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The geometry of surfaces contact

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Abstract

This contribution deals with a geometrical exact description of contact between two given surfaces which are defined by the vector functions. These surfaces are substituted at a contact point by approximate surfaces of the second order in accordance with the Taylor series and consequently there is derived a differential surface of these second order surfaces. Knowledge of principal normal curvatures, their directions and the tensor (Dupin) indicatrix of this differential surface are necessary for description of contact of these surfaces. For description of surface geometry the first and the second surface fundamental tensor and a further methods of the differential geometry are used. A geometrical visualisation of obtained results of this analysis is made. Method and results of this study will be applied to contact analysis of tooth screw surfaces of screw machines.

Keywords: contact mechanics, differential geometry, first and second fundamental tensor, Gaussian and mean curvature, principal curvatures and their directions, screw machine, Taylor series, tensor indicatrix

1. Introduction

The aim of this paper, which creates the first part of contact analysis of two bodies in accordance with the Hertz theory [2], [3], is the determination of differential surface and its curvatures at this contact point. In this study the simplified surfaces, which represent the complicated technical surface, are considered. The surfaces are given by vector functions. Both surfaces are replaced in the contact point by approximate surfaces of second order in accordance with the Taylor series, [4], pp. 205. At the contact point on this differential surface the principal normal curvatures and their principal directions, which define a *contact base*, are determined. The principle normal curvatures at the contact point must be known in order to describe the contact of surfaces in the manner of the Hertz theory. Therefore the derivatives of this differential surface are necessary up to the second order made. All descriptions are shown for the surface σ_3 only. For the surface σ_2 is valid the same procedure. Obtained method and results of the solution will be applied for determination of contact of tooth surfaces of screw compressor rotors or screw machine with, as consequence of force and heat deformation of machine housing, skew axes. After that it is possible to deal with the displacements field and stress field in a neighbourhood of the contact point depending on geometry of tooth surfaces.

2. Problem description and input parameters

Two screw surfaces σ_3 and σ_2, fig. 1, create a general kinematic couple in the space. Their rotation axes are o_3 and o_2. The initial global coordinate system R_1 given by $\{O_1;\ \mathbf{e}_{iR_1}\}$ is placed on the o_{30} axis, fig. 1. It is considered with respect to a change of its position that the axis

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o_{30} is displaced into the position o_3. This axis displacement is determined by radius vectors $\mathbf{r}_{A_{30}}$, $\mathbf{r}_{B_{30}}$ and displacement vectors $\mathbf{u}_{A_3}$, $\mathbf{u}_{B_3}$. The radius vectors are expressed by a homogeneous coordinates. Let these surfaces contact itself at the point $C \equiv C_2 \equiv C_3$ which are given by the following radius vectors

$$_{R_{\sigma_3}}\mathbf{r}_{C_3} = \begin{bmatrix} \theta_{C_3} & \varphi_{C_3} & 1 \end{bmatrix}^{\mathrm{T}}, \qquad _{R_{\sigma_2}}\mathbf{r}_{C_2} = \begin{bmatrix} \theta_{C_2} & \varphi_{C_2} & 1 \end{bmatrix}^{\mathrm{T}}, \tag{1}$$

where R_{σ_i}, $i \in \{3,\ 2\}$ is the surface σ_i coordinate system of the Euclidean affine space E_2 and θ_{C_i}, φ_{C_i} are curvilinear coordinates of this point C_i on the surface σ_i. The determination of the contact point of surfaces σ_3 and σ_2 is not subject of this solution. This problem will be solved separately with the creation of individual surfaces.

For the solution these following parameters are selected. The initial position of the axis o_3, which is marked as o_{30}, is coincident with the third base vector $\mathbf{e}_{3R_1}$, fig. 1. The position of the axis o_3 determine these following parameters $_{R_1}\mathbf{r}_{A_{30}} = \begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix}^{\mathrm{T}}$, $_{R_1}\mathbf{r}_{B_{30}} = \begin{bmatrix} 0 & 0 & 3 & 1 \end{bmatrix}^{\mathrm{T}}$, $_{R_1}\mathbf{u}_{A_3} = = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}^{\mathrm{T}}$, $_{R_1}\mathbf{u}_{B_3} = \begin{bmatrix} 3 & -2 & 2 \end{bmatrix}^{\mathrm{T}}$. The surface σ_3 is defined with these parameters R_{31} = 1; R_{3r} = 0,5; R_{3z} = 1; H_3 = 3; n_{R3} = 1,75 and similary the surface σ_2 has these parameters R_2 = 1; H_2 = 4; n_{R2} = 1,75. The explanation of these parameters is in the next chapter. For determination of the axis o_2 position are used following two parameters. The first one is $C_{rotation}$ = 80 [°] and means a rotation arround the normal line at the contact point, the second one $C_{distance}$ has a function for a visual demonstration only which defines the distance between contact points on the contact normal line n, fig. 1. A turning of the surface σ_3 towards the fixed coordinate system R_{3f}, fig. 1, is given by the coordinate φ_3 = 120 [°]. The contact point C_3 on the surface σ_3 is determined by $_{R_{\sigma_3}}C_3 = \begin{bmatrix} \theta_{C_3},\ \varphi_{C_3} \end{bmatrix} = [3\pi/2,\ 0] \in \Omega_{\sigma_3} \subset \mathrm{R}^2$ and on the surface σ_2 by $_{R_{\sigma_2}}C_2 = \begin{bmatrix} \theta_{C_2},\ \varphi_{C_2} \end{bmatrix} = [3{,}75\pi/2,\ 0] \in \Omega_{\sigma_2} \subset \mathrm{R}^2$.

3. Geometry definition of problem

The fixed coordinate system R_{3f} was introduced. This system is determined by the vectors $\mathbf{r}_{A_{30}}$, $\mathbf{r}_{B_{30}}$, $\mathbf{u}_{A_3}$, $\mathbf{u}_{B_3}$, fig. 1. Because the surface σ_3 has one degree of freedom there is created an actual coordinate system R_3 which coordinate is φ_3. The equation for transformation of the coordinate system R_3 into R_1 is

$$_{R_1}\mathbf{r} = \mathbf{T}_{R_{3f},R_1}\mathbf{T}_{R_3,R_{3f}}(\varphi_3)\,_{R_3}\mathbf{r}\,, \tag{2}$$

where

$$\mathbf{T}_{R_{3f},R_1} = \begin{bmatrix} _{R_1}\mathbf{e}_{1R_{3f}} & _{R_1}\mathbf{e}_{2R_{3f}} & _{R_1}\mathbf{e}_{3R_{3f}} & _{R_1}\mathbf{r}_{O_{3f}} \\ 0 & 0 & 0 & \end{bmatrix}, \qquad \mathbf{T}_{R_3,R_{3f}}(\varphi_3) = \begin{bmatrix} \cos\varphi_3 & -\sin\varphi_3 & 0 & 0 \\ \sin\varphi_3 & \cos\varphi_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

$\mathbf{T}_{R_bR_a}$ is the transformation matrix of the vector coordinates in the coordinate system R_b into the coordinate system R_a, $^{\sigma_a}_{R_a}\mathbf{r}_A$ is the radius vector of the point A on the surface σ_a in the coordinate system R_a and $_{R_a}\mathbf{e}_{iR_b(j)}$ is the j-th coordinate of the i-th base vector of the coordinate system R_b expressed in the coordinate system R_a. The displacement of the origin O_{30} is

$$ {}_{R_1}\mathbf{r}_{O_{3f}} = {}_{R_1}\mathbf{r}_{A_{30}} + {}_{R_1}\mathbf{u}_{A_3} + \left| {}_{R_1}\mathbf{r}_{A_{30}} \right| {}_{R_1}\mathbf{e}_{3R_{3f}} \,. \tag{4} $$

Both surfaces σ_3 and σ_2 are defined in the actual coordinate systems, R_3 and R_2, by vector functions as follows

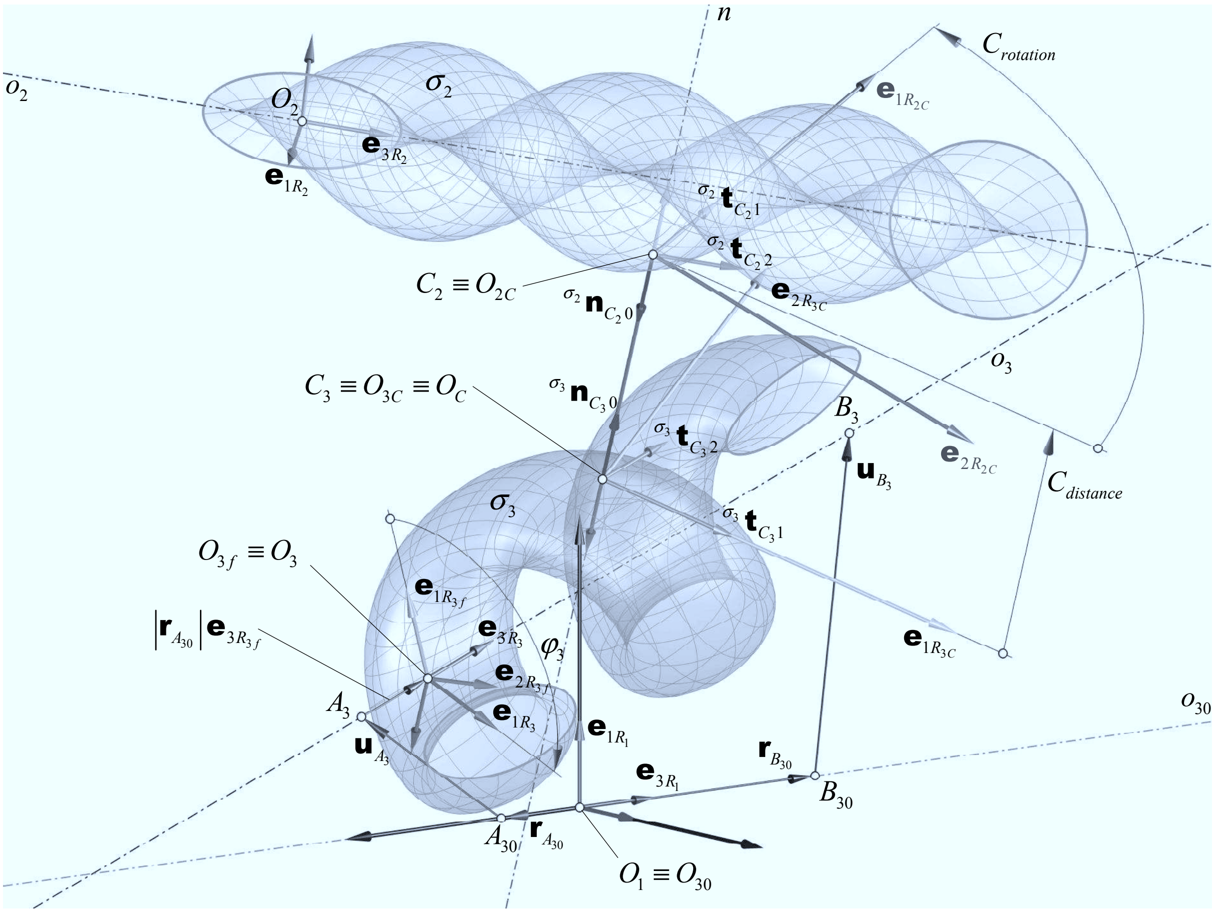

Fig. 1. Visualisation of definite surfaces and coordinate systems.

$$ {}^{\sigma_3}_{R_3}\mathbf{r} = {}^{\sigma_3}_{R_3}\mathbf{r}(\theta,\ \varphi) = \left[R_{31}\cos\theta + R_{3r}\cos\varphi\cos\theta,\ R_{31}\sin\theta + R_{3r}\cos\varphi\sin\theta,\ R_{3z}\sin\varphi + \frac{H_3}{2\pi}\theta,\ 1 \right]^{\mathrm{T}}, \tag{5} $$
$$ [\theta,\ \varphi] \in \Omega_{\sigma_3} \subset \mathrm{R}^2,\ \ \Omega_{\sigma_3} = \langle 0,\ 2\pi) \times \langle 0,\ 2\pi n_{R3} \rangle, $$

where R_{31}, R_{3r}, R_{3z} are radiuses of an anuloid, H_3 is a height per one rotate in the direction of the third coordinate, n_{R3} is a revolution multiplicator and the Ω is the 2D range on which the vector function $\mathbf{r}$ is given, its input. The special cases of the surface described by the (5) are for example ellipsoid, anuloid, circle surface, Corkscrew surface, helicoid, screw anuloid etc. The surface σ_2 is defined by

$$ {}^{\sigma_2}_{R_2}\mathbf{r} = {}^{\sigma_2}_{R_2}\mathbf{r}(\theta,\ \varphi) = \left[R_2\cos\varphi\cos\theta,\ R_2\cos\varphi\sin\theta,\ R_2\sin\varphi + \frac{H_2}{2\pi}\theta,\ 1 \right]^{\mathrm{T}}, \tag{6} $$
$$ [\theta,\ \varphi] \in \Omega_{\sigma_2} \subset \mathrm{R}^2,\ \ \Omega_{\sigma_2} = \langle 0,\ 2\pi) \times \langle 0,\ 2\pi n_{R2} \rangle, $$

where R_2 is a radius of this screw surface, H_2 is a height per one rotate in the direction of the third coordinate, n_{R2} is a revolution multiplicator. This surface is sometimes called as the Corkscrew surface. The θ, φ are curvilinear coordinates on the surface σ_i. The tangent vectors fields determining a base of curvilinear coordinates in every point of surface σ_3 are

$$ {}^{\sigma_3}_{R_3}\mathbf{t}_1 = {}^{\sigma_3}_{R_3}\mathbf{t}_1(\theta,\varphi) = \frac{\partial\left[{}^{\sigma_3}_{R_3}\mathbf{r}(\theta,\varphi)\right]}{\partial\theta} = \left[-R_{31}\sin\theta - R_{3r}\cos\varphi\sin\theta,\ R_{31}\cos\theta + R_{3r}\cos\varphi\cos\theta,\ \frac{H_3}{2\pi},\ 0\right]^{\mathrm{T}}, \quad (7) $$

$$ {}^{\sigma_3}_{R_3}\mathbf{t}_2 = {}^{\sigma_3}_{R_3}\mathbf{t}_2(\theta,\varphi) = \frac{\partial\left[{}^{\sigma_3}_{R_3}\mathbf{r}(\theta,\varphi)\right]}{\partial\varphi} = \left[-R_{3r}\sin\varphi\cos\theta,\ -R_{3r}\sin\varphi\sin\theta,\ R_{3z}\cos\varphi,\ 0\right]^{\mathrm{T}}. \quad (8) $$

The normal and the unit normal vector is

$$ {}^{\sigma_3}_{R_3}\mathbf{n}(\theta,\ \varphi) = {}^{\sigma_3}_{R_3}\mathbf{t}_1 \times {}^{\sigma_3}_{R_3}\mathbf{t}_2, \qquad {}^{\sigma_3}_{R_3}\mathbf{n}_0(\theta,\ \varphi) = {}^{\sigma_3}_{R_3}\mathbf{n}\left|{}^{\sigma_3}_{R_3}\mathbf{n}\right|^{-1}. \quad (9) $$

On the surface σ_3 is selected arbitrary point C_3, its radius vector in the R_3 is ${}^{\sigma_3}_{R_3}\mathbf{r}_{C_3} = {}^{\sigma_3}_{R_3}\mathbf{r}_{C_3}\left(\theta_{C_3},\ \varphi_{C_3}\right)$, which determines the contact point. At this point is established a coordinate system $R_{3C} \equiv \left\{O_{3C};\ \mathbf{e}_{iR_{3C}}\right\}$, the transformation matrix of this system into R_3 is

$$ \mathbf{T}_{R_{3C},R_3} = \begin{bmatrix} {}_{R_3}\mathbf{e}_{1R_{3C}} & {}_{R_3}\mathbf{e}_{2R_{3C}} & {}_{R_3}\mathbf{e}_{3R_{3C}} & {}^{\sigma_3}_{R_3}\mathbf{r}_{C_3} \\ 0 & 0 & 0 & \end{bmatrix} = \left[\begin{array}{c|c|c|c} {}^{\sigma_3}_{R_3}\mathbf{t}_{C_31}\left|{}^{\sigma_3}_{R_3}\mathbf{t}_{C_31}\right|^{-1} & {}^{\sigma_3}_{R_3}\mathbf{n}_{C_30} \times {}^{\sigma_3}_{R_3}\mathbf{t}_{C_31}\left|{}^{\sigma_3}_{R_3}\mathbf{t}_{C_31}\right|^{-1} & {}^{\sigma_3}_{R_3}\mathbf{n}_{C_30} & {}^{\sigma_3}_{R_3}\mathbf{r}_{C_3} \\ 0 & 0 & 0 & \end{array}\right]. \quad (10) $$

A similar coordinate system $R_{2C} \equiv \left\{O_{2C};\ \mathbf{e}_{iR_{2C}}\right\}$ is created for the surface σ_2 in which origin will be to lie the contact point C_2 of the surface σ_2. This system is defined with the transformation matrix $\mathbf{T}_{R_{2C},R_{3C}}\left(C_{distance},\ C_{rotation},\ \pi\right)$, fig. 1. On the surface σ_2 is determined arbitrary point C_2, its radius vector in the R_2 is ${}^{\sigma_2}_{R_2}\mathbf{r}_{C_2} = {}^{\sigma_2}_{R_2}\mathbf{r}_{C_2}\left(\theta_{C_2},\ \varphi_{C_2}\right)$, which determines the contact point on the surface σ_2. At this point is determined a general coordinate system on the surface σ_2 $\left\{C_2;\ {}^{\sigma_2}\mathbf{t}_{C_21},\ {}^{\sigma_2}\mathbf{t}_{C_22},\ {}^{\sigma_2}\mathbf{n}_{C_20}\right\}$, which the first base ${}^{\sigma_2}\mathbf{t}_{C_21}$ is colinear with the $\mathbf{e}_{1R_{2C}}$ and the third base ${}^{\sigma_2}\mathbf{n}_{C_20}$ is colinear with the $\mathbf{e}_{3R_{2C}}$, fig. 1. The transformation matrix of the coordinate system R_{2C} into the coordinate system R_2 has thus the form

$$ \mathbf{T}_{R_{2C},R_2} = \begin{bmatrix} {}_{R_2}\mathbf{e}_{1R_{2C}} & {}_{R_2}\mathbf{e}_{2R_{2C}} & {}_{R_2}\mathbf{e}_{3R_{2C}} & {}^{\sigma_2}_{R_2}\mathbf{r}_{C_2} \\ 0 & 0 & 0 & \end{bmatrix} = \left[\begin{array}{c|c|c|c} {}^{\sigma_2}_{R_2}\mathbf{t}_{C_21}\left|{}^{\sigma_2}_{R_2}\mathbf{t}_{C_21}\right|^{-1} & {}^{\sigma_2}_{R_2}\mathbf{n}_{C_20} \times {}^{\sigma_2}_{R_2}\mathbf{t}_{C_21}\left|{}^{\sigma_2}_{R_2}\mathbf{t}_{C_21}\right|^{-1} & {}^{\sigma_2}_{R_2}\mathbf{n}_{C_20} & {}^{\sigma_2}_{R_2}\mathbf{r}_{C_2} \\ 0 & 0 & 0 & \end{array}\right], \quad (11) $$

the inverse matrix $\mathbf{T}_{R_2,R_{2C}} = \mathbf{T}^{-1}_{R_{2C},R_2}$. The contact points C_3 and C_2 are identical by the parameter $C_{distance} = 0$ and therefore they create the contact point C.

4. The approximation of surfaces with second order surfaces at the contact points

The surface σ_i, $i \in \{3,\ 2\}$, is substituted at the point ${}_{R_{\sigma_i}}C_i = \left[\theta_{C_i} \quad \varphi_{C_i}\right]$ by the *Taylor series* of the vector function defining the surface σ_i up to the second order, fig. 2, 3. As an illustration the surface σ_3 at the contact point ${}_{R_{\sigma_3}}C_3$ is thus substituted with following approximate surface

$$ \begin{aligned} {}^{\sigma_3^T}_{R_3}\mathbf{r}\ (\theta,\ \varphi) = {}^{\sigma_3}_{R_3}\mathbf{r}\left(\theta_{C_3},\varphi_{C_3}\right) + \frac{\partial\, {}^{\sigma_3}_{R_3}\mathbf{r}\left(\theta_{C_3},\varphi_{C_3}\right)}{\partial\theta}\left(\theta-\theta_{C_3}\right) + \frac{\partial\, {}^{\sigma_3}_{R_3}\mathbf{r}\left(\theta_{C_3},\varphi_{C_3}\right)}{\partial\varphi}\left(\varphi-\varphi_{C_3}\right) + \\ + \frac{1}{2}\left[\frac{\partial^2\, {}^{\sigma_3}_{R_3}\mathbf{r}\left(\theta_{C_3},\varphi_{C_3}\right)}{\partial\theta\partial\theta}\left(\theta-\theta_{C_3}\right)^2 + 2\frac{\partial^2\, {}^{\sigma_3}_{R_3}\mathbf{r}\left(\theta_{C_3},\varphi_{C_3}\right)}{\partial\theta\partial\varphi}\left(\theta-\theta_{C_3}\right)\left(\varphi-\varphi_{C_3}\right) + \frac{\partial^2\, {}^{\sigma_3}_{R_3}\mathbf{r}\left(\theta_{C_3},\varphi_{C_3}\right)}{\partial\varphi\partial\varphi}\left(\varphi-\varphi_{C_3}\right)^2\right]. \end{aligned} \quad (12) $$

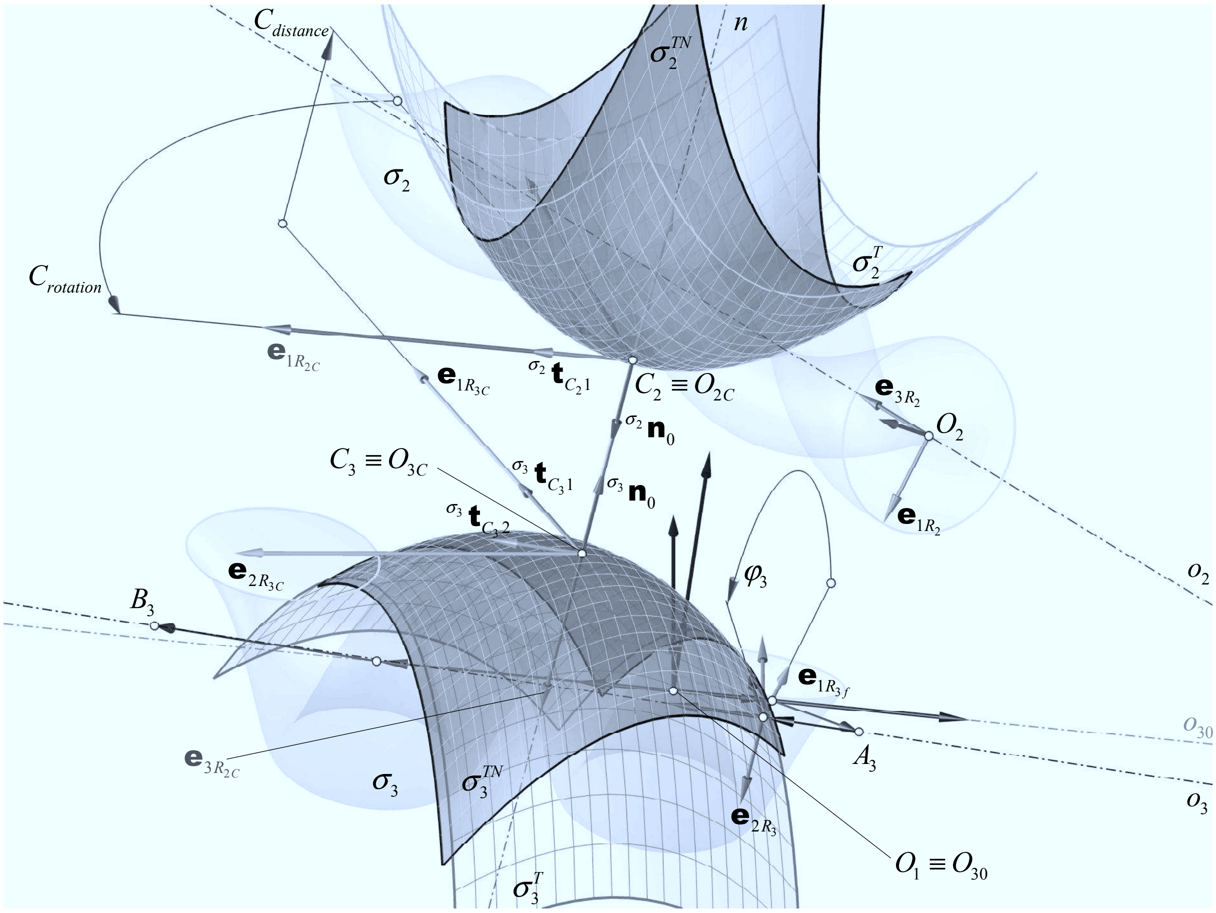

Fig. 2. Approximate surfaces with remeshed approximate surfaces.

This approximate surface σ_3^T is expressed in the coordinate system R_{3C} with the equation

$$ {}^{\sigma_3^T}_{R_{3C}}\mathbf{r} = \mathbf{T}^{-1}_{R_{3C},R_3}\left({}^{\sigma_3^T}_{R_3}\mathbf{r} - {}^{\sigma_3^T}_{R_3}\mathbf{r}_{C_3} \right). \tag{13} $$

The approximate surface σ_2^T is expressed in the coordinate system R_{3C} likewise

$$ {}^{\sigma_2^T}_{R_{3C}}\mathbf{r} = \mathbf{T}_{R_{2C},R_{3C}}\mathbf{T}^{-1}_{R_{2C},R_2}\left({}^{\sigma_2^T}_{R_2}\mathbf{r} - {}^{\sigma_2^T}_{R_2}\mathbf{r}_{C_2} \right). \tag{14} $$

For determination of the differential surface σ_D at the contact point C the first and the second coordinate of these approximate surfaces σ_3^T and σ_2^T have to be selected on an orthonormal plain grid in the coordinate system R_{3C}, fig. 3. This transformation is given with the system of three nonlinear equations

$$ {}^{\sigma_3^{TN}}_{R_{3C}}\mathbf{r} = \mathbf{T}^{-1}_{R_{3C},R_3}\left({}^{\sigma_3^T}_{R_3}\mathbf{r} - {}^{\sigma_3^T}_{R_3}\mathbf{r}_{C_3} \right), \tag{15} $$

where coordinates $\left\{ {}^{\sigma_3^{TN}}_{R_{3C}}\mathbf{r}_{(1)},\ {}^{\sigma_3^{TN}}_{R_{3C}}\mathbf{r}_{(2)} \right\} \in \langle -M, M \rangle$, M is a boundary of the discrete interval, N index indicates the new surface. The (15) is rewrited into the form $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, so

$$ \mathbf{T}_{R_3,R_{3C}}\left({}^{\sigma_3^T}_{R_3}\mathbf{r} - {}^{\sigma_3^T}_{R_3}\mathbf{r}_{C_3} \right) - {}^{\sigma_3^{TN}}_{R_{3C}}\mathbf{r} = \mathbf{0}, \tag{16} $$

where the unknows vector $\mathbf{x} = \left[{}_{R_{\sigma_3^{TN}}}\theta \quad {}_{R_{\sigma_3^{TN}}}\varphi \quad {}^{\sigma_3^{TN}}_{R_{3C}}\mathbf{r}_{(3)} \right]^{\mathrm{T}}$. For the solution of this equations system the *Newton's method* is used. This new approximate surface σ_i^{TN}, $i\in\{3,\ 2\}$, fig. 4, is actually given in the form ${}^{\sigma_i^{TN}}_{R_{3C}}\mathbf{r}=\left[u^1,\ u^2,\ f_i\left(u^1,\ u^2\right)\right]^{\mathrm{T}}$, $\left[u^1,\ u^2\right]\in\Omega_{\sigma_i^{TN}}$, where $\Omega_{\sigma_i^{TN}}$ is the two-dimensional discrete region and f_i is a function of two variables on the $\Omega_{\sigma_i^{TN}}$. The surface σ_i^{TN} is thus identical with the σ_i^{T}.

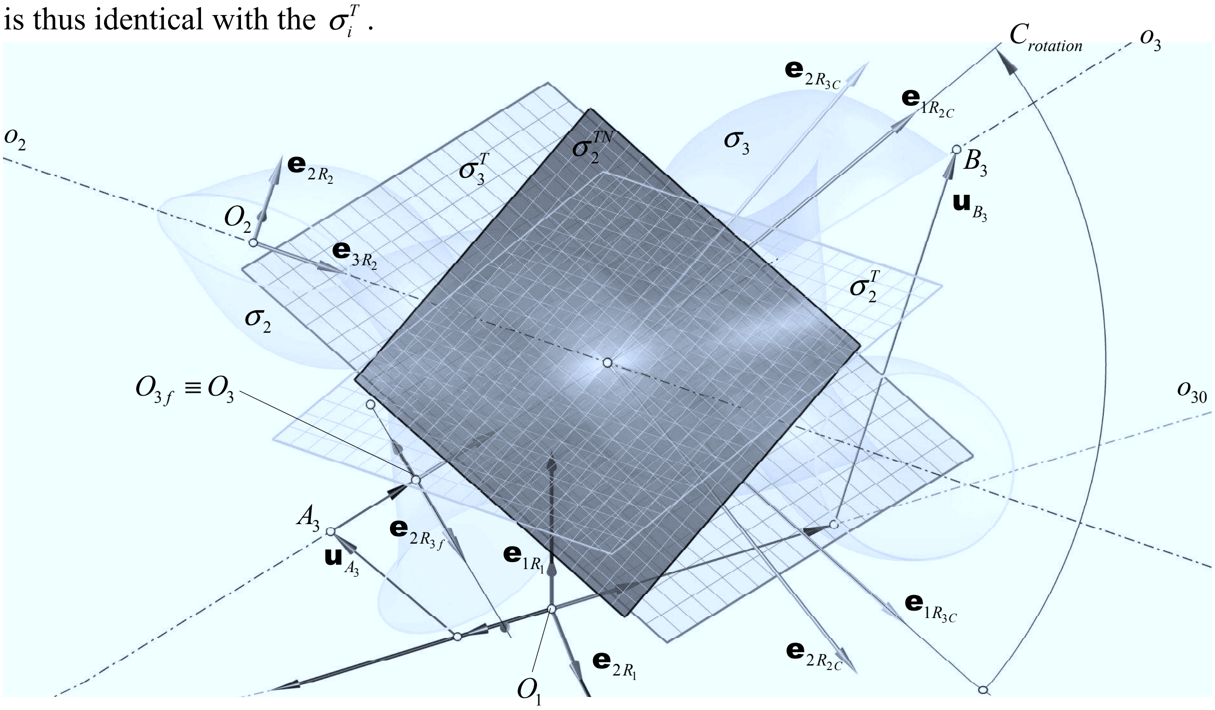

Fig. 3. View of approximate surfaces with remeshed approximate surfaces along the normal line *n*.

5. Differential surface and inner geometry at the contact point

The differential surface σ_D is described in the coordinate system R_{3C} with the equation

$${}^{\sigma_D}_{R_{3C}}\mathbf{r}={}^{\sigma_D}_{R_{3C}}\mathbf{r}\left(u^1,\ u^2\right)=\left(\sum_{i=1}^{2}{}^{\sigma_3^{TN}}_{R_{3C}}\mathbf{r}_{(i)}\ {}_{R_{3C}}\mathbf{e}_{iR_{3C}}\right)+\left({}^{\sigma_2^{TN}}_{R_{3C}}\mathbf{r}_{(3)}-{}^{\sigma_3^{TN}}_{R_{3C}}\mathbf{r}_{(3)}\right){}_{R_{3C}}\mathbf{e}_{3R_{3C}}+{}_{R_{3C}}\mathbf{e}_{4R_{3C}}\,. \tag{17}$$

The differentiations have to be performed numerically, thus differentiation ${}^{\sigma_D}_{R_{3C}}\mathbf{r}\left(u^1,\ u^2\right)$ with respect to u^i, $i\in\{1,\ 2\}$ gives two tangent vector fields determining the base of the local curvilinear coordinates and four vector fields given with the second derivatives

$${}^{\sigma_D}_{R_{3C}}\mathbf{t}_i\left(u^1,\ u^2\right)=\frac{\partial\,{}^{\sigma_D}_{R_{3C}}\mathbf{r}\left(u^1,\ u^2\right)}{\partial u^i},\quad {}^{\sigma_D}_{R_{3C}}\mathbf{r}_{ij}\left(u^1,\ u^2\right)=\frac{\partial\,{}^{\sigma_D}_{R_{3C}}\mathbf{r}\left(u^1,\ u^2\right)}{\partial u^i\partial u^j},\qquad i,j\in\{1,\ 2\}\,. \tag{18}$$

The normal and the unit normal vector is

$${}^{\sigma_D}_{R_{3C}}\mathbf{n}\left(u^1,\ u^2\right)={}^{\sigma_D}_{R_{3C}}\mathbf{t}_1\times{}^{\sigma_D}_{R_{3C}}\mathbf{t}_2\,,\qquad {}^{\sigma_D}_{R_{3C}}\mathbf{n}_0\left(u^1,\ u^2\right)=\frac{{}^{\sigma_D}_{R_{3C}}\mathbf{n}}{\left|{}^{\sigma_D}_{R_{3C}}\mathbf{n}\right|}\,. \tag{19}$$

The covariant coordinates of the *first fundamental tensor* g_{ij}, [1], pp. 186, on the surface σ_D at the contact point ${}_{R_{\sigma_D}}C=\left[u_C^1,\ u_C^2\right]=[0,\ 0]\in\Omega_{\sigma_D}\subset\mathrm{R}^2$ are defined by the dot product of the tangent vectors ${}^{\sigma_D}\mathbf{t}_{Ci}$

$$\mathbf{G} = \left[g_{ij} \right] = {}^{\sigma_D}_{R_{3C}}\mathbf{t}_{Ci} \cdot {}^{\sigma_D}_{R_{3C}}\mathbf{t}_{Cj}, \qquad i, j \in \{1,\ 2\}. \tag{20}$$

The covariant coordinates of the *second fundamental tensor* h_{ij}, [1], pp. 199, on the surface σ_D at the contact point C are defined by

$$\mathbf{H} = \left[h_{ij} \right] = {}^{\sigma_D}_{R_{3C}}\mathbf{n}_{C0} \cdot {}^{\sigma_D}_{R_{3C}}\mathbf{r}_{Cij}, \qquad i, j \in \{1,\ 2\}. \tag{21}$$

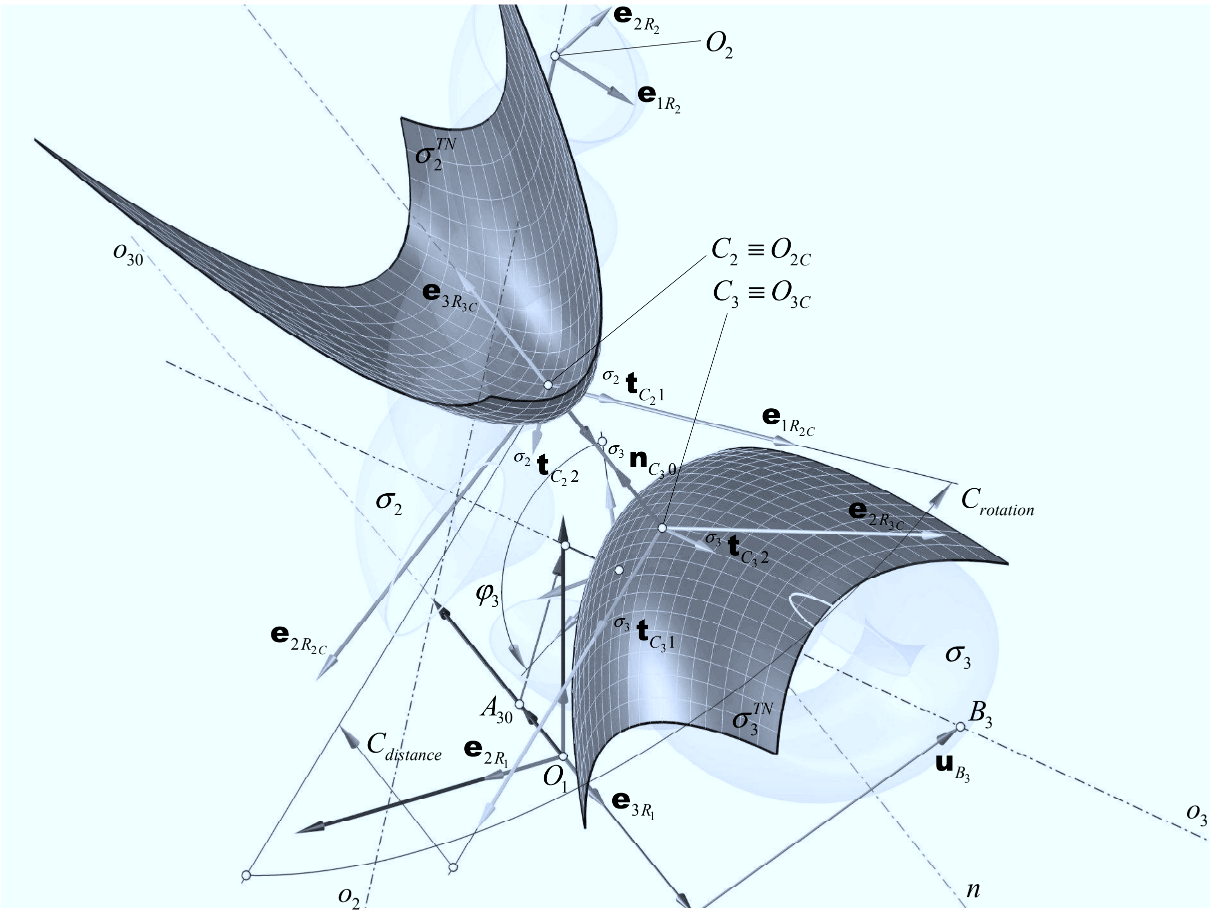

Fig. 4. The remeshed approximate surfaces.

The Gaussian curvature K and the mean curvature H of this surface at the contact point C is given, [1], pp. 215, by

$$K = \frac{\det(\mathbf{H})}{\det(\mathbf{G})} = \frac{h_{11}h_{22} - (h_{12})^2}{g_{11}g_{22} - (g_{12})^2}, \qquad H = \frac{1}{2}\frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{g_{11}g_{22} - (g_{12})^2}. \tag{22}$$

The principal normal curvatures $\kappa_{1,2}$ are determined from the following equations system

$$K = \kappa_1\kappa_2,\ H = \frac{1}{2}(\kappa_1 + \kappa_2) \quad \Rightarrow \quad \kappa_{1,2} = H \pm \sqrt{H^2 - K}. \tag{23}$$

Determination of the principal curvatures and the directions of their normal planes leads up to the generalized problem of eigen values which is described, [6], pp. 288, with the equation

$$\mathbf{Hx} = \lambda\mathbf{Gx}. \tag{24}$$

The solution of this equation gives the eigen values λ_i, that are principal normal curvatures or extrem curvatures actually and eigen vectors $\mathbf{v}_i$. These eigen values and vectors are writed down as follows

$$\mathbf{D} = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}, \quad \mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2]. \tag{25}$$

The local coordinate system at a point on a surface is generally an affine coordinate system. In this case at the contact point C there is the local coordinate system $R_{DCg} \equiv \{C;\ {}^{\sigma_D}_{R_{3C}}\mathbf{t}_{C1},\ {}^{\sigma_D}_{R_{3C}}\mathbf{t}_{C2},\ {}^{\sigma_D}_{R_{3C}}\mathbf{n}_{C0}\}$ orthogonal. The eigen vectors expressed in the coordinate system R_{3C} are

$${}_{R_{3C}}\mathbf{V} = \mathbf{T}_{R_{DCg},R_{3C}} \left[{}_{R_{DCg}}\mathbf{v}_1 \quad {}_{R_{DCg}}\mathbf{v}_2\right] = \left[{}^{\sigma_D}_{R_{3C}}\mathbf{t}_{C1} \quad {}^{\sigma_D}_{R_{3C}}\mathbf{t}_{C2}\right]\left[{}_{R_{DCg}}\mathbf{v}_1 \quad {}_{R_{DCg}}\mathbf{v}_2\right] \tag{26}$$

and the angle between eigen vectors $\mathbf{v}_i$ is at each point $[u^1, u^2]$ of σ_D

$$\varphi_{\mathbf{v}_1\mathbf{v}_2} = \varphi_{\mathbf{v}_1\mathbf{v}_2}(u^1, u^2) = \operatorname{acos}\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1||\mathbf{v}_2|} = \frac{\pi}{2}. \tag{27}$$

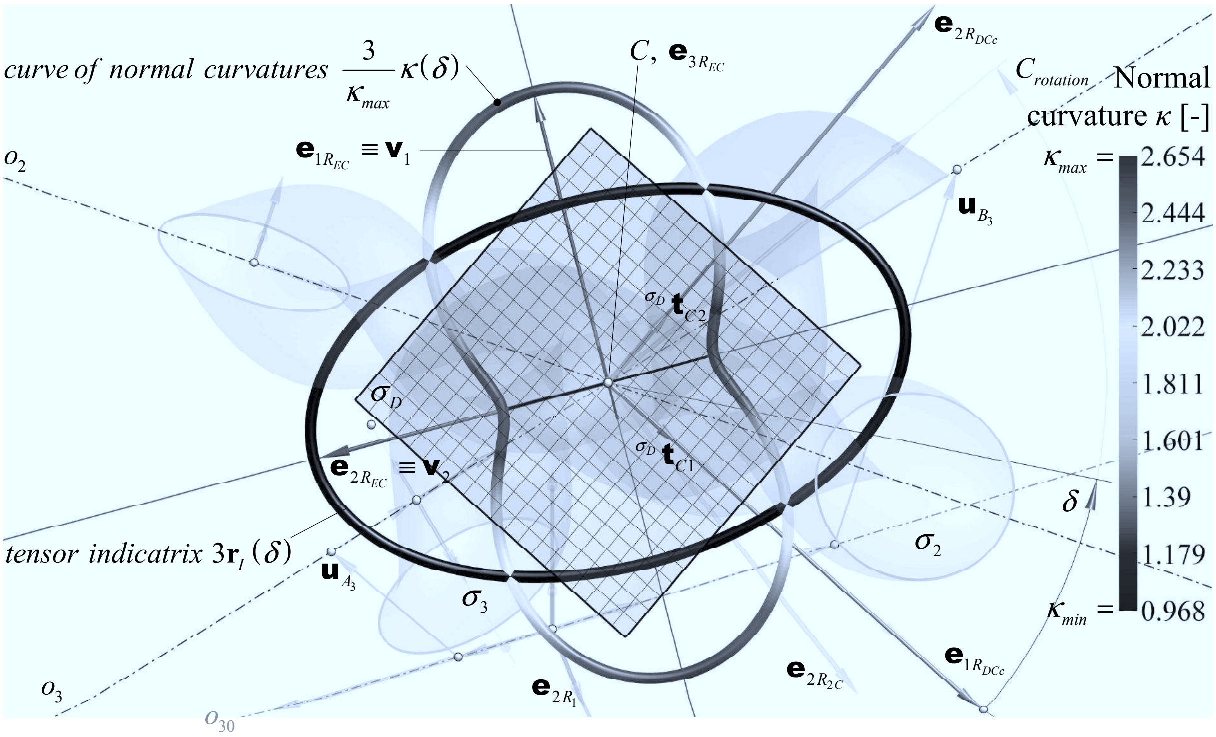

Fig. 5. View along the normal line on tensor indicatrix, curve of normal curvatures and their directions of differential surface at contact point C.

These unit eigen vectors define the new orthonormal coordinate system R_{EC} (extreme curvatures) at the contact point C given by $\{C;\ \mathbf{e}_{iR_{EC}}\}$ that it is called *the contact base* as well. The third base vector is defined with

$${}_{R_{3C}}\mathbf{e}_{3R_{EC}} = {}_{R_{3C}}\mathbf{v}_1 \times {}_{R_{3C}}\mathbf{v}_2\,. \tag{28}$$

For the next step it is necessary at the point C to create an auxiliary orthonormal coordinate system $R_{DCc} \equiv \{C;\ \mathbf{e}_{iDCc}\}$. The transformation matrix of this system into R_{3C} is

$$\mathbf{T}_{R_{DCc},R_{3C}}=\begin{bmatrix} {}_{R_{3C}}\mathbf{e}_{1DCc} & {}_{R_{3C}}\mathbf{e}_{2DCc} & {}_{R_{3C}}\mathbf{e}_{3DCc} & \mathbf{o} \\ 0 & 0 & 0 & 1 \end{bmatrix}=\left[\begin{array}{c|c|c|c} {}^{\sigma_D}_{R_{3C}}\mathbf{t}_{C1}\left|{}^{\sigma_D}_{R_{3C}}\mathbf{t}_{C1}\right|^{-1} & {}^{\sigma_D}_{R_{3C}}\mathbf{n}_{C0}\times{}^{\sigma_D}_{R_{3C}}\mathbf{t}_{C1}\left|{}^{\sigma_D}_{R_{3C}}\mathbf{t}_{C1}\right|^{-1} & {}^{\sigma_D}_{R_{3C}}\mathbf{n}_{C0} & \mathbf{o} \\ \hline 0 & 0 & 0 & 1 \end{array}\right],\quad(29)$$

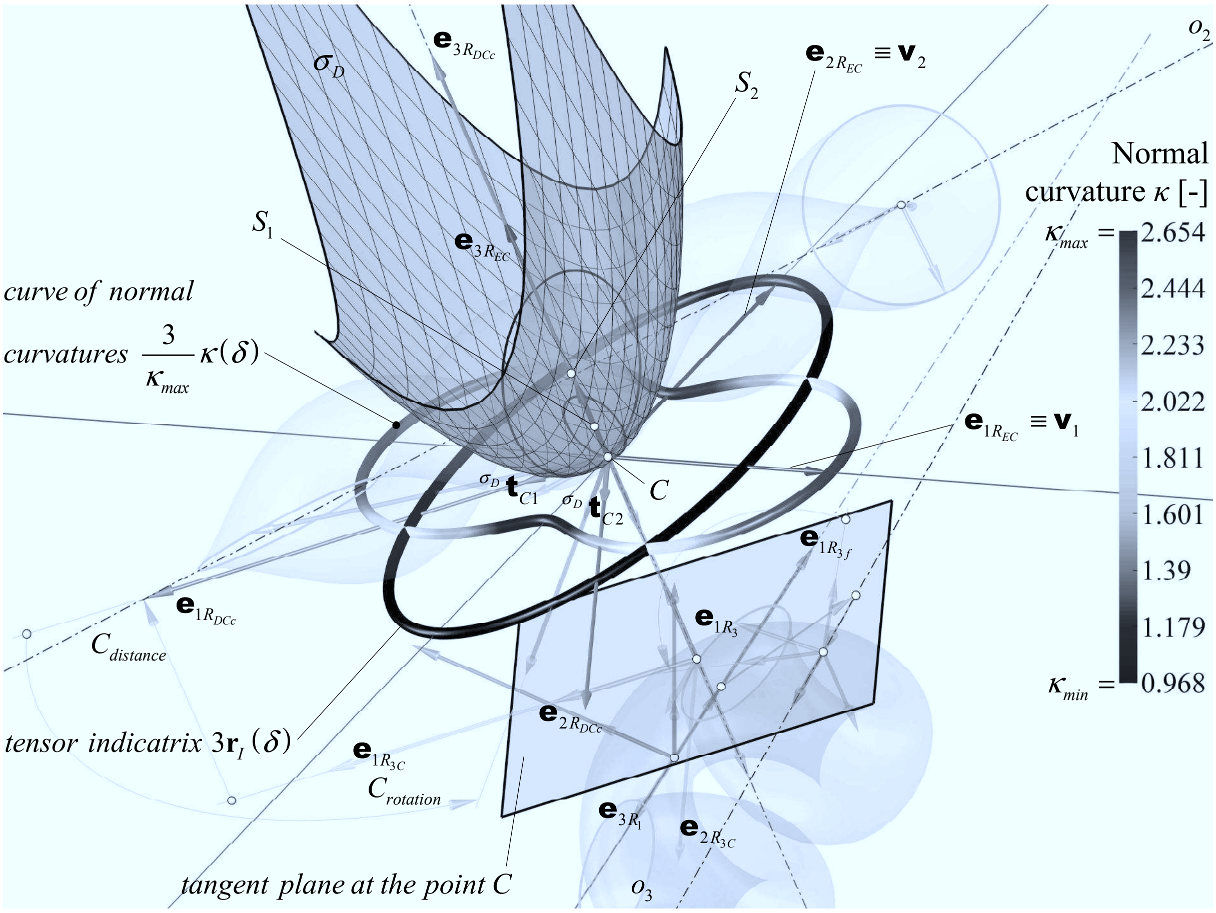

Fig. 6. Tensor indicatrix, curve of normal curvatures and osculate circles in the principle directions, the point S_i is the circle center.

where $\mathbf{o} = [0\ 0\ 0]^{\mathrm{T}}$. The affine transformation of an orthonormal coordinate system R_{DCc} into an affine coordinate system R_{DCg} is given by

$${}_{R_{DCg}}\mathbf{r}=\mathbf{T}_{R_{DCc},R_{DCg}}(\varphi)\,{}_{R_{DCc}}\mathbf{r}=\begin{bmatrix}\left|{}^{\sigma_D}\mathbf{t}_{C1}\right|^{-1} & 0\\ 0 & \left|{}^{\sigma_D}\mathbf{t}_{C2}\right|^{-1}\end{bmatrix}\begin{bmatrix}1 & -1/\tan(\varphi)\\ 0 & 1/\sin(\varphi)\end{bmatrix}\cdot\begin{bmatrix}{}_{R_{DCc}}x^1\\ {}_{R_{DCc}}x^2\end{bmatrix},\qquad(30)$$

where φ is the angle between ${}^{\sigma_D}\mathbf{t}_{Ci}$ vectors and x^i is a vector coordinate. Because the base vectors ${}^{\sigma_D}\mathbf{t}_{Ci}$ are orthogonal in the solved case the angle $\varphi = \pi/2$. The relation for the normal curvature $\kappa(u^1,\ u^2,\ \delta)$, [1], pp. 207, in the normal plane given by vectors ${}^{\sigma_D}\mathbf{n}_{C0}$ and $\mathbf{t}(\delta)$ at the contact point C is

$$\kappa=\kappa\left(u^1,\ u^2,\ \delta\right)=\frac{h_{ij}t^it^j}{g_{ij}t^it^j}=\frac{\left({}_{R_{DCg}}\mathbf{t}\right)^{\mathrm{T}}\mathbf{H}\left({}_{R_{DCg}}\mathbf{t}\right)}{\left({}_{R_{DCg}}\mathbf{t}\right)^{\mathrm{T}}\mathbf{G}\left({}_{R_{DCg}}\mathbf{t}\right)},\qquad(31)$$

where

$$_{R_{DCg}}\mathbf{t}(\delta) = \mathbf{T}_{R_{DCc},R_{DCg}}(\varphi)\, _{R_{DCc}}\mathbf{t}(\delta), \qquad _{R_{DCc}}\mathbf{t}(\delta) = \begin{bmatrix} \cos\delta \\ \sin\delta \end{bmatrix}, \quad \delta \in \langle 0,\ 2\pi), \quad i, j \in \{1,\ 2\}, \tag{32}$$

$\mathbf{t}(\delta)$ is an unit vector in the tangential plane. The equation of the *tensor indicatrix*, i.e. Dupin indicatrix, [1], pp. 209, is

$$\left|h_{ij} r^i r^j\right| = 1 = \left|\left(_{R_{DCg}}\mathbf{r}\right)^{\mathrm{T}} \mathbf{H}\left(_{R_{DCg}}\mathbf{r}\right)\right|, \tag{33}$$

where r^i is a point coordinate on the indicatrix curve, fig. 5 and 6. In these pictures there is the curve of the tensor indicatrix illustrated on a scale 3 and the curve of normal curvatures on a scale $3/\kappa_{max}$. The parametric expression of the indicatrix curve at the point C can be

$$_{R_{3C}}\mathbf{r}_I = {_{R_{3C}}\mathbf{r}_I}(\delta) = {^{\sigma_D}_{R_{3C}}\mathbf{r}_{C_2}} + \frac{1}{\sqrt{|\kappa(\delta)|}}\left[_{R_{DCg}} t^i(\delta)\right]\left[^{\sigma_D}_{R_{3C}}\mathbf{t}_{Ci}\right], \quad \delta \in \langle 0,\ 2\pi),\ i \in \{1,\ 2\}, \tag{34}$$

where $t^i(\delta)$ is a coordinate of the unit vector $\mathbf{t}(\delta)$.

6. Conclusion

This work, which occupies by the geometry of surfaces and their differential surface at the contact point, is the preliminary part of the contact analysis of two surfaces based on the Hertz theory. The aim of this presented analysis is the determination of the differential surface of both surfaces and its curvatures at the contact point. Consequently the contact base, that creates coordinate system, is determined. This theoretical study of the contact geometry will be implemented to the contact of tooth surfaces of screw machines in operation mode when the axes of tooth surfaces are skew. In this case the original contact curve between tooth surfaces changes into the point contact, which causes an increase of the value of normal force at this point of more then eighty times, [5], with respect to normal force at general point of contact curve in case of non-deformed, parallel, position of rotors. This effect can be the cause of a damage of tooth surfaces. The differential surface σ_D, which describes the relative distance of approximate surfaces σ_3^T and σ_2^T, in the neighbourhood of the contact point, is fundamental to the solution of the contact analysis. The next step of this work will be the determination of the displacement field and stress field in the neighbourhood of the contact point of tooth screw surfaces of screw machines.

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