}$ is the continuous strictly decreasing function on $[0, 1+\tau_{\alpha,\beta}]$, which maps this interval onto itself, and it is an involution. Indeed, for $0 \le s \le \tau_{\alpha,\beta}$, we have

$$\mathcal{N}_{\alpha,\beta}(\mathcal{N}_{\alpha,\beta}(s)) = \bar{\bar{M}}_{\alpha,\beta}\left(\bar{\bar{M}}_{\alpha,\beta}(s) + 1 - 1\right) = s,$$

and for $1 \le s \le 1 + \tau_{\alpha,\beta}$, we have

$$\mathcal{N}_{\alpha,\beta}(\mathcal{N}_{\alpha,\beta}(s)) = \bar{\bar{M}}_{\alpha,\beta}\left(\bar{\bar{M}}_{\alpha,\beta}(s-1)\right) + 1 = s - 1 + 1 = s. \quad \square$$

Now, let us focus on $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ introduced in Definition 1 (see Figs. 27 and 28). Using Lemma 7, we get that $0 \le \eta_{\alpha,\beta} < 1$ and $0 < \mu_{\alpha,\beta} < 1$ since $(V^{\beta}_{\kappa_\beta+1} - 1)/V^{\beta}_{\kappa_\beta}$ is negative or equal to $\infty$ and $V^{\beta}_{\kappa_\beta+1}/(V^{\beta}_{\kappa_\beta} + 1)$ is negative.

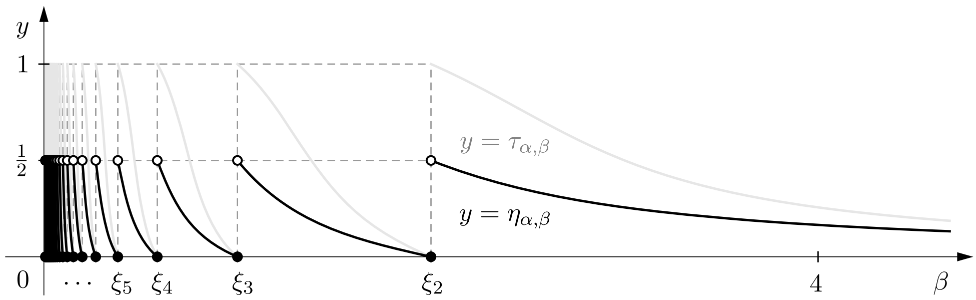


**Fig. 28.** The graph of the function $\beta \mapsto \eta_{\alpha,\beta}$ for fixed $\alpha = 2.9$.

**Lemma 21.** *The points $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ are fixed points of $\bar{\bar{M}}_{\alpha,\beta}$ and $\bar{M}_{\alpha,\beta}$, respectively. Moreover, we have*

$$\mathcal{N}_{\alpha,\beta}(\eta_{\alpha,\beta}) = \eta_{\alpha,\beta} + 1, \qquad \mathcal{N}_{\alpha,\beta}(\mu_{\alpha,\beta}) = \mu_{\alpha,\beta}.$$

**Proof.** Using (29), we obtain

$$\bar{\bar{M}}_{\alpha,\beta}(\eta_{\alpha,\beta}) = T^{\alpha}\left(W^{\beta}_{\kappa_\beta+2}\left(\frac{V^{\beta}_{\kappa_\beta}}{V^{\beta}_{\kappa_\beta+1}-1}\right)\right) = T^{\alpha}\left(\frac{V^{\beta}_{\kappa_\beta+1}-1}{V^{\beta}_{\kappa_\beta}}\right) = \eta_{\alpha,\beta},$$

where we used (40) to simplify

$$W^{\beta}_{\kappa_\beta+2}\left(\frac{V^{\beta}_{\kappa_\beta}}{V^{\beta}_{\kappa_\beta+1}-1}\right) = \frac{V^{\beta}_{\kappa_\beta+1} - \left(V^{\beta}_{\kappa_\beta+1}V^{\beta}_{\kappa_\beta+1} - V^{\beta}_{\kappa_\beta}V^{\beta}_{\kappa_\beta+2}\right)}{V^{\beta}_{\kappa_\beta}} = \frac{V^{\beta}_{\kappa_\beta+1}-1}{V^{\beta}_{\kappa_\beta}}.$$

In a similar way, we show that $\bar{M}_{\alpha,\beta}(\mu_{\alpha,\beta}) = \mu_{\alpha,\beta}$. Finally, we have $\mathcal{N}_{\alpha,\beta}(\eta_{\alpha,\beta}) = \bar{\bar{M}}_{\alpha,\beta}(\eta_{\alpha,\beta}) + 1 = \eta_{\alpha,\beta} + 1$ and $\mathcal{N}_{\alpha,\beta}(\mu_{\alpha,\beta}) = \bar{M}_{\alpha,\beta}(\mu_{\alpha,\beta}) = \mu_{\alpha,\beta}$, which finishes the proof. $\square$

**Lemma 22.** *Let $0 < \alpha < 4$.*

1. *If $\beta = \xi_k$, $k \in \mathbb{N}$, $k \geq 2$, then $\mathcal{N}_{\alpha,\beta}(s) = 1 - s$.*
2. *If $\beta = \alpha$ then $\mathcal{N}_{\alpha,\beta}(s) = 1 - s + \tau_{\beta,\beta} = 1 - s + \frac{\pi}{\omega_\beta} - \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor$.*

**Proof.** At first, let us assume that $\beta = \xi_k$, $k \in \mathbb{N}$, $k \geq 2$. Using Lemma 16, we obtain that $\tau_{\alpha,\beta} = 0$ and thus, we have that $\mathcal{N}_{\alpha,\beta}(0) = 1$ and $\mathcal{N}_{\alpha,\beta}(1) = 0$. Moreover, since $W^{\beta}_{\kappa_\beta+1}$ is the identity function (recall (42)), we have for $0 < s < 1$ that

$$\mathcal{N}_{\alpha,\beta}(s) = \bar{M}_{\alpha,\beta}(s) = T^{\alpha}\left(W^{\beta}_{\kappa_\beta+1}\left(Q^{\alpha}(1-s)\right)\right) = 1 - s.$$

At second, let us assume that $0 < \beta = \alpha < 4$. Then using (64), we obtain

$$\tfrac{\pi}{\omega_\beta} = t_2 - t_1 = s + \kappa_\beta + \mathcal{N}_{\alpha,\beta}(s), \quad 0 \le s \le 1 + \tau_{\beta,\beta}, \tag{67}$$

where $\tau_{\beta,\beta} = \frac{\pi}{\omega_\beta} - \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor$ due to Lemma 16. Finally, using (67), we get

$$\mathcal{N}_{\alpha,\beta}(s) = \tfrac{\pi}{\omega_\beta} - s - \kappa_\beta = \tfrac{\pi}{\omega_\beta} - s - \left\lfloor \tfrac{\pi}{\omega_\beta} \right\rfloor + 1 = 1 - s + \tau_{\beta,\beta}. \quad \square$$

In the following lemma, we show that values $\eta_{\alpha,\beta}$, $\tau_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ are always ordered in one way (see Figs. 27 and 28).

**Lemma 23.** *For $0 < \alpha < 4$ and $\beta > 0$, we have that*

$$0 < \eta_{\alpha,\beta} < \tau_{\alpha,\beta} < \mu_{\alpha,\beta} < 1 \qquad \text{if } \beta \neq \xi_k,\ k \in \mathbb{N},\ k \ge 2. \tag{68}$$

*Moreover, we have*

1. $\eta_{\alpha,\beta} = 0$ *if and only if* $\beta = \xi_k$ *for some* $k \in \mathbb{N}$, $k \ge 2$,
2. $\mu_{\alpha,\beta} = \frac{1}{2}$ *if and only if* $\beta = \xi_k$ *for some* $k \in \mathbb{N}$.

**Proof.** At first, for $\beta \neq \xi_k$, $k \in \mathbb{N}$, $k \ge 2$, we have that

$$\frac{V^\beta_{\kappa_\beta+1} - 1}{V^\beta_{\kappa_\beta}} < \frac{V^\beta_{\kappa_\beta+1}}{V^\beta_{\kappa_\beta}} < \frac{V^\beta_{\kappa_\beta+1}}{V^\beta_{\kappa_\beta} + 1} < 0,$$

which implies $0 < \eta_{\alpha,\beta} < \tau_{\alpha,\beta} < \mu_{\alpha,\beta} < 1$. At second, for $\beta = \xi_k$, $k \in \mathbb{N}$, $k \ge 2$, we have $V^\beta_{\kappa_\beta} = 0$, $V^\beta_{\kappa_\beta+1} = -1$, and thus, $\eta_{\alpha,\beta} = \tau_{\alpha,\beta} = T^\alpha(\infty) = 0$ and $\mu_{\alpha,\beta} = T^\alpha(-1) = \frac{1}{2}$. At third, for $\beta = 4$, we have $\kappa_\beta = 0$, $V^\beta_{\kappa_\beta} = 1$, $V^\beta_{\kappa_\beta+1} = 2 - \beta = -2$ and thus, $\mu_{\alpha,\beta} = T^\alpha(-1) = \frac{1}{2}$. Finally, let us assume that $\beta \neq 4$ and $\mu_{\alpha,\beta} = \frac{1}{2}$. This means that $\frac{V^\beta_{\kappa_\beta+1}}{V^\beta_{\kappa_\beta}+1} = -1$ and thus, we have

$$V^\beta_{\kappa_\beta} + V^\beta_{\kappa_\beta+1} = -1.$$

Thus, using Lemma 7, we obtain that $\beta = \xi_k$ for some $k \in \mathbb{N}$, $k \ge 2$. $\square$

Let us reveal a close connection among values $\eta_{\alpha,\beta}$, $\tau_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ using the function $G^\beta : \mathbb{R} \to \mathbb{R}^*$ defined in the following way

$$G^\beta(q) := \frac{2q - (2-\beta)q^2}{1 - q^2} \quad \text{for } q \neq \pm 1, \qquad G^\beta(\pm 1) := \infty.$$

**Lemma 24.** *For* $0 < \alpha, \beta < 4$*, we have*

$$\tau_{\alpha,\beta} = 1 - T^{\alpha}\left(G^{\beta}\left(Q^{\alpha}(1-\eta_{\alpha,\beta})\right)\right), \tag{69}$$

$$\tau_{\alpha,\beta} = T^{\alpha}\left(G^{\beta}\left(Q^{\alpha}(\mu_{\alpha,\beta})\right)\right), \tag{70}$$

$$G^{\beta}(Q^{\alpha}(1-\eta_{\alpha,\beta})) = \frac{1}{G^{\beta}(Q^{\alpha}(\mu_{\alpha,\beta}))}. \tag{71}$$

**Proof.** Firstly, let us assume that $\beta = \xi_k$ for some $k \in \mathbb{N}$, $k \geq 2$. According to Lemmas 16 and 23, we have $\eta_{\alpha,\beta} = \tau_{\alpha,\beta} = 0$ and $\mu_{\alpha,\beta} = \frac{1}{2}$. Moreover, we have

$$\begin{aligned} 1 - T^{\alpha}\left(G^{\beta}\left(Q^{\alpha}(1-\eta_{\alpha,\beta})\right)\right) &= 1 - T^{\alpha}\left(G^{\beta}\left(Q^{\alpha}(1)\right)\right) = 1 - T^{\alpha}\left(G^{\beta}(0)\right) = 1 - T^{\alpha}(0) \\ &= 0 = \tau_{\alpha,\beta} \end{aligned}$$

and

$$T^{\alpha}\left(G^{\beta}\left(Q^{\alpha}(\mu_{\alpha,\beta})\right)\right) = T^{\alpha}\left(G^{\beta}\left(Q^{\alpha}\left(\tfrac{1}{2}\right)\right)\right) = T^{\alpha}\left(G^{\beta}(-1)\right) = T^{\alpha}(\infty) = 0 = \tau_{\alpha,\beta}.$$

Secondly, let us assume that $\beta \neq \xi_k$, $k \in \mathbb{N}$, $k \geq 2$. We claim that $\eta_{\alpha,\beta} \neq \frac{1}{2}$. Indeed, if we assume that $\eta_{\alpha,\beta} = \frac{1}{2}$ then we get $(V^{\beta}_{\kappa_\beta+1} - 1)/V^{\beta}_{\kappa_\beta} = -1$ and

$$V^{\beta}_{\kappa_\beta} + V^{\beta}_{\kappa_\beta+1} = 1,$$

which is a contradiction since $V^{\beta}_{\kappa_\beta} + V^{\beta}_{\kappa_\beta+1} < 1$ according to Lemma 7. Thus, $0 < \eta_{\alpha,\beta} < 1$, $0 < 1 - \eta_{\alpha,\beta} < 1$, $Q^{\alpha}(1-\eta_{\alpha,\beta}) < 0$ and

$$Q^{\alpha}(1-\eta_{\alpha,\beta}) \neq -1, \qquad Q^{\alpha}(\eta_{\alpha,\beta}) \neq -1.$$

Now, using (29), we have

$$G^{\beta}\left(Q^{\alpha}(1-\eta_{\alpha,\beta})\right) = G^{\beta}\left(\frac{1}{Q^{\alpha}(\eta_{\alpha,\beta})}\right) = \frac{2Q^{\alpha}(\eta_{\alpha,\beta}) - (2-\beta)}{(Q^{\alpha}(\eta_{\alpha,\beta}))^2 - 1}$$

and thus, using (39), we obtain

$$\begin{aligned} G^{\beta}\left(Q^{\alpha}(1-\eta_{\alpha,\beta})\right) &= V^{\beta}_{\kappa_\beta} \frac{2(V^{\beta}_{\kappa_\beta+1} - 1) - (2-\beta)V^{\beta}_{\kappa_\beta}}{(V^{\beta}_{\kappa_\beta+1} - 1)^2 - (V^{\beta}_{\kappa_\beta})^2} \\ &= V^{\beta}_{\kappa_\beta} \frac{2(V^{\beta}_{\kappa_\beta+1} - 1) - (V^{\beta}_{\kappa_\beta+1} + V^{\beta}_{\kappa_\beta-1})}{(V^{\beta}_{\kappa_\beta+1} - 1)^2 - (1 + V^{\beta}_{\kappa_\beta+1}V^{\beta}_{\kappa_\beta-1})} \\ &= V^{\beta}_{\kappa_\beta} \frac{V^{\beta}_{\kappa_\beta+1} - 2 - V^{\beta}_{\kappa_\beta-1}}{(V^{\beta}_{\kappa_\beta+1})^2 - 2V^{\beta}_{\kappa_\beta+1} - V^{\beta}_{\kappa_\beta+1}V^{\beta}_{\kappa_\beta-1}} = \frac{V^{\beta}_{\kappa_\beta}}{V^{\beta}_{\kappa_\beta+1}}. \end{aligned} \tag{72}$$

Using (27) and (72), we get

$$1 - T^\alpha\left(G^\beta\left(Q^\alpha(1-\eta_{\alpha,\beta})\right)\right) = T^\alpha\left(\frac{1}{G^\beta\left(Q^\alpha(1-\eta_{\alpha,\beta})\right)}\right) = T^\alpha\left(\frac{V^\beta_{\kappa_\beta+1}}{V^\beta_{\kappa_\beta}}\right) = \tau_{\alpha,\beta},$$

which justifies (69). According to Lemma 23, we have that $0 < \mu_{\alpha,\beta} < 1$ and $\mu_{\alpha,\beta} \neq \frac{1}{2}$, which means that $Q^\alpha(\mu_{\alpha,\beta}) < 0$ and $Q^\alpha(\mu_{\alpha,\beta}) \neq -1$. As in the previous case, using (39), we simplify $G^\beta\left(Q^\alpha(\mu_{\alpha,\beta})\right)$ as $V^\beta_{\kappa_\beta+1}/V^\beta_{\kappa_\beta}$, which justifies (70). Finally, if we combine (69) and (70) and use (27) then we obtain (71). □

Let us note that for $0 < \alpha < 4$, we have

$$\begin{aligned}
0 < \eta_{\alpha,\beta} < \tau_{\alpha,\beta} < \mu_{\alpha,\beta} < \tfrac{1}{2} \qquad & \text{for } \beta > 4,\\
0 < \eta_{\alpha,\beta} < \tau_{\alpha,\beta} < \mu_{\alpha,\beta} = \tfrac{1}{2} \qquad & \text{for } \beta = 4,\\
0 < \eta_{\alpha,\beta} < \tfrac{1}{2} < \mu_{\alpha,\beta} < 1 \qquad & \text{for } 2 < \beta < 4,
\end{aligned} \tag{73}$$

since $Q^\alpha(\eta_{\alpha,\beta}) = 1-\beta$, $Q^\alpha(\tau_{\alpha,\beta}) = 2-\beta$ and $Q^\alpha(\mu_{\alpha,\beta}) = \frac{2-\beta}{2}$ for $\beta > 2$. The following lemma indicates that the values of $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ are separated by $\frac{1}{2}$ for $0 < \alpha, \beta < 4$.

**Lemma 25.** *For $0 < \alpha, \beta < 4$, we have that*

$$0 \le \eta_{\alpha,\beta} < \tfrac{1}{2} \le \mu_{\alpha,\beta} < 1.$$

**Proof.** Firstly, for $2 < \beta < 4$, we have the inequalities in (73). Secondly, for $0 < \beta < 2$ such that $\beta \neq \xi_k$, $k \in \mathbb{N}$, $k > 2$, we have $0 < \tau_{\alpha,\beta} < 1$, which implies $G^\beta(Q^\alpha(\mu_{\alpha,\beta})) < 0$ according to (70). Thus, we obtain that $-1 < Q^\alpha(\mu_{\alpha,\beta}) < 0$, which leads to $\frac{1}{2} < \mu_{\alpha,\beta} < 1$. Similarly, using (71), we get $G^\beta(Q^\alpha(1-\eta_{\alpha,\beta})) < 0$, which implies $0 < \eta_{\alpha,\beta} < \frac{1}{2}$. Finally, for $\beta = \xi_k$, $k \in \mathbb{N}$, $k \ge 2$, we have $\eta_{\alpha,\beta} = 0$ and $\mu_{\alpha,\beta} = \frac{1}{2}$ due to Lemma 23. □

**Proof of Theorem 5.** The proof is based on Theorem 26 in [20]. Let us find the description of the Fučík curve $\mathcal{C}^+_{2j-1}$, $j \in \mathbb{N}$, in terms of functions $t^+_j = t^+_j(\alpha,\beta)$ and $t^-_j = t^-_j(\alpha,\beta)$ defined in (24) and (25), respectively. Using Theorem 26 in [20], we obtain that the Fučík curve $\mathcal{C}^+_{2j-1}$ has in $\mathcal{D}$ the following implicit description

$$\sum_{i=1}^{j} p_i(\alpha,\beta) + \sum_{i=1}^{j} p_i(\beta,\alpha) + T^\alpha(\vartheta_j(\alpha,\beta)) + T^\alpha(\vartheta_j(\beta,\alpha)) = n+1 \tag{74}$$

and moreover, the corresponding non-trivial solution $u$ has exactly $(2j-1)$ generalized zeros on $\mathbb{T}$ and has exactly $j$ positive semi-waves as continuous extensions. These positive continuous semi-waves have the zeros $t_i$ and $\tilde{t}_{-i}$, which can be reconstructed from left and right endpoints of $\widehat{\mathbb{T}}$, respectively, in the following way

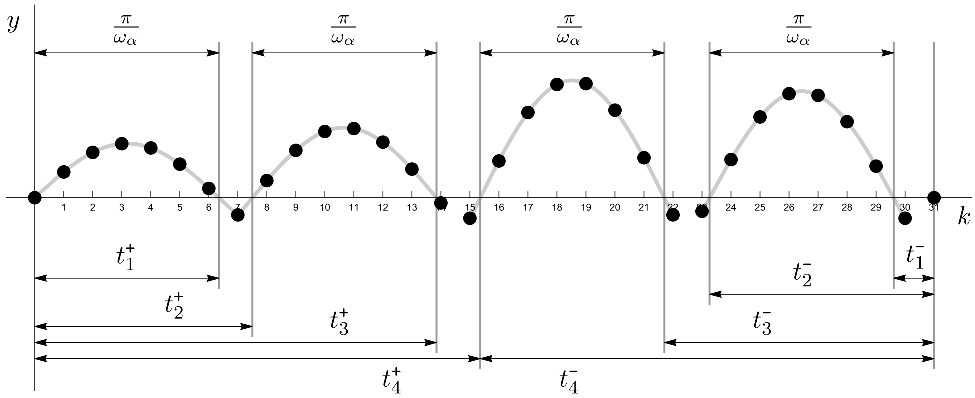


**Fig. 29.** A non-trivial solution of the problem (7) with 7 generalized zeros on $\mathbb{T}$ for $(\alpha, \beta) \in \mathcal{C}_7^+$ ($n = 30$, $\alpha \doteq 0.240$, $\beta \doteq 3.534$).

$$t_i = \sum_{k=1}^{i} p_k(\alpha, \beta) + T^\alpha(\vartheta_i(\alpha, \beta)), \qquad \tilde{t}_{-i} = n + 1 - \sum_{k=1}^{i} p_k(\beta, \alpha) - T^\alpha(\vartheta_i(\beta, \alpha)),$$

for $i = 1, \ldots, j$. The condition (74) means that $t_j = \tilde{t}_{-j}$. Now, according to Lemma 14, Remark 18 and (66), we have that

$$\begin{aligned} t_i &= t_i^+(\alpha, \beta), \\ \tilde{t}_{-i} &= n + 1 - t_i^-(\alpha, \beta), \end{aligned} \tag{75}$$

and thus, the condition (74) reads as

$$t_j^+(\alpha, \beta) + t_j^-(\alpha, \beta) = n + 1.$$

To justify (75), it remains to show that $\varrho_{\alpha,\beta}(0) = p_1(\beta, \alpha) + T^\alpha(\vartheta_1(\beta, \alpha))$. Indeed, using Lemma 20, we obtain

$$\varrho_{\alpha,\beta}(0) = \kappa_\beta + \mathcal{N}_{\alpha,\beta}(0) = \kappa_\beta + 1 + \tau_{\alpha,\beta},$$

and using Lemma 12, we get

$$\begin{aligned} p_1(\beta, \alpha) &= \kappa_\beta + 1, \\ T^\alpha(\vartheta_1(\beta, \alpha)) &= T^\alpha\left(W^\beta_{p_1(\beta,\alpha)}(\infty)\right) = T^\alpha\left(\frac{V^\beta_{\kappa_\beta+1}}{V^\beta_{\kappa_\beta}}\right) = \tau_{\alpha,\beta}. \end{aligned}$$

Now, the description of Fučík curves $\mathcal{C}^-_{2j-1}$, $\mathcal{C}^+_{2j}$ and $\mathcal{C}^-_{2j}$ in terms of functions $t_j^+ = t_j^+(\alpha, \beta)$ and $t_j^- = t_j^-(\alpha, \beta)$ can be obtained analogously (see Figs. 30 and 31). Let us only mention here the implicit description of curves $\mathcal{C}^\pm_{2j}$ similar to (74)

$$\mathcal{C}^+_{2j} : \sum_{i=1}^{j+1} p_i(\alpha, \beta) + \sum_{i=1}^{j} p_i(\alpha, \beta) + T^\alpha(\vartheta_{j+1}(\alpha, \beta)) + T^\alpha(\vartheta_j(\alpha, \beta)) = n + 1,$$

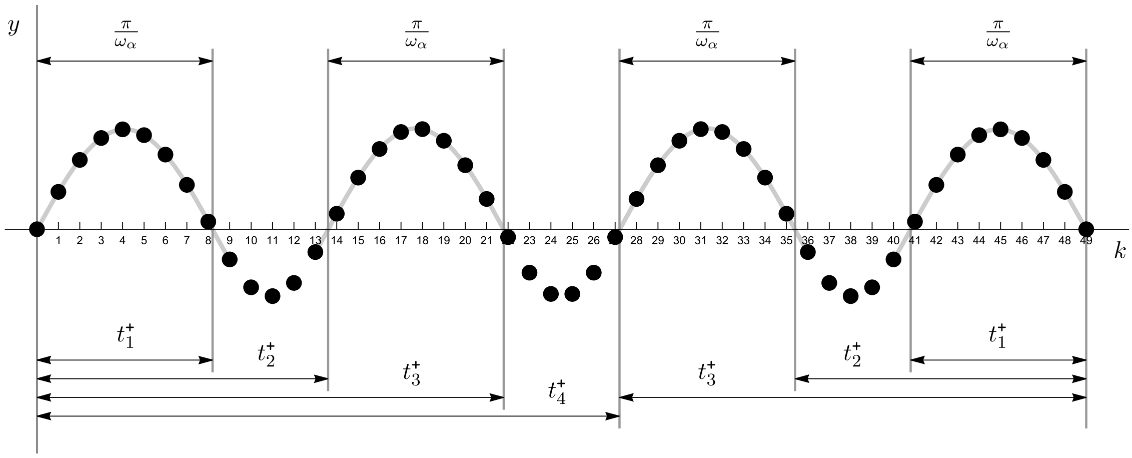


**Fig. 30.** A non-trivial solution of the problem (7) with 6 generalized zeros on $\mathbb{T}$ for $(\alpha, \beta) \in \mathcal{C}_6^+$ ($n = 48$, $\alpha \doteq 0.145$, $\beta \doteq 0.329$).

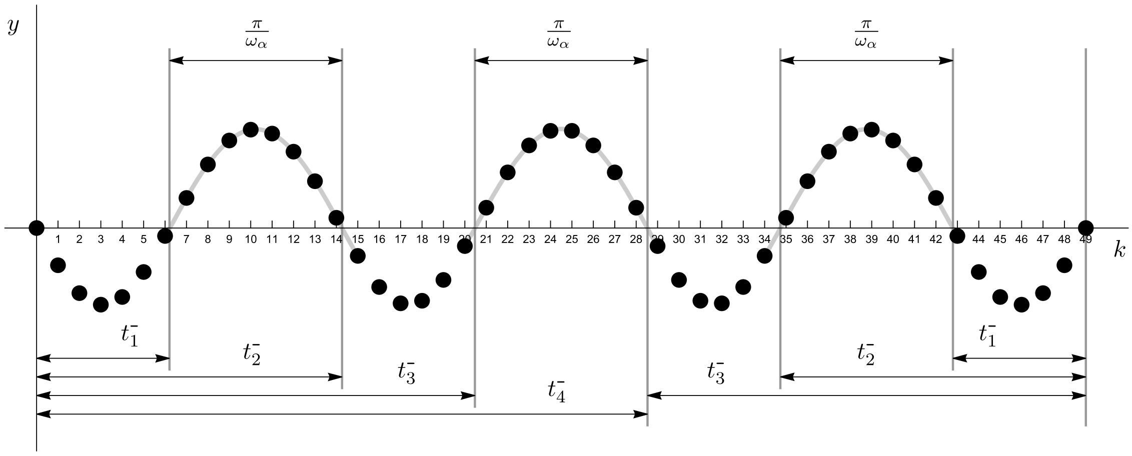


**Fig. 31.** A non-trivial solution of the problem (7) with 6 generalized zeros on $\mathbb{T}$ for $(\alpha, \beta) \in \mathcal{C}_6^-$ ($n = 48$, $\alpha \doteq 0.150$, $\beta \doteq 0.251$).

$$\mathcal{C}_{2j}^- : \sum_{i=1}^{j+1} p_i(\beta, \alpha) + \sum_{i=1}^{j} p_i(\beta, \alpha) + T^\alpha(\vartheta_{j+1}(\beta, \alpha)) + T^\alpha(\vartheta_j(\beta, \alpha)) = n + 1. \quad \square$$

## 6. Improved bounds for Fučík curves

In this last section, we focus on the function $\rho_{\alpha,\beta}$ introduced in Definition 19 which measures the distance between every two consecutive zeros of two different continuous positive semi-waves. In Theorem 31, we prove that $\rho_{\alpha,\beta}$ attains its global extrema at $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ (see Fig. 32). Since $\rho_{\alpha,\beta}$ is defined by $\mathcal{N}_{\alpha,\beta}$, we express the first derivative $(\mathcal{N}_{\alpha,\beta})'$ in Lemma 28 and then in Lemma 30, we determine where this derivative is less or greater than $-1$. Let us note that at the end of this section, the proof of the main Theorem 3 is available.

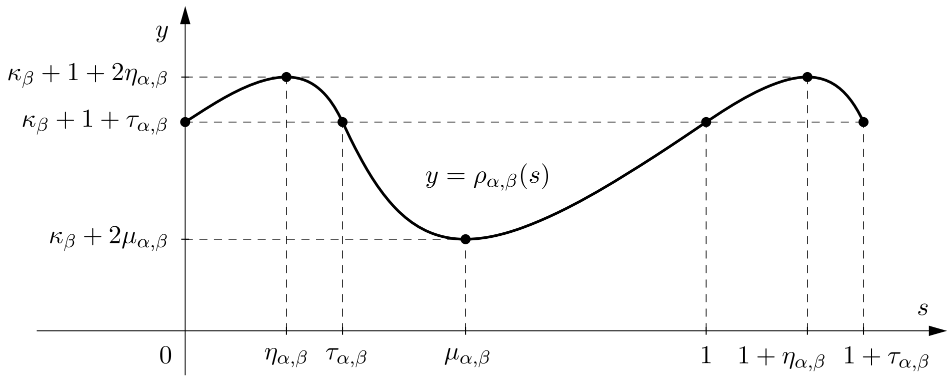


**Fig. 32.** The graph of the function $\rho_{\alpha,\beta} = \rho_{\alpha,\beta}(s)$ for fixed $\alpha = 2.6$ and $\beta = 3.8$.

Let us introduce the function $\mathcal{S}_k^{\alpha,\beta}$, which we use to express the first derivative $(\mathcal{N}_{\alpha,\beta})'$ of the function $\mathcal{N}_{\alpha,\beta}$ given in Definition 17.

**Definition 26.** Let $0 < \alpha < 4$ and $\beta > 0$, $k \in \mathbb{Z}$. Let us define the function $\mathcal{S}_k^{\alpha,\beta} : \mathbb{R}^* \to \mathbb{R}$ as

$$\mathcal{S}_k^{\alpha,\beta}(q) := V_k^\beta \cdot \frac{q^2 V_{k+1}^\beta - 2q V_k^\beta + V_{k-1}^\beta}{q^2 - (2-\alpha)q + 1} \quad \text{for } q \in \mathbb{R}, \tag{76}$$

$$\mathcal{S}_k^{\alpha,\beta}(\infty) := V_k^\beta \cdot V_{k+1}^\beta. \tag{77}$$

Let us note that the denominator $q^2 - (2-\alpha)q + 1$ in (76) is always positive since its discriminant is $\alpha(\alpha - 4) < 0$.

**Lemma 27.** *Let $0 < \alpha < 4$, $\beta > 0$ and $k = \kappa_\beta$ or $k = \kappa_\beta + 1$. Then for $0 < t \le 1$, we have*

$$\left(T^\alpha \circ W_{k+1}^\beta \circ Q^\alpha\right)'(t) = \frac{1}{1 - (\beta - \alpha)\mathcal{S}_k^{\alpha,\beta}\left(Q^\alpha(t)\right)}. \tag{78}$$

*Moreover, we have*

$$\left(T^\alpha \circ W_{k+1}^\beta \circ Q^\alpha\right)_+'(0) = \frac{1}{1 - (\beta - \alpha)\mathcal{S}_k^{\alpha,\beta}(\infty)}. \tag{79}$$

**Proof.** For $0 \le t \le 1$, let us denote $q = Q^\alpha(t)$. Thus, we have that $q$ is finite and non-positive for $0 < t \le 1$ and $q = \infty$ for $t = 0$.

At first, in the special case of $V_k^\beta = 0$, we have that $\mathcal{S}_k^{\alpha,\beta}(q) = 0$ and $W_{k+1}^\beta(q) = q$ and thus, we obtain

$$T^\alpha\left(W_{k+1}^\beta\left(Q^\alpha(t)\right)\right) = T^\alpha\left(Q^\alpha(t)\right) = t,$$

which justifies (78) and (79).

Now, for the rest of the proof, let us assume that $V_k^\beta \neq 0$. Using (12), we obtain

$$(T^\alpha)'(q) = \frac{\sin\omega_\alpha}{\omega_\alpha} \cdot \frac{1}{q^2 - (2-\alpha)q + 1}, \tag{80}$$
$$(Q^\alpha)'(t) = \frac{1}{(T^\alpha)'(q)}.$$

Moreover, we have

$$\left(T^\alpha\left(W_{k+1}^\beta\left(Q^\alpha(t)\right)\right)\right)' = \frac{(T^\alpha)'\left(W_{k+1}^\beta(q)\right)}{(T^\alpha)'(q)} \cdot \left(W_{k+1}^\beta\right)'(q). \tag{81}$$

Let us point out that $W_{k+1}^\beta(q)$ is a finite number due to $q = Q^\alpha(t)$ for $0 \leq t \leq 1$ and $k = \kappa_\beta$ or $k = \kappa_\beta + 1$. Now, let us expand the factor $(T^\alpha)'\left(W_{k+1}^\beta(q)\right)$ in (81). Thus, let us write the denominator in (80) for $q$ equal to $W_{k+1}^\beta(q)$ as

$$\left(W_{k+1}^\beta(q)\right)^2 - (2-\alpha)W_{k+1}^\beta(q) + 1 = \frac{Aq^2 + Bq + C}{\left(qV_k^\beta - V_{k-1}^\beta\right)^2}, \tag{82}$$

where the polynomial $Aq^2 + Bq + C$ has the form of

$$\left(qV_{k+1}^\beta - V_k^\beta\right)^2 - (2-\alpha)\left(qV_{k+1}^\beta - V_k^\beta\right)\left(qV_k^\beta - V_{k-1}^\beta\right) + \left(qV_k^\beta - V_{k-1}^\beta\right)^2.$$

Moreover, the coefficients $A$, $B$ and $C$ of this polynomial can be identified as

$$\begin{aligned} A &= V_{k+1}^\beta V_{k+1}^\beta - (2-\alpha)V_k^\beta V_{k+1}^\beta + V_k^\beta V_k^\beta = 1 + (2-\beta)V_k^\beta V_{k+1}^\beta - (2-\alpha)V_k^\beta V_{k+1}^\beta \\ &= 1 - (\beta-\alpha)V_k^\beta V_{k+1}^\beta, \end{aligned} \tag{83}$$

$$\begin{aligned} C &= V_k^\beta V_k^\beta - (2-\alpha)V_k^\beta V_{k-1}^\beta + V_{k-1}^\beta V_{k-1}^\beta = 1 + (2-\beta)V_k^\beta V_{k-1}^\beta - (2-\alpha)V_k^\beta V_{k-1}^\beta \\ &= 1 - (\beta-\alpha)V_k^\beta V_{k-1}^\beta, \end{aligned} \tag{84}$$

$$\begin{aligned} B &= -2V_k^\beta\left(V_{k+1}^\beta + V_{k-1}^\beta\right) + (2-\alpha)\left(V_k^\beta V_k^\beta - V_{k+1}^\beta V_{k-1}^\beta\right) \\ &= -2(2-\beta)V_k^\beta V_k^\beta + (2-\alpha)\left(2V_k^\beta V_k^\beta - 1\right) \\ &= -(2-\alpha) + 2(\beta-\alpha)V_k^\beta V_k^\beta. \end{aligned} \tag{85}$$

If we combine (83), (84), (85) and (82), we obtain

$$\frac{(T^\alpha)'\left(W^\beta_{k+1}(q)\right)}{(T^\alpha)'(q)} = \frac{\left(q^2-(2-\alpha)q+1\right)\left(qV^\beta_k - V^\beta_{k-1}\right)^2}{q^2-(2-\alpha)q+1-(\beta-\alpha)V^\beta_k\left(q^2V^\beta_{k+1}-2qV^\beta_k+V^\beta_{k-1}\right)}$$
$$= \frac{\left(qV^\beta_k - V^\beta_{k-1}\right)^2}{1-(\beta-\alpha)\mathcal{S}^{\alpha,\beta}_k(q)},$$

which means that (81) has the form of (78) since we have that

$$\left(W^\beta_{k+1}\right)'(q) = \frac{V^\beta_{k+1}\left(qV^\beta_k - V^\beta_{k-1}\right) - \left(qV^\beta_{k+1}-V^\beta_k\right)V^\beta_k}{\left(qV^\beta_k - V^\beta_{k-1}\right)^2} = \frac{1}{\left(qV^\beta_k - V^\beta_{k-1}\right)^2}. \qquad \square$$

Using the function $\mathcal{S}^{\alpha,\beta}_k$, we express the first derivative $(\mathcal{N}_{\alpha,\beta})'$ (see Fig. 33).

**Lemma 28.** *For $0<\alpha<4$ and $\beta>0$, we have*

$$(\mathcal{N}_{\alpha,\beta})'(s) = \begin{cases} \dfrac{-1}{1-(\beta-\alpha)\mathcal{S}^{\alpha,\beta}_{\kappa_\beta+1}\left(Q^\alpha(1-s)\right)} & \text{for } s\in(0,\tau_{\alpha,\beta}], \\[2ex] \dfrac{-1}{1-(\beta-\alpha)\mathcal{S}^{\alpha,\beta}_{\kappa_\beta}\left(Q^\alpha(1-s)\right)} & \text{for } s\in(\tau_{\alpha,\beta},1), \\[2ex] \dfrac{-1}{1-(\beta-\alpha)\mathcal{S}^{\alpha,\beta}_{\kappa_\beta}(\infty)} & \text{for } s=1, \end{cases}$$

*where the functions $\mathcal{S}^{\alpha,\beta}_{\kappa_\beta}$ and $\mathcal{S}^{\alpha,\beta}_{\kappa_\beta+1}$ are defined in (76), (77) as $\mathcal{S}^{\alpha,\beta}_k$ for $k=\kappa_\beta$ and $k=\kappa_\beta+1$, respectively.*

**Proof.** Let us split the proof according to the value of the variable $s$.

At first, let us assume that $0<s<\tau_{\alpha,\beta}$. Then we have

$$\mathcal{N}_{\alpha,\beta}(s) = \bar{\bar{M}}_{\alpha,\beta}(s)+1 = T^\alpha\left(W^\beta_{\kappa_\beta+2}\left(Q^\alpha(1-s)\right)\right)+1$$

and thus, the expression of the first derivative $(\mathcal{N}_{\alpha,\beta})'(s)$ follows directly from (78) for $t=1-s$ and $k=\kappa_\beta+1$.

At second, let us assume that $\tau_{\alpha,\beta}<s<1$. Then we have

$$\mathcal{N}_{\alpha,\beta}(s) = \bar{M}_{\alpha,\beta}(s) = T^\alpha\left(W^\beta_{\kappa_\beta+1}\left(Q^\alpha(1-s)\right)\right)$$

and thus, the expression of $(\mathcal{N}_{\alpha,\beta})'(s)$ follows from (78) for $t=1-s$ and $k=\kappa_\beta$.

At third, let us assume that $s=\tau_{\alpha,\beta}$. If we take into account that

$$V^\beta_{\kappa_\beta+2}V^\beta_{\kappa_\beta} - \left(V^\beta_{\kappa_\beta+1}\right)^2 = 1 = V^\beta_{\kappa_\beta+1}V^\beta_{\kappa_\beta-1} - \left(V^\beta_{\kappa_\beta}\right)^2$$

then we obtain

$$\begin{aligned}
\mathcal{S}^{\alpha,\beta}_{\kappa_\beta+1}\left(Q^\alpha(1-\tau_{\alpha,\beta})\right) &= \mathcal{S}^{\alpha,\beta}_{\kappa_\beta+1}\left(\frac{V^\beta_{\kappa_\beta}}{V^\beta_{\kappa_\beta+1}}\right)\\
&= V^\beta_{\kappa_\beta+1}V^\beta_{\kappa_\beta}\frac{V^\beta_{\kappa_\beta+2}V^\beta_{\kappa_\beta}-\left(V^\beta_{\kappa_\beta+1}\right)^2}{\left(V^\beta_{\kappa_\beta}\right)^2-(2-\alpha)V^\beta_{\kappa_\beta}V^\beta_{\kappa_\beta+2}+\left(V^\beta_{\kappa_\beta+1}\right)^2}\\
&= V^\beta_{\kappa_\beta+1}V^\beta_{\kappa_\beta}\frac{V^\beta_{\kappa_\beta+1}V^\beta_{\kappa_\beta-1}-\left(V^\beta_{\kappa_\beta}\right)^2}{\left(V^\beta_{\kappa_\beta}\right)^2-(2-\alpha)V^\beta_{\kappa_\beta}V^\beta_{\kappa_\beta+2}+\left(V^\beta_{\kappa_\beta+1}\right)^2}\\
&= \mathcal{S}^{\alpha,\beta}_{\kappa_\beta}\left(\frac{V^\beta_{\kappa_\beta}}{V^\beta_{\kappa_\beta+1}}\right)\\
&= \mathcal{S}^{\alpha,\beta}_{\kappa_\beta}\left(Q^\alpha(1-\tau_{\alpha,\beta})\right).
\end{aligned}$$

And thus, we obtain that the one-sided derivatives of $\mathcal{N}_{\alpha,\beta}$ at $\tau_{\alpha,\beta}$ coincide. Indeed, we have

$$\begin{aligned}
\left(\mathcal{N}_{\alpha,\beta}\right)'_-(\tau_{\alpha,\beta}) &= \frac{-1}{1-(\beta-\alpha)\mathcal{S}^{\alpha,\beta}_{\kappa_\beta+1}\left(Q^\alpha(1-\tau_{\alpha,\beta})\right)} = \frac{-1}{1-(\beta-\alpha)\mathcal{S}^{\alpha,\beta}_{\kappa_\beta}\left(Q^\alpha(1-\tau_{\alpha,\beta})\right)}\\
&= \left(\mathcal{N}_{\alpha,\beta}\right)'_+(\tau_{\alpha,\beta}). \quad \square
\end{aligned}$$

**Remark 29.** If we take into account that $\mathcal{S}^{\alpha,\beta}_{\kappa_\beta}(\infty)=\mathcal{S}^{\alpha,\beta}_{\kappa_\beta+1}(0)$, we obtain using Lemma 28 that

$$\left(\mathcal{N}_{\alpha,\beta}\right)'(1) = \frac{-1}{1-(\beta-\alpha)\mathcal{S}^{\alpha,\beta}_{\kappa_\beta+1}(0)} = \left(\mathcal{N}_{\alpha,\beta}\right)'_+(0).$$

Moreover, due to Lemma 20, the function $\mathcal{N}_{\alpha,\beta}$ is an involution and thus, for all $s\in(0,1+\tau_{\alpha,\beta})$, we have that $\left(\mathcal{N}_{\alpha,\beta}\right)'\left(\mathcal{N}_{\alpha,\beta}(s)\right)\cdot\left(\mathcal{N}_{\alpha,\beta}\right)'(s)=1$. And thus, we have

$$\frac{1}{\left(\mathcal{N}_{\alpha,\beta}\right)'(\tau_{\alpha,\beta})} = \left(\mathcal{N}_{\alpha,\beta}\right)'(1) = \left(\mathcal{N}_{\alpha,\beta}\right)'_+(0) = \frac{1}{\left(\mathcal{N}_{\alpha,\beta}\right)'_-(1+\tau_{\alpha,\beta})}.$$

Let us examine where the value of the first derivative $\left(\mathcal{N}_{\alpha,\beta}\right)'(s)$ is equal, less or greater than $-1$ (see Fig. 33).

**Lemma 30.** *Let $0<\alpha<4$ and $\beta>0$. For $\beta=\xi_k$, $k\in\mathbb{N}$, $k\geq 2$, we have that*

$$\left(\mathcal{N}_{\alpha,\beta}\right)'(s) = -1 \quad \textit{for } s\in(0,1). \tag{86}$$

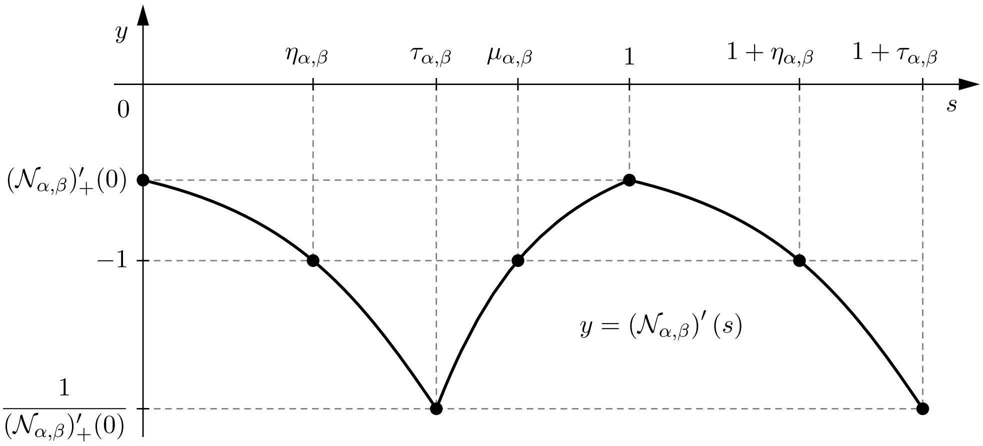


**Fig. 33.** The graph of the first derivative $s \mapsto (\mathcal{N}_{\alpha,\beta})'(s)$ for fixed $\alpha = 1.5$ and $\beta = 2.7$.

*In the case of $\beta \neq \xi_k$ for all $k \in \mathbb{N}$, $k \geq 2$, we have for $\alpha \lesseqgtr \beta$ that*

$$(\mathcal{N}_{\alpha,\beta})'(s) \begin{cases} \gtreqless -1 & \textit{for } s \in (0, \eta_{\alpha,\beta}) \cup (\mu_{\alpha,\beta}, 1), \\ = -1 & \textit{for } s = \eta_{\alpha,\beta} \textit{ and for } s = \mu_{\alpha,\beta}, \\ \lesseqgtr -1 & \textit{for } s \in (\eta_{\alpha,\beta}, \mu_{\alpha,\beta}), \end{cases}$$

*and the statement (86) holds for $\alpha = \beta$.*

**Proof.** In the case of $\beta = \xi_k$, $k \in \mathbb{N}$, $k \geq 2$, we have $\mathcal{N}_{\alpha,\beta}(s) = 1 - s$ according to Lemma 22 and thus, (86) holds. For the rest of the proof, let us assume that $\beta \neq \xi_k$ for all $k \in \mathbb{N}$, $k \geq 2$. According to Lemma 23, we have that $0 < \eta_{\alpha,\beta} < \tau_{\alpha,\beta} < \mu_{\alpha,\beta} < 1$. Now, let us denote $q = Q^\alpha(1-s)$ and split the proof according to the value of $s$.

At first, let us assume that $\tau_{\alpha,\beta} < s < 1$. Thus, we have $q < V^\beta_{\kappa_\beta}/V^\beta_{\kappa_\beta+1} < 0$. Using Lemma 28, we obtain

$$(\mathcal{N}_{\alpha,\beta})'(s) \gtreqqless -1 \qquad \text{if and only if} \qquad (\alpha-\beta)\cdot \mathcal{S}^{\alpha,\beta}_{\kappa_\beta}(q) \gtreqqless 0. \tag{87}$$

If we take into account that $V^\beta_{\kappa_\beta-1}V^\beta_{\kappa_\beta+1} = \left(V^\beta_{\kappa_\beta}\right)^2 - 1$, it is possible to write $\mathcal{S}^{\alpha,\beta}_{\kappa_\beta}(q)$ in the following form

$$\mathcal{S}^{\alpha,\beta}_{\kappa_\beta}(q) = V^\beta_{\kappa_\beta}V^\beta_{\kappa_\beta+1}\frac{\left(q - \frac{V^\beta_{\kappa_\beta}-1}{V^\beta_{\kappa_\beta+1}}\right)\left(q - \frac{V^\beta_{\kappa_\beta}+1}{V^\beta_{\kappa_\beta+1}}\right)}{q^2-(2-\alpha)q+1}.$$

Thus, the sign of $\mathcal{S}^{\alpha,\beta}_{\kappa_\beta}(q)$ is equal to the sign of the factor $\left(q - (V^\beta_{\kappa_\beta}+1)/V^\beta_{\kappa_\beta+1}\right)$ due to

$$V^\beta_{\kappa_\beta}V^\beta_{\kappa_\beta+1} < 0, \qquad q - \frac{V^\beta_{\kappa_\beta}-1}{V^\beta_{\kappa_\beta+1}} < 0, \qquad q^2-(2-\alpha)q+1 > 0.$$

Now, since $q = Q^\alpha(1-s)$ and $(V^\beta_{\kappa_\beta}+1)/V^\beta_{\kappa_\beta+1} = 1/Q^\alpha(\mu_{\alpha,\beta}) = Q^\alpha(1-\mu_{\alpha,\beta})$, we conclude that

$$\mathcal{S}^{\alpha,\beta}_{\kappa_\beta}(q) \gtreqless 0 \qquad \text{if and only if} \qquad Q^\alpha(1-s) \gtreqless Q^\alpha(1-\mu_{\alpha,\beta}). \tag{88}$$

If we combine (87) and (88) and take into account that $Q^\alpha$ is a strictly increasing function, we obtain

$$(\mathcal{N}_{\alpha,\beta})'(s) \gtreqless -1 \qquad \text{if and only if} \qquad (\alpha-\beta)\cdot(\mu_{\alpha,\beta}-s) \gtreqless 0. \tag{89}$$

At second, let us assume that $0 < s \le \tau_{\alpha,\beta}$ and $0 < \beta < 2$. Thus, we have $V^\beta_{\kappa_\beta}/V^\beta_{\kappa_\beta+1} \le q < 0$. Using Lemma 28, we obtain

$$(\mathcal{N}_{\alpha,\beta})'(s) \gtreqless -1 \qquad \text{if and only if} \qquad (\alpha-\beta)\cdot\mathcal{S}^{\alpha,\beta}_{\kappa_\beta+1}(q) \gtreqless 0. \tag{90}$$

It is possible to write $\mathcal{S}^{\alpha,\beta}_{\kappa_\beta+1}(q)$ in the following form

$$\mathcal{S}^{\alpha,\beta}_{\kappa_\beta+1}(q) = V^\beta_{\kappa_\beta+1}\frac{\left(qV^\beta_{\kappa_\beta+2} - V^\beta_{\kappa_\beta+1}+1\right)\left(q - \frac{V^\beta_{\kappa_\beta}}{V^\beta_{\kappa_\beta+1}-1}\right)}{q^2-(2-\alpha)q+1}. \tag{91}$$

Now, the factor $\left(qV^\beta_{\kappa_\beta+2} - V^\beta_{\kappa_\beta+1}+1\right)$ is positive since for $0<\beta<2$, we have

$$V^\beta_{\kappa_\beta} > 0, \qquad V^\beta_{\kappa_\beta+1} < 0, \qquad V^\beta_{\kappa_\beta+2} = (2-\beta)V^\beta_{\kappa_\beta+1} - V^\beta_{\kappa_\beta} < 0.$$

Thus, $\mathcal{S}^{\alpha,\beta}_{\kappa_\beta+1}(q)$ has the opposite sign than the factor $\left(q - V^\beta_{\kappa_\beta}/(V^\beta_{\kappa_\beta+1}-1)\right)$. Moreover, since $q = Q^\alpha(1-s)$ and $V^\beta_{\kappa_\beta}/(V^\beta_{\kappa_\beta+1}-1) = 1/Q^\alpha(\eta_{\alpha,\beta}) = Q^\alpha(1-\eta_{\alpha,\beta})$, we conclude that

$$\mathcal{S}^{\alpha,\beta}_{\kappa_\beta+1}(q) \gtreqless 0 \qquad \text{if and only if} \qquad Q^\alpha(1-s) \lesseqgtr Q^\alpha(1-\eta_{\alpha,\beta}). \tag{92}$$

If we combine (90) and (92) and take into account that $Q^\alpha$ is a strictly increasing function, we obtain

$$(\mathcal{N}_{\alpha,\beta})'(s) \gtreqless -1 \qquad \text{if and only if} \qquad (\alpha-\beta)\cdot(\eta_{\alpha,\beta}-s) \lesseqgtr 0. \tag{93}$$

At third, let us assume that $0 < s \le \tau_{\alpha,\beta}$ and $\beta > 2$. In this case, we have $\kappa_\beta = 0$, $\frac{1}{2-\beta} \le q < 0$ and (90) holds. Moreover, the factor $\left(qV^\beta_{\kappa_\beta+2} - V^\beta_{\kappa_\beta+1}+1\right)$ in (91) reads as $(\beta-1)(q(\beta-3)+1)$ and thus, it is positive due to $0 \le q(\beta-2)+1$. As in the previous case, we obtain that (93) holds.

Finally, the statement now follows from (89) and (93). □

The following theorem provides the values of the global extrema of the function $\rho_{\alpha,\beta}$ as well as points where these extrema are attained: $\mu_{\alpha,\beta}$ and $\eta_{\alpha,\beta}$.

**Theorem 31.** *Let $0 < \alpha < 4$ and $\beta > 0$. Then the function $\rho_{\alpha,\beta}$ attains its global extrema at $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$. More precisely, we have that*

$$\min_{s\in[0,1+\tau_{\alpha,\beta}]} \rho_{\alpha,\beta}(s) = \begin{cases} \rho_{\alpha,\beta}(\mu_{\alpha,\beta}) = 2\mu_{\alpha,\beta} + \kappa_\beta & \text{for } \alpha \le \beta, \\ \rho_{\alpha,\beta}(\eta_{\alpha,\beta}) = 2\eta_{\alpha,\beta} + \kappa_\beta + 1 & \text{for } \alpha > \beta, \end{cases}$$

$$\max_{s\in[0,1+\tau_{\alpha,\beta}]} \rho_{\alpha,\beta}(s) = \begin{cases} \rho_{\alpha,\beta}(\eta_{\alpha,\beta}) = 2\eta_{\alpha,\beta} + \kappa_\beta + 1 & \text{for } \alpha \le \beta, \\ \rho_{\alpha,\beta}(\mu_{\alpha,\beta}) = 2\mu_{\alpha,\beta} + \kappa_\beta & \text{for } \alpha > \beta. \end{cases}$$

**Proof.** Using Lemma 28 and Remark 29, we get that the function $\rho_{\alpha,\beta}$ is continuously differentiable on $[0, 1+\tau_{\alpha,\beta}]$. Let us point out that $\rho_{\alpha,\beta}(s) = \rho(s-1)$ for $1 \le s \le 1+\tau_{\alpha,\beta}$. Indeed, for $1 \le s \le 1 + \tau_{\alpha,\beta}$, we have

$$\begin{aligned} \rho_{\alpha,\beta}(s) &= s + \kappa_\beta + \bar{\bar{M}}_{\alpha,\beta}(s-1) = s - 1 + \kappa_\beta + \bar{\bar{M}}_{\alpha,\beta}(s-1) + 1 \\ &= (s-1) + \kappa_\beta + \mathcal{N}_{\alpha,\beta}(s-1) = \rho_{\alpha,\beta}(s-1). \end{aligned}$$

Thus, we obtain for $0 < s < 1 + \tau_{\alpha,\beta}$ that

$$(\rho_{\alpha,\beta})'(s) = 1 + (\mathcal{N}_{\alpha,\beta})'(s).$$

Moreover, using Lemma 21 we have that

$$\begin{aligned} \rho_{\alpha,\beta}(\eta_{\alpha,\beta}) &= \eta_{\alpha,\beta} + \kappa_\beta + \mathcal{N}_{\alpha,\beta}(\eta_{\alpha,\beta}) = 2\eta_{\alpha,\beta} + \kappa_\beta + 1, \\ \rho_{\alpha,\beta}(\mu_{\alpha,\beta}) &= \mu_{\alpha,\beta} + \kappa_\beta + \mathcal{N}_{\alpha,\beta}(\mu_{\alpha,\beta}) = 2\mu_{\alpha,\beta} + \kappa_\beta. \end{aligned}$$

For $\beta = \xi_k$, $k \in \mathbb{N}$, $k \ge 2$, we have that $\mathcal{N}_{\alpha,\beta}(s) = 1 - s$ according to Lemma 22 and thus $\rho_{\alpha,\beta}(s) \equiv 1 + \kappa_\beta$, which means that $\rho_{\alpha,\beta}$ is a constant function. Let us note that in this case, we have $\eta_{\alpha,\beta} = 0$ and $\mu_{\alpha,\beta} = \frac{1}{2}$ due to Lemma 23.

Now, let us assume that $\beta \ne \xi_k$ for all $k \in \mathbb{N}$, $k \ge 2$. Using Lemma 30, we determine the monotonic intervals of $\rho_{\alpha,\beta}$. The points $\mu_{\alpha,\beta}$ and $\eta_{\alpha,\beta}$ are stationary points of $\rho_{\alpha,\beta}$. Firstly, let us assume that $\alpha < \beta$. Then $\rho_{\alpha,\beta}$ is strictly increasing on intervals $(0, \eta_{\alpha,\beta})$, $(\mu_{\alpha,\beta}, 1)$ and strictly decreasing on the interval $(\eta_{\alpha,\beta}, \mu_{\alpha,\beta})$. Thus, $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ are points of the global extrema of $\rho_{\alpha,\beta}$ on the interval $[0, 1 + \tau_{\alpha,\beta}]$ (the global maximum and the global minimum, respectively). Secondly, let us assume that $\alpha > \beta$. In this case, we similarly obtain that $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ are points of the global minimum and the global maximum, respectively. Finally, in the case of $\alpha = \beta$, we get using Lemma 22 that $\mathcal{N}_{\alpha,\beta}(s) = 1 - s + \tau_{\beta,\beta} = 1 - s + \frac{\pi}{\omega_\beta} - \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor$. Thus, we obtain that $\rho_{\alpha,\beta}$ is a constant function such that $\rho_{\alpha,\beta}(s) \equiv 1 + \kappa_\beta + \tau_{\beta,\beta} = \frac{\pi}{\omega_\beta}$. Let us note that $\tau_{\beta,\beta} = 2\eta_{\beta,\beta} = 2\mu_{\beta,\beta} - 1$ in this case. $\square$

**Remark 32.** Due to Theorem 31, we have for all $(\alpha,\beta)\in\mathcal{D}$ and $s\in[0,1+\tau_{\alpha,\beta}]$ that $\rho^{\mathtt{min}}_{\alpha,\beta}\le\rho_{\alpha,\beta}(s)\le\rho^{\mathtt{max}}_{\alpha,\beta}$, where $\rho^{\mathtt{min}}_{\alpha,\beta}$ and $\rho^{\mathtt{max}}_{\alpha,\beta}$ are given in Definition 2. Moreover, using Lemma 25, we get the following bounds

$$\left\lfloor\tfrac{\pi}{\omega_\beta}\right\rfloor=1+\kappa_\beta\le\rho^{\mathtt{min}}_{\alpha,\beta}\le\rho_{\alpha,\beta}(s)\le\rho^{\mathtt{max}}_{\alpha,\beta}<2+\kappa_\beta=\left\lfloor\tfrac{\pi}{\omega_\beta}\right\rfloor+1$$

for $0<\alpha,\beta<4$.

**Proof of Theorem 3.** At first, let us prove that $\left(\mathcal{C}^{\pm}_{2j-1}\cap\mathcal{D}\right)\subset\Upsilon^{\pm}_{2j-1}$, $j\in\mathbb{N}$, where

$$\Upsilon^{\pm}_{2j-1}=\left\{(\alpha,\beta)\in\mathcal{D}:\ \rho^{\mathtt{min}}_{\alpha,\beta}\ \le\ \tfrac{n+1}{j}-\tfrac{\pi}{\omega_\alpha}\ \le\ \rho^{\mathtt{max}}_{\alpha,\beta}\right\}.$$

Thus, let us assume that $(\alpha,\beta)\in\mathcal{C}^{\pm}_{2j-1}\cap\mathcal{D}$. Then using Theorem 5, we get

$$t^+_j(\alpha,\beta)+t^-_j(\alpha,\beta)=n+1, \tag{94}$$

where $t^+_j$ and $t^-_j$ are defined in (24) and (25), respectively. The corresponding non-trivial solution $u$ consists of $j$ positive and $j$ negative semi-waves (as continuous extensions) and the equation (94) reads as

$$j\cdot\tfrac{\pi}{\omega_\alpha}+\sum_{i=1}^{j}\rho_{\alpha,\beta}\left(\lceil t_{2i-1}\rceil-t_{2i-1}\right)=n+1, \tag{95}$$

where $t_i$, $i=1,\dots,2j-1$, are zeros of positive semi-waves

$$0<t_1=t^+_1<\dots<t_j=t^+_j=n+1-t^-_j<\dots<t_{2j-1}=n+1-t^-_1<n+1.$$

Now, using Theorem 31, we obtain for $i=1,\dots,j$ that

$$\begin{aligned}
2\mu_{\alpha,\beta}+\kappa_\beta\le\rho_{\alpha,\beta}\left(\lceil t_{2i-1}\rceil-t_{2i-1}\right)&\le 2\eta_{\alpha,\beta}+\kappa_\beta+1 &&\text{for } \alpha\le\beta,\\
1+2\eta_{\alpha,\beta}+\kappa_\beta\le\rho_{\alpha,\beta}\left(\lceil t_{2i-1}\rceil-t_{2i-1}\right)&\le 2\mu_{\alpha,\beta}+\kappa_\beta &&\text{for } \alpha>\beta.
\end{aligned}$$

And thus, we have

$$j\cdot\rho^{\mathtt{min}}_{\alpha,\beta}\le\sum_{i=1}^{j}\rho_{\alpha,\beta}\left(\lceil t_{2i-1}\rceil-t_{2i-1}\right)\le j\cdot\rho^{\mathtt{max}}_{\alpha,\beta}, \tag{96}$$

where $\rho^{\mathtt{min}}_{\alpha,\beta}$ and $\rho^{\mathtt{max}}_{\alpha,\beta}$ are given in Definition 2. Finally, if we combine (95) and (96), we obtain $\rho^{\mathtt{min}}_{\alpha,\beta}\le\frac{n+1}{j}-\frac{\pi}{\omega_\alpha}\le\rho^{\mathtt{max}}_{\alpha,\beta}$.

At second, let us show that $\left(\mathcal{C}^{+}_{2j}\cap\mathcal{D}\right)\subset\Upsilon^{+}_{2j}$, $j\in\mathbb{N}$, where

$$\Upsilon^{+}_{2j}=\left\{(\alpha,\beta)\in\mathcal{D}:\ \rho^{\mathtt{min}}_{\alpha,\beta}\ \le\ \tfrac{n+1}{j}-\tfrac{j+1}{j}\cdot\tfrac{\pi}{\omega_\alpha}\ \le\ \rho^{\mathtt{max}}_{\alpha,\beta}\right\}.$$

Using Theorem 5, we obtain for $(\alpha, \beta) \in \mathcal{C}^+_{2j} \cap \mathcal{D}$ that

$$t^+_{j+1}(\alpha, \beta) + t^+_j(\alpha, \beta) = n + 1, \tag{97}$$

and the corresponding non-trivial solution $u$ consists of $(j + 1)$ positive and $j$ negative semi-waves. The equation (97) can be also written as

$$(j + 1) \cdot \tfrac{\pi}{\omega_\alpha} + \sum_{i=1}^{j} \rho_{\alpha,\beta} \left(\lceil t_{2i-1}\rceil - t_{2i-1}\right) = n + 1, \tag{98}$$

where $t_i$, $i = 1, \dots, 2j - 1$, are zeros of positive semi-waves. Using Theorem 31, we obtain the same inequalities as in (96). Now, if we combine (96) and (98), we get $\rho^{\mathtt{min}}_{\alpha,\beta} \le \frac{n+1}{j} - \frac{j+1}{j} \cdot \frac{\pi}{\omega_\alpha} \le \rho^{\mathtt{max}}_{\alpha,\beta}$.

At third, the last type of the inclusion $\left(\mathcal{C}^-_{2j} \cap \mathcal{D}\right) \subset \Upsilon^-_{2j}$, $j \in \mathbb{N}$, can be proved similarly as in the previous two cases. Let us only note that for $(\alpha, \beta) \in \mathcal{C}^-_{2j} \cap \mathcal{D}$, we obtain $(t_0 = 0)$

$$j \cdot \tfrac{\pi}{\omega_\alpha} + \sum_{i=0}^{j} \rho_{\alpha,\beta} \left(\lceil t_{2i}\rceil - t_{2i}\right) = n + 1,$$

which leads to $\rho^{\mathtt{min}}_{\alpha,\beta} \le \frac{n+1}{j+1} - \frac{j}{j+1} \cdot \frac{\pi}{\omega_\alpha} \le \rho^{\mathtt{max}}_{\alpha,\beta}$. $\square$

## 7. Conclusion

In the paper, we improve and extend known results for the Fučík spectrum of the discrete Dirichlet operator. In Theorem 5, we present a new simple implicit description of all non-trivial Fučík curves $\mathcal{C}^\pm_l$. Moreover, for each non-trivial Fučík curve $\mathcal{C}^\pm_l$, we provide the suitable bound $\Upsilon^\pm_l$ by two simple curves in Theorem 3. These results are based on Lemma 10 concerning the detailed analysis of nodal properties of the solution $u$ of the discrete initial value problem (30). Generalized zeros of the solution $u$ can be described by the sequence of functions $p_i$ which are given recurrently and were introduced in [20]. In Lemma 12, we provide a new simpler expression of these functions $p_i$, which can be used to obtain the basic bound $\Theta^\pm_l$ for each non-trivial Fučík curve $\mathcal{C}^\pm_l \subset \Theta^\pm_l$ (see Theorem 13).

In this paper, we mainly focus on positive semi-waves of $u$ as continuous extensions and investigate the distribution of zeros of these extensions with respect to the integer lattice. More precisely, if $t_1$ and $t_2$ are two consecutive zeros of two different positive semi-waves (as continuous extensions) then we have

$$t_2 = t_1 + \rho_{\alpha,\beta}(\lceil t_1\rceil - t_1),$$

where the function $\rho_{\alpha,\beta}$ is given explicitly in Definition 19 using Chebyshev polynomials of the second kind. We use this function $\rho_{\alpha,\beta}$ in Theorem 5 to describe implicitly all non-trivial Fučík curves $\mathcal{C}_l^{\pm}$. Let us emphasize that this new description using $\rho_{\alpha,\beta}$ does not require complicated construction of sequences of functions $p_i$ and $\vartheta_i$ as it was done in [20].

Let us note that in the case of $\alpha = \beta$, the discrete initial value problem (30) is linear and $\rho_{\alpha,\beta}$ is the constant function $\rho_{\alpha,\beta}(s) \equiv \frac{\pi}{\omega_\beta}$. Now, for $0 < \alpha < 4$ and $\beta > 0$, the function $\rho_{\alpha,\beta}$ is a differentiable bounded function and its global extrema are given in Theorem 31. Since the global extrema of $\rho_{\alpha,\beta}$ are available in an explicit form, we provide the improved bound $\Upsilon_l^{\pm}$ for $\mathcal{C}_l^{\pm}$ in Theorem 3 with the boundary given by two simple curves, which are described similarly to the first non-trivial Fučík curve $\mathcal{C}_1^{\pm}$.

## Declaration of competing interest

The authors declare that there is no competing interest.

## Acknowledgement

The authors were supported by the Grant Agency of the Czech Republic, grant no. 18-03253S.

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For the published article

[25] I. Looseová (Sobotková), P. Nečesal, The Fučík spectrum of the discrete Dirichlet operator, Linear Algebra Appl. 553 (2018) 58–103

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*Plzeň, August 20, 2021*

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For the published article

[31] P. Nečesal, I. Sobotková, Localization of Fučík curves for the second order discrete Dirichlet operator, Bulletin des Sciences Mathématiques 171 (2021) 103014

co-author Petr Nečesal confirms that the contribution of Iveta Sobotková is approximately 50%.

*Plzeň, August 20, 2021*

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