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Growing and remodeling material as a dynamical system

J. Rosenberg [a,*], L. Hynčík [a]

[a] Faculty of Applied Sciences, UWB in Pilsen, Univerzitní 22, 306 14 Plzeň, Czech Republic

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Abstract

Contribution contains the short description of the general theory of growth and remodeling based on Di'Carlo's approach. This theory is applied to the one dimensional continuum using the quadratic form of the free energy function. Two different forms of loading are dealt – isometric and isotonic one. For both cases the corresponding equations describe the dynamical system. Its properties are analyzed using the methods of nonlinear dynamics. It's shown the influence of the constant growth and remodeling parameters on the stability of the equilibrium point.

Keywords: growth, remodeling, 1D continuum, isometric loading, isotonic loading, dynamical system, stability, equilibrium point

1. Introduction

Further we will assume growth as a volume change and remodeling as a change of material parameters or anisotropy. Then a lot of different changes in the material behavior can be thought as a growth or remodeling. As example we can take the growth of tissue during development and aging but also the changes in muscle during its stimulation. In the last case we can observe also some changes in its stiffness – kind of remodeling. Another example is e.g. the piezoelectric material – under influence of electric potential it changes its length.

In literature we can find different mathematical models of growth and remodeling. This contribution is based on the DiCarlo's theory [1]. This theory leads to the system of evolution differential equations for parameters describing volume, deformation or material properties. The form of these equations depends on the particular material. This is expressed by a set of material parameter (some of these are constants, some are time dependent). In any cases this system can be thought as a dynamical system and be analyzed with the adequate means. The properties of this system, e.g. the stability, depend on the above parameters. To find the critical values of these parameters is crucial from two points of view: At first same instabilities can occur in the modeled system and we can see, where is the source of this behavior. This can happen e.g. in the modeling of muscles using the mentioned theory. The question is if the causes of the observed instability of muscle under some outer conditions are the mechanical properties of the muscle tissue or the nerve control. The second reason is that when we identify the parameters using some results of experiment we need to know their limits.

This is the goal of this contribution. To be simple as possible the 1D continuum is taken in account. In [1] and [3] can be found some concrete application of the used growth theory and therefore here is introduced only very brief introduction and summarization of this theory. Main attempt is devoted to the analysis of some chosen types of dynamical systems.

* Corresponding author. Tel.: +420 377 632 325, e-mail: rosen@kme.zcu.cz.

2. Basic equations

The process of growth and remodeling can be expressed at first by the tensor **P** (further growth tensor) that relates the initial configuration to the relaxed one B_r with zero inner stress. To the real configuration B_t where the inner stress invoked by growth and geometrical remodeling and external loading can already exists, it is related by deformation tensor $\mathbf{F}_r$. The whole deformation gradient between configurations B_0 a B_t can be written as ($\mathbf{p}$ is the placement)

$$\nabla \mathbf{p} = \mathbf{F}_r \mathbf{P} \ . \tag{1}$$

For simplicity we will further take into account only the small deformations and therefore we will not distinguish between Lagrangian and Eulerian approach. Applying the principle of virtual working the following equations are obtained

$$Div\ \boldsymbol{\tau} + \mathbf{b} = 0 \text{ on } B_0 \ , \ \mathbf{B} + \mathbf{C} = \mathbf{0} \text{ on } B_0 \ , \ \hat{\boldsymbol{\tau}}\ \mathbf{n} = \boldsymbol{\tau}\ \mathbf{n} \text{ on } \partial B_0 , \tag{2}$$

where $\boldsymbol{\tau}$ is the Cauchy stress tensor, **b** is the volume force, z is the vector of inner effects, **B** the inner remodeling generalized force and **C** is the generalized external remodeling force, $\hat{\boldsymbol{\tau}}\mathbf{n}$ is prescribed stress on boundary and **n** is the vector of outer normal. The stress $\boldsymbol{\tau}$ can be decomposed into the elastic part $\boldsymbol{\tau}_{el}$ and the dissipative part $\boldsymbol{\tau}_{dis}$. If we assume the specific free energy related to the relaxed volume in form $\psi(\mathbf{F},\mathbf{K})$ where $\mathbf{K}$ represents the material parameters occurring in the expression of the free energy, which can be changing during the material remodeling and $\dot{\mathbf{K}}$ is the corresponding velocity, we can obtain from the 2. thermodynamical law (see e.g. [4]) the following constitutive equations.

$$\boldsymbol{\tau}_{el} = \frac{\partial \psi}{\partial \mathbf{F}}, \boldsymbol{\tau}_{dis} = \mathbf{H}\dot{\mathbf{F}} \ , \ \mathbf{C} - \mathbf{E} = \mathbf{G}\ \mathbf{V} \ ; \quad \mathbf{E} = \psi\ \mathbf{I} - \mathbf{F}\boldsymbol{\tau} \ , \ R - \frac{\partial \psi}{\partial \mathbf{K}} = \mathbf{M} \cdot \dot{\mathbf{K}} \ . \tag{3}$$

M, **H** and **G** are generally positively definite matrices[1]. Let we have now 1D continuum of the initial length l_0. Its actual length after growth, remodeling and loading will be l. The relaxed length – it means after growth and remodeling- is l_r. For the corresponding deformation gradients we can write

$$P = \frac{l_r}{l_0}, \quad F = \frac{l}{l_r}, \quad \nabla p = \frac{l}{l_0} . \tag{4}$$

The deformation energy will have the simplest form

$$\psi = \frac{1}{2}\ k(F-1)^2 , \tag{5}$$

where k is the material parameter (stiffness). Than we can write for the relaxation or isometric stimulation ($l = const$) the following system of equations

[1] It is necessary to mention that the above property of **M, H** and **G** sufficient condition of the fulfilling of the second law of thermodynamics. In [3] was shown, that under certain circumstances this condition need not be fulfilled.

$$\dot{k} = \frac{1}{m}\left[r - \frac{1}{2}\left(\frac{l}{l_r} - 1\right)^2\right], \tag{6}$$

$$\dot{l}_r = l_r \frac{\frac{k}{2}l^2 - \left(C + \frac{k}{2}\right)l_r^{\,2}}{gl_r^{\,2} + hl^2}, \tag{7}$$

$$\tau = k\left(\frac{l}{l_r} - 1\right) + h\frac{l}{l_r}\frac{\frac{k}{2}l^2 - \left(C + \frac{k}{2}\right)l_r^{\,2}}{gl_r^{\,2} + hl^2} \tag{8}$$

on the state space l_r, k. We obtained these equations putting (4) and (5) into (3). Instead of matrices **M**, H and **G** we have here the real parameters of growth and remodeling m, h and g. They needn't be constant but further we will assume they are constant.

For creep or isotonic loading ($\tau = const$) we have the system

$$\dot{k} = \frac{1}{m}\left[r - \frac{1}{2}\left(\frac{l}{l_r} - 1\right)^2\right], \tag{9}$$

$$\dot{l}_r = \frac{l_r}{g}\left[\frac{l}{l_r}\tau - \frac{1}{2}k\left(\frac{l}{l_r} - 1\right)^2 - C\right], \tag{10}$$

$$\dot{l} = \frac{l_r}{h}\left\{\tau - k\left(\frac{l}{l_r} - 1\right) + h\frac{l}{gl_r}\left[\frac{l}{l_r}\tau - \frac{1}{2}k\left(\frac{l}{l_r} - 1\right)^2 - C\right]\right\} \tag{11}$$

on the state space l_r, k, l.

3. Relaxation

We start with the equations (6), (7), (8). Further we will assume $C = const$. Equilibrium point coordinates are

$$l_r = \frac{l}{\sqrt{2\left(r \pm \sqrt{2r}\right) + 1}} = \frac{l}{1 \pm \sqrt{2r}}, \tag{12}$$

$$k = \frac{C}{r \pm \sqrt{2r}}. \tag{13}$$

It can be shown that the physical meaning has the upper sign +.

In the special case when $k = const$ ($m \to \infty$), then

$$l_r = \frac{l}{\sqrt{\frac{2}{k}\left(C + \frac{k}{2}\right)}}. \tag{14}$$

To analyze the stability conditions of this equilibrium points we constructed the Jacobian matrix of this system and evaluated its eigenvalues. As a result of numerical calculation we can declare that the stability can be achieved only for $m < 0$. According footnote at page 2, this is allowed. In the Fig. 1 the phase portrait and stress relaxation of this system for $m = -0.2$ ($l = 1.1$; $g = 1$; $h = 1$; $C = 0.1$; $r = 0.02$; $l_0 = 1$) are shown. Corresponding eigenvalues in an equilibrium point $l_r = 0.9167$, $k = 0.4545$ are -0.2750±0.1805i. The influence of r is visible in Fig. 2. With $r = 0.01$ the intermitent burst are occuring and $r = 0$ is the bifurcation point.
It's necessary to mention that the equilibrium point depends only on C and r! Its stability for $C \neq 0$ depends only on the sign of m. For $m < 0$ this point is stable. $C = 0$ is unstable if m is not infinite, it means k is changing.

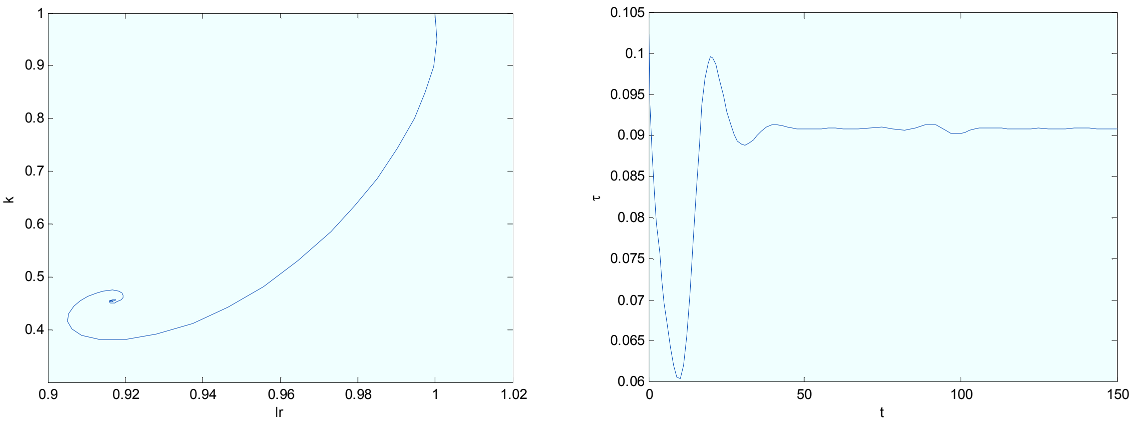

Fig. 1. Phase portrait and stress relaxation process.

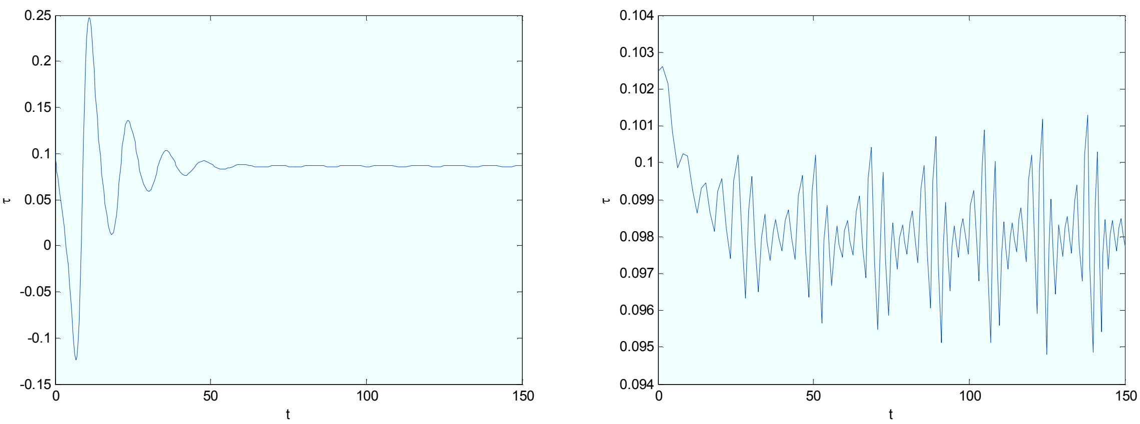

Fig. 2. Stress relaxation for $r = 0.04$ (left) and 0.01 (right).

The above mentioned criteria of stability can be confirmed analytically. For this reason we rewrite the basic equations into the form ($h = 0$, $x = \left(\frac{l_r}{l}\right)^2$) to

$$\dot{k} = \frac{1}{m}\left[r - \frac{1}{2}\left(x^{-1/2} - 1\right)^2\right], \tag{15}$$

$$\dot{x} = \frac{k}{g} - x\frac{2}{g}\left(C + \frac{k}{2}\right). \tag{16}$$

The corresponding system of equation for perturbations in the neighbourhood of fix point with the coordinates (13) and $x = \left(1+\sqrt{2r}\right)^{-2}$ is

$$\dot{\xi}_k = 0 \cdot \xi_k + \frac{1}{2m}\sqrt{2r}\left(1+\sqrt{2r}\right)^3 \xi_x , \tag{17}$$

$$\dot{\xi}_x = \frac{2}{g}\frac{\sqrt{2r}+r}{\left(1+\sqrt{2r}\right)^2}\xi_k - \frac{C}{g}\frac{\left(1+\sqrt{2r}\right)^2}{r+\sqrt{2r}}\xi_x . \tag{18}$$

The eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left[-\frac{C}{g}\frac{\left(1+\sqrt{2r}\right)^2}{r+\sqrt{2r}} \pm \sqrt{\left[\frac{C}{g}\frac{\left(1+\sqrt{2r}\right)^2}{r+\sqrt{2r}}\right]^2 + \frac{4\sqrt{2r}}{gm}\left(r+\sqrt{2r}\right)\left(1+\sqrt{2r}\right)}\right]. \tag{19}$$

Immediately we can see, that the fix point will be stable if $m < 0$ but $m \neq -\infty$. For $r = 0$ then we obtain the non-hyperbolicity and according the Grobman-Hartman theorem the system will be structural non-stable – the bifurcation is occurring. This corresponds with results obtained numerically.

Till now we took in account $C = const$ what corresponds with $f(t) = const \cdot u(t)$ where $u(t)$ is the unite jump. Now we try to analyze the case, when $f(t)$ is periodical function what corresponds with the skeletal muscle stimulation. This stimulation will be approximated with the function

$$C = C_0 + C_1 \sin \omega t . \tag{20}$$

Then we have a non-autonomous dynamical system

$$\dot{k} = \frac{1}{m}\left[r - \frac{1}{2}\left(\frac{l}{l_r} - 1\right)^2\right], \tag{21}$$

$$\dot{l}_r = l_r \frac{\frac{k}{2}l^2 - \left(C_0 + C_1 \sin \omega t + \frac{k}{2}\right)l_r^{\,2}}{g l_r^{\,2} + h l^2} .$$

We will again suppose $h = 0$.

For the most simple case $k = const$ we can obtain only one equation for $x = \left(\frac{l_r}{l}\right)^2$ as

$$\dot{x} = \frac{k}{g} - x\frac{2}{g}\left(C_0 + C_1 \sin \omega t + \frac{k}{2}\right). \tag{22}$$

For the perturbation ξ gilt

$$\dot{\xi} = -\frac{2}{g}\left(C_0 + C_1 \sin \omega t + \frac{k}{2}\right)\xi . \tag{23}$$

The fundamental solution of this equation is $exp(\lambda t)$. Putting this into (23) we will see that it must be

$$\lambda = -\frac{2}{g}\left(C_0 + C_1 \sin \omega t + \frac{k}{2}\right). \tag{24}$$

Setting $t = \dfrac{2\pi}{\omega}$ into the fundamental solution we obtain the Floquet multiplier (F.m.)

$$F.m. = e^{-\frac{2}{g}\left(C_0 + \frac{k}{2}\right)\frac{2\pi}{\omega}}. \tag{25}$$

The same expression can be obtained if we find the solution of (23) in the known form (see e.g. [2])

$$\xi = A \exp \int_0^t \left[-\frac{2}{g}\left(C_0 + C_1 \sin \omega t + \frac{k}{2}\right)\right] dt. \tag{26}$$

After integration we obtain

$$\xi = A \exp \frac{2}{g}\left[-\left(C_0 + \frac{k}{2}\right)t + \frac{C_1}{\omega}\left(\cos \omega t - 1\right)\right]. \tag{27}$$

Putting $t = \dfrac{2\pi}{\omega}$ we obtain again (25).

The condition for the stability of the periodical solution is therefore if we assume $g > 0$

$$\left(C_0 + \frac{k}{2}\right)\frac{2\pi}{\omega} > 0 \Rightarrow C_0 > -\frac{k}{2}. \tag{28}$$

A numerical example is shown in Fig. 3.

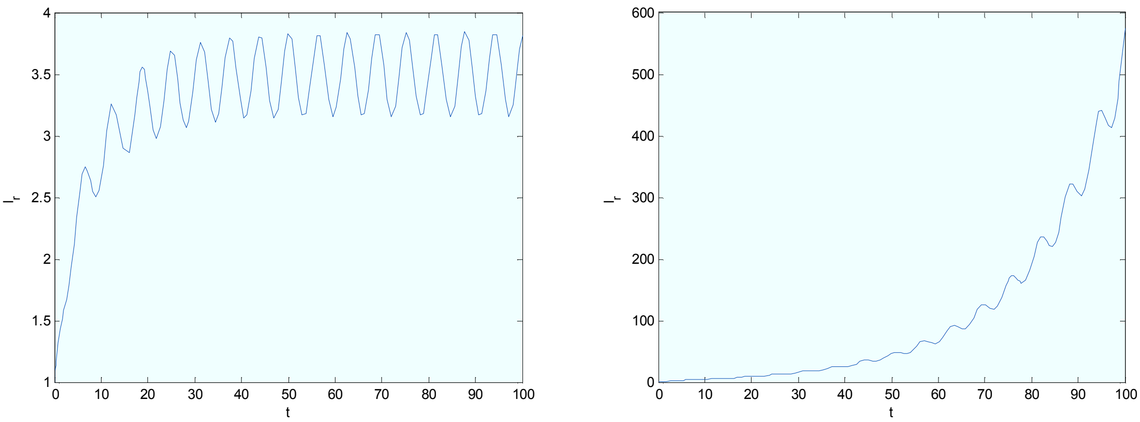

Fig. 3. Time dependence of l_r for l = 1.1, g = 1, h = 0, m = -10000000, C = -0.45 (right), C = -0.55 (left), r = 0.02, $l_0 = l_{r0}$ = 1, k = 1.

Now let us analyze the case when *k(t)*. The system is than

$$\dot{x} = \frac{k}{g} - x\frac{2}{g}\left(C_0 + C_1 \sin \omega t + \frac{k}{2}\right), \tag{29}$$

$$\dot{k} = \frac{1}{m}\left[r - \frac{1}{2}\left(x^{-\frac{1}{2}} - 1\right)^2\right]. \tag{30}$$

For the perturbation we obtain

$$\dot{\xi}_x = -\frac{2}{g}\left(C_0 + C_1 \sin \omega t + \frac{k}{2}\right)\xi_x + \frac{1}{g}(1-x)\xi_k \,, \tag{31}$$

$$\dot{\xi}_k = \frac{1}{2m}\left(x^{-\frac{1}{2}} - 1\right)x^{-\frac{3}{2}}\xi_x + 0\cdot\xi_k \,. \tag{32}$$

We will analyze the autonomous system ($C_1 = 0, C_0 = C$). We put the coordinates of the fix point (12) and (13) into (31) and (32)

$$x = \frac{1}{\left(1+\sqrt{2r}\right)^2}, k = \frac{C}{r+\sqrt{2r}} \,. \tag{33}$$

We obtain the system of equations

$$\dot{\xi}_x = -\frac{C}{g}\left(2 + \frac{1}{r+\sqrt{2r}}\right)\xi_x + \frac{1}{g}\left[1 - \frac{1}{\left(1+\sqrt{2r}\right)^2}\right]\xi_k \,, \tag{34}$$

$$\dot{\xi}_k = \frac{\sqrt{2r}}{2m}\left(1+\sqrt{2r}\right)^3 \xi_x + 0\cdot\xi_k \,. \tag{35}$$

Eigenvalues of the corresponding matrix are

$$\lambda_{1,2} = \frac{1}{2}\left\{-\frac{C}{g}\left(2+\frac{1}{r+\sqrt{2r}}\right) \pm \sqrt{\left[\frac{C}{g}\left(2+\frac{1}{r+\sqrt{2r}}\right)\right]^2 + \frac{4r}{mg}\left(1+\sqrt{2r}\right)\left(2+\sqrt{2r}\right)}\right\}. \tag{36}$$

Non-hyperbolic point will occure if $r = 0$. This is the bifurcation point.

4. Creep – isotonic loading

Similar analysis can be done in case of creep. Here we can observe the special situation – in three equations for the calculation of the equilibrium point (RHS of (9), (10) and (11) equal zero) are only two unknowns – k and l/l_r. To fulfill all three equations, the relation between C and r has to be satisfied

$$C = \tau\left(\frac{r}{\sqrt{2r}} + 1\right). \tag{37}$$

Then the coordinates of the equilibrium point in the phase space are

$$k = \frac{\tau}{\sqrt{2r}} \,, \tag{38}$$

$$\frac{l}{l_r} = \sqrt{2r} + 1 \,. \tag{39}$$

The situation for m = -1 (τ = 1; g = 1; h = 2; r = 0.5; l_0 = 1) is shown in fig. 4. C is then equal according (37) 1.5.

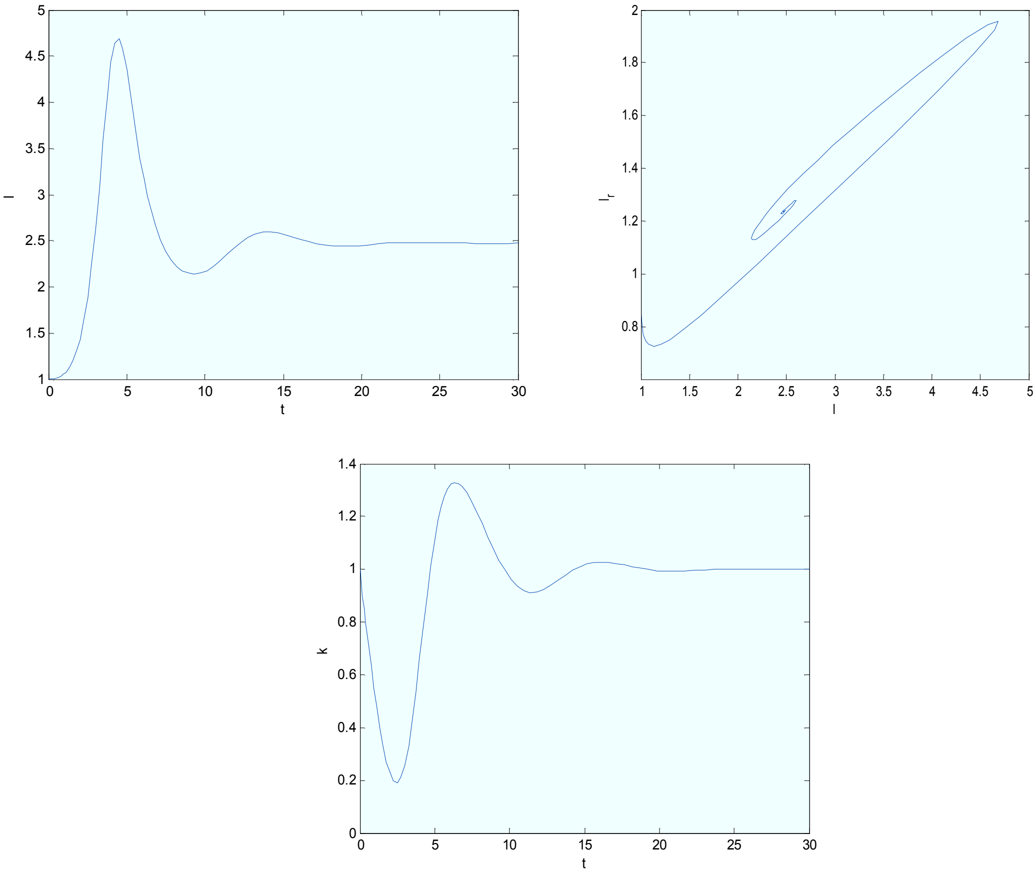

Fig. 4. $l - t$, l_r - l and k - t dependence for creep for m = -1.

When we slightly increase C than we will observe the shortening of this 1D continuum – see Fig. 5. It corresponds to the elevation of load through the muscle stimulation.

For comparison you can see the same situation for $k = 1 = const$ ($m \to \infty$) in Fig. 6. Here the fix point coordinates are

$$C = \frac{\tau^2}{2k} + \tau, \frac{l}{l_r} = \frac{\tau}{k} + 1 . \tag{40}$$

Because the analytical integration is here possible, we obtain the corresponding value of l

$$l = \frac{l_0}{k}(\tau + k)\exp\left(\frac{\tau^2 h}{4gk^2}\right). \tag{41}$$

Now we can analyze the stability of the fix point (37), (38), (39). Instead of analyzing the system (9), (10), (11) we use the dependence only on $x = l/l_r$. Then the corresponding system of equation will be

$$\dot{k} = \frac{1}{m}\left[r - \frac{1}{2}(x-1)^2\right], \tag{42}$$

$$\dot{x} = \frac{1}{h}[\tau - k(x-1)]. \tag{43}$$

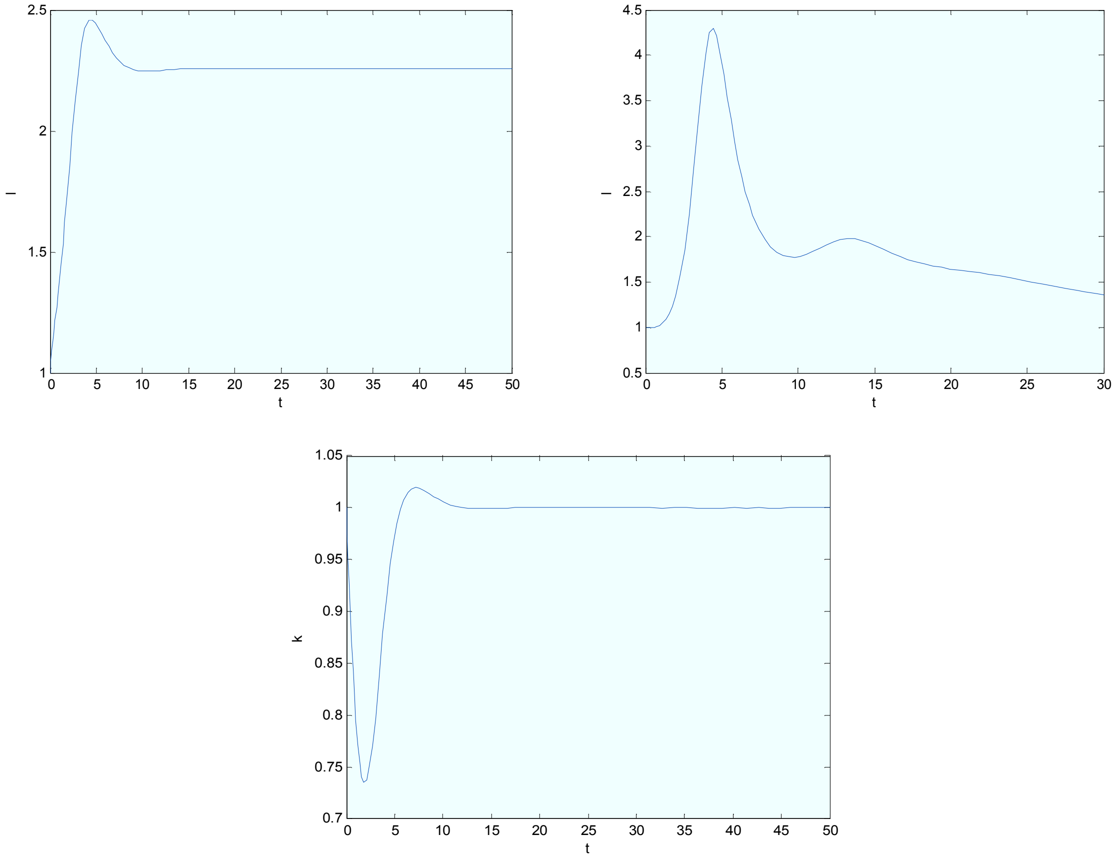

Fig. 5. l - t and k - t dependence for $C = \dfrac{\tau\sqrt{2r}}{2} + \tau + 0.02$.

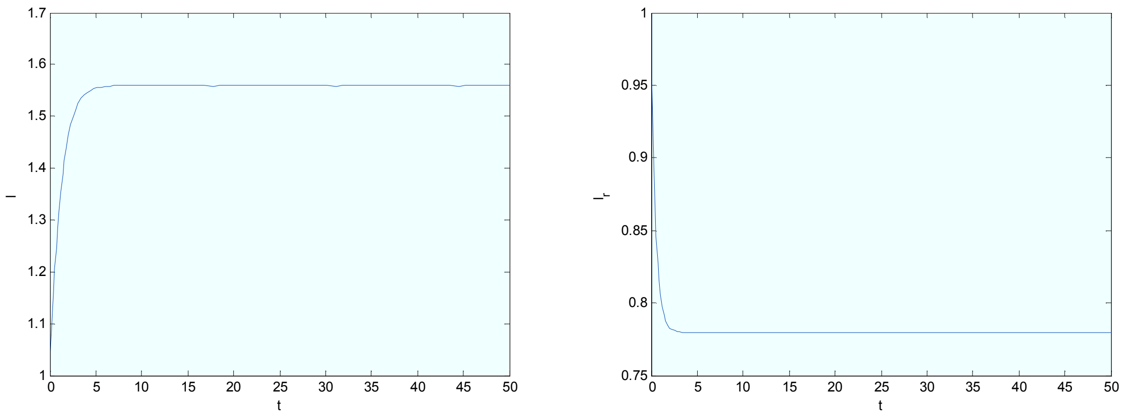

Fig. 6. l_r - t and l - t dependence for $k = 1 = const.$

The equations for perturbation in the given fix point have the form

$$\dot{\xi}_x = -\frac{\tau}{h\sqrt{2r}}\xi_x - \frac{\sqrt{2r}}{h}\xi_k , \tag{44}$$

$$\dot{\xi}_k = -\frac{\sqrt{2r}}{m}\xi_x + 0.\xi_k \,. \tag{45}$$

Corresponding eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left[-\frac{\tau}{h\sqrt{2r}} \pm \sqrt{\left(\frac{\tau}{h\sqrt{2r}}\right)^2 + \frac{8r}{mh}} \right]. \tag{46}$$

Both eigenvalues will be non-positive if $m < 0$ for positive values of r and h. That is the same condition as above in case of relaxation.

5. Conclusion

The aim of this contribution was introduce the model of growth and remodeling as a dynamical system and show some of its properties. For simplicity was chosen 1D continuum with the quadratic form of free energy function. From the analysis can be seen the importance of the growth and remodeling parameters for stability of equilibrium points. In the chosen approach were these parameters chosen as constants. Important result is the necessity of negativity of m for the stability of the whole system. This is mainly in application on living tissues, e.g. muscles, not the case. This will be the direction of further research.

Acknowledgement

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