DECOMPOSING ALLOCATIVE EFFICIENCY FOR MULTI-PRODUCT PRODUCTION SYSTEMS

Tao Zhang

Introduction

Data envelopment analysis (DEA), the non-parametric approach to measuring efficiency, was first introduced in the literature as a linear programming model by Charnes *et al.* [1], following on Farrell's [3] posing of the question of relative technical efficiency in the form of a unit isoquant model. Generally, the DEA approach defines the technical efficiency in terms of a minimum set of inputs needed to produce a given output known as input-orientated model or maximum output obtainable from a given set of inputs known as output-orientated model. [2]

However, since the DEA is non-parametric linear program model, the estimated efficiency might be biased if there is data aggregation in DEA. A series of articles debates on the technical efficiency bias caused by data aggregation in DEA. Fare *et al.* [4, 5] propose that both the inter-input aggregation and inter-output aggregation will make the estimated technical efficiency biased downwardly. Barnum *et al.* [6, 7] discuss the intra-input aggregation and intra-output aggregation caused by the linear aggregation of the same types of inputs and outputs. Barnum *et al.* [8] propose the intra-input allocative efficiency which measures the efficiency in allocating each type of input among outputs using input oriented DEA. But they do not study the normal allocative efficiency bias caused by inter-output or inter-input aggregation. This paper will concern the inter-output allocative efficiency bias which comes from output aggregation.

However, nearly all the observed studies are focused on the efficiency bias caused by data aggregation and blame the DEA method for its bad performance in front of data aggregation. Hitherto, we have not observed any studies on considering how to utilize this character of DEA in methodology extensions for multi-output production system. As presented in the existing literature, the reason of the efficiency bias estimated from DEA is that one part of allocative efficiency will be incorporated into the estimated technical efficiency because of output aggregation. Then, we can use this character to decompose the allocative efficiency according to empirical requirements. This paper hopes to shed new light on the methodology extensions of DEA for decomposing allocative efficiency using output aggregation. In fact, the meaning of results from this paper is outside the DEA method, and the generality of the findings in this paper can provide useful information for researchers who concern the decision-making process in allocating resources for multi-product production system.

1. Decomposing Allocative Efficiency

To explain the theoretical underpinning for decomposing allocative efficiency by data aggregation, we use output oriented technology $\Re$ with i (i=1,...,I) observations. Suppose that for each observation i there is M inputs $X_i=(X_{i1},\dots,X_{iM})\in\Re_+^M$ and J outputs $Y_i=(Y_{i1},\dots,Y_{iJ})\in\Re_+^J$ with corresponding output prices $P_i=(P_{i1},\dots,P_{iJ})\in\Re_+^J$.

The output oriented technical efficiency DEA with fully disaggregated outputs and inputs can be written as:

$$\theta_i^A(X_i, Y_i)=\max\{\theta \mid \sum_i \lambda_i X_{i,m} \le X_{i,m} \quad m=1,\dots,M;$$
$$\sum_i \lambda_i Y_{i,j} \ge \theta Y_{i,j} \quad j=1,\dots,J; \qquad (1)$$
$$\sum_i \lambda_i = 1,\ \lambda_i \ge 0 \quad i=1,\dots,I\}$$

$\theta_i^A(X_i, Y_i)$ is pure technical inefficiency. The pure technical efficiency (TE) can be computed by $TE_i^A = 1/\theta_i^A(X_i, Y_i)$.

Then, consider the following linear program to maximize revenue $A(X_i, Y_i)$:

$$A(X_i, Y_i) = max\{\sum_j P_j Y_j \mid \sum_i \lambda_i X_{i,m} \le X_{i,m} \quad m=1,...,M;$$
$$\sum_i \lambda_i Y_{i,j} \ge Y_j \quad j=1,...,J; \quad (2)$$
$$\sum_i \lambda_i = 1,\ \lambda_i \ge 0 \quad i=1,...,I\}$$

Specifically $EE(X_i, Y_i) = \sum_j P_j Y_j / A(X_i, Y_i)$ is economic efficiency (EE) or aggregate technical and allocative efficiency. Normally, allocative efficiency (AE) is calculated by

$$AE_i(X_i, Y_i) = EE(X_i, Y_i) / TE_i^A(X_i, Y_i) \quad (3)$$

when there is no output aggregation. This allocative efficiency calculated from economic efficiency and pure technical efficiency can be defined as *whole allocative efficiency which measures the efficiency in allocating all resources among all fully disaggregated and undividable outputs*.

But, if the outputs (or inputs in input oriented DEA) are not fully disaggregated and estimated technical efficiency is biased, then the allocative efficiency calculated by Equation (3) is also biased.

We first consider a sub-vector of output which is linearly aggregated with prices as:

$$C_{i,j} = \sum_{j=1}^{\hat{j}} P_j, Y_{i,j} \quad i = 1,...I \quad \hat{j} \le J \quad (4)$$

When some outputs are aggregated using Equation (4), the output oriented technical inefficiency DEA can be expressed as:

$$\theta_{ci}^B(X_i, C_{i,\hat{j}}, Y_{i,\hat{j}+1}, ..., Y_{i,J}) \quad (5.0)$$
$$= \max\{\theta \mid \sum_i \lambda_i X_{i,m} \le X_{i,m} \quad m=1,...,M; \quad (5.1)$$
$$\sum_i \lambda_i Y_{i,j} \ge \theta Y_{i,j} \quad j = \hat{j}+1,...,J; \quad (5.2)$$
$$\sum_i \lambda_i C_{i,\hat{j}} \ge \theta C_{i,\hat{j}} \quad (5.3)$$
$$\sum_i \lambda_i = 1,\ \lambda_i \ge 0 \quad i=1,...,I\} \quad (5.4)$$

and those obtained from the same measure but if all outputs are aggregated into one output variable use Equation (4)

$$\theta_c^C(X_i, C_{i,\hat{j}}) \quad (6.0)$$
$$= \max\{\theta \mid \sum_i \lambda_i X_{i,m} \le X_{i,m} \quad m=1,...,M; \quad (6.1)$$
$$\sum_i \lambda_i C_{i,\hat{j}} \le \theta C_{i,\hat{j}} \quad (6.2)$$
$$\sum_i \lambda_i = 1,\ \lambda_i \ge 0 \quad i=1,...,I\} \quad (6.3)$$

The technical efficiency for aggregated data can be computed by $TE_{ci}^B = 1/\theta_{ci}^B$ and $TE_c^C = 1/\theta_c^C$.

According to Fare et al. [4][5], it is obvious that θ_{ci}^B and θ_c^C are biased. Therefore, the technical efficiencies computed by them are also downwardly biased because the allocative efficiencies are incorporated in the technical efficiency scores.

We start to answer the questions proposed in the **Introduction** from exploring the bias bounds of allocative efficiency. As showed by Fare et al.[5], the bias bounds of technical efficiency can be given as:

$$TE_c^C(X_i, C_i) \le TE_{ci}^B(X_i, C_{i,\hat{j}}, Y_{i,\hat{j}+1}, ..., Y_{i,J}) \le TE_i^A(X_i, Y_i) \quad (7)$$

Because normal allocative efficiency is calculated by dividing economics efficiency by technical efficiency, if economic efficiency is fixed, then we can give:

$$AE_c(X_i, C_i) \ge AE_{ci}(X_i, C_{i,\hat{j}}, Y_{i,\hat{j}+1}, ..., Y_{i,J}) \ge AE_i(X_i, Y_i) \quad (8)$$

Banker et al. [9] propose and proof that the estimated technical efficiency TE_c^C calculated using Equation (6) is identical to economic efficiency $EE(X_i, Y_i)$ calculated using Equation (2). Then, $AE_c(X_i, C_i)$ is equal to 1, because $TE_c^C(X_i, C_i) = EE(X_i, C_i)$ when all the outputs are aggregated into one variable. In addition, $AE_i(X_i, Y_i)$ is the *whole allocative efficiency* which is calculated from the pure technical efficiency $TE_i(X_i, Y_i)$.

According to the above proofed proposition, it is intuitively to know that incorporating the linearly aggregated output using Equation (4) in technical efficiency DEA will incorporate the allocative efficiency (relative to the aggregated outputs) into the technical efficiency. Here, the incorporated allocative efficiency only measures the efficiency in allocating resources among those outputs which are aggregated using Equation (4).

In other words, the estimated technical efficiencies using Equation (5) include the allocative efficiencies for the aggregated outputs in Equation (5.3).

As a result, the estimated allocative efficiency $AE_{ci}(X_i, C_{i,j}, Y_{i,j+1}, ..., Y_{i,J})$ only measures the efficiency in allocating resources among the outputs in Equation (5.2). Here, it should be noted that AE_{ci} also includes the allocative efficiency for Equation (5.3) as a whole output choice but not the individual outputs aggregated in Equation (5.3).

Consequently, the whole allocative efficiency is decomposed into two components. It is also easy to find the individual allocative efficiency $AE_{ic}(X_i,$

$C_{i,\hat{j}}, Y_{i,\hat{j}+1}, ..., Y_{i,J})$ for the aggregated outputs in Equation (5.3) by dividing the estimated technical efficiency TE^B_{ci} by pure technical efficiency TE^A_i. The relationship of these allocative efficiency components and technical efficiency can be expressed as:

$$AE_i(X_i, Y_i) = \frac{EE(X_i, C_i)}{TE^A_i(X_i, Y_i)}$$

$$= \frac{EE(X_i, C_i)}{TE^B_{ci}(X_i, C_{i,\hat{j}}, Y_{i,\hat{j}+1}, ..., Y_{i,J})} \times \frac{TE^B_{ci}(X_i, C_{i,j}, Y_{i,j+1}, ..., Y_{i,J})}{TE^A_i(X_i, Y_i)}$$

$$= AE_{ci}(X_i, C_{i,\hat{j}}, Y_{i,\hat{j}+1}, ..., Y_{i,J}) \times AE_{ic}(X_i, C_{i,\hat{j}}, Y_{i,\hat{j}+1}, ..., Y_{i,J}) \quad (9)$$

and:

$$AE_{ic}(X_i, C_{i,\hat{j}}, Y_{i,\hat{j}+1}, ..., Y_{i,J}) = TE^B_{ci}(X_i, C_{i,\hat{j}}, Y_{i,\hat{j}+1}, ..., Y_{i,J}) / TE^A_i(X_i, Y_i) \quad (10)$$

Above functions can be used in the specific application for measuring allocating resources. For example, if we focus on the allocative efficiency component for some specific outputs which we are interested in, we can aggregate all the other outputs and then calculate the allocative efficiency component which we want.

2. An Empirical Example for the Composition of Allocative Efficiency

According to the previous section, the allocative efficiency can be composed through functions (9) and (10) in the empirical analysis. This paper provides an empirical example to show how to use the above method in allocating resources for multi-product production process. Normally, when measuring the allocative efficiency for multi-output production, the 'traditional' DEA only gives one allocative efficiency score indicating the efficiency for whole production. However, with the increasing complexity of production system, the 'whole' AE can not provide enough or clear information in decision-making process for allocating resources.

For example, in the agricultural sector, since the reliance of farm households on non-farm income is an increasing phenomenon in transforming economies, the effects of off-farm job on the farms and farm households become a growing area of research. These studies should mainly concern the efficiency in allocating household resources between on-farm work and off-farm job while excluding the AE for on-farm outputs. However, the 'traditional' allocative efficiency only gives the AE for 'whole' outputs including all farm outputs and off-farm income. In this case, the method developed in this paper can give us the allocative efficiency component only for on-farm and off-farm choice. The logic routine for the above method can be explained easily as follows: The individual farmer firstly has to decide whether he will take off-farm job or not and if yes, how much time he will input in the off-farm job; Then, he will allocate the household resources for on-farm inputs for different farm products. This process, in fact, gives two stages in allocating household resources. If we want to know the allocative efficiencies in the first stage and the second stage respectively, the method in the previous section can satisfy us.

As for the industrial sector, this paper will show an empirical example in detail with a data set and the application result. The data set comes from 20 bio-chemical companies in China's Jiangsu province in 2001. The data are collected by Jiangsu Statistics Bureau for small and medium size enterprises (SMEs). The main objective of this application is to act as an example for the method developed in this paper. The data set includes two inputs and 3 outputs. The inputs are the capital input measured by 100 thousand Yuan CNY (Chinese Yuan), and labour input measured by the number of employees. The data of inputs and outputs are listed in the Table 1.

The chosen companies have the similar outputs. The main outputs can be classified into two types: One is the pesticide; the other is the animal pharmaceuticals. In addition, with the improvement of bio-chemical technologies, the pesticide can be further divided into two categories: 'traditional' chemical pesticide and bio-pesticide. Therefore, the two-stage allocation process includes: The first, allocating the resources between the pesticide and animal pharmaceuticals; the second, allocating the resources, which have been decided to be allocated in the pesticide production, between the chemical pesticide and bio-pesticide. This paper will provide the 'traditional' 'whole' allocative efficiency, allocative efficiency component for allocating all resources between the pesticide and animal pharmaceuticals, and the allocative efficiency component for allocating the first-stage-decided resources between the

Tab. 1: The data for the empirical example

	Capital	Labour	Chemical Pesticide	Bio-pesticide	Animal Pharmaceuticals
1	21	66	32	63	0
2	49	399	97	28	0
3	77	340	0	0	120
4	280	1370	171	153	30
5	57	206	44	32	0
6	9	58	0	0	21
7	237	1366	539	125	26
8	50	846	90	20	0
9	20	102	21	35	0
10	12	133	26	9	4
11	21	157	20	37	0
12	15	35	15	21	0
13	50	544	54	33	0
14	43	267	40	17	0
15	59	216	85	65	0
16	21	125	26	51	0
17	8	27	21	19	5
18	43	178	29	14	26
19	26	97	17	69	0
20	12	57	16	12	0

Data source: Jiangsu Statistics Bureau

chemical pesticide and bio-pesticide. The GAMS software (please refer to www.gams.com) is used to estimate the empirical example.

The Table 2 depicts the whole and composed allocative efficiencies, technical efficiencies and economic efficiencies for 20 companies.

In the Table 2, EE is the economic efficiency estimated using function (2) or (6). Pure TE is the 'traditional' technical efficiency estimated by function (1) using fully disaggregated outputs. Whole AE is the 'traditional' 'whole' allocative efficiency estimated by dividing EE by pure TE. TE-aggregated is the technical efficiency estimated by function (5) where some of outputs are aggregated. (Here, we aggregate the bio-pesticide and chemical pesticide.) AE for the first-stage allocation indicates the allocative efficiency measuring the efficiency in allocating all resources between the pesticide production and animal pharmaceuticals production. AE for the second-stage allocation measures the efficiency in allocating the determined resources between the chemical pesticide and bio-pesticide.

The estimated economic efficiencies change from 0.366 to 1 with the geometric mean at 0.629, indicating a relatively low economic efficiency as a whole. The pure technical efficiencies range from 0.411 to 1 with a geometric mean at 0.769. The technical efficiencies estimated from aggregated outputs range from 0.373 to 1 with the geometric mean at 0.704. Since the second-stage AE is incorporated into the aggregated technical efficiency, the aggregated TE is probably biased from pure technical efficiency. For example, the pure TE of company 19 is 1 while its aggregated TE is 0.795 which is equal to the

Tab. 2: The estimated efficiencies for the empirical example

	EE	Pure TE	Whole AE	TE-aggregated	AE for the first--stage allocation	AE for the second--stage allocation
1	1.000	1.000	1.000	1.000	1.000	1.000
2	0.726	0.853	0.852	0.741	0.980	0.869
3	0.544	1.000	0.544	1.000	0.544	1.000
4	0.513	1.000	0.513	0.632	0.811	0.632
5	0.478	0.504	0.947	0.486	0.982	0.964
6	0.430	1.000	0.430	1.000	0.430	1.000
7	1.000	1.000	1.000	1.000	1.000	1.000
8	0.629	0.776	0.811	0.642	0.980	0.827
9	0.614	0.632	0.972	0.617	0.996	0.976
10	0.646	0.865	0.746	0.682	0.947	0.789
11	0.600	0.607	0.989	0.600	1.000	0.989
12	0.652	0.749	0.869	0.702	0.928	0.937
13	0.497	0.520	0.957	0.508	0.980	0.976
14	0.366	0.411	0.890	0.373	0.983	0.906
15	0.917	0.938	0.977	0.934	0.982	0.995
16	0.811	0.811	0.999	0.811	1.000	0.999
17	1.000	1.000	1.000	1.000	1.000	1.000
18	0.472	0.705	0.669	0.622	0.758	0.882
19	0.791	1.000	0.791	0.795	0.994	0.795
20	0.464	0.543	0.853	0.492	0.943	0.905
Geomean	0.629	0.769	0.819	0.704	0.894	0.916

Data source: Jiangsu Statistics Bureau

product of pure TE and the second-stage AE. But, for the company 1, 3, 6, 7, and 17, because the second-stage AEs of them are 1, the pure TEs and the aggregated TEs are the same.

The first-stage allocative efficiencies range from 0.43 to 1 with the mean at 0.894. The second-stage allocative efficiencies range from 0.632 to 1 with the mean at 0.916. It is clear that there are different efficiencies in allocating all resources between the pesticide production and animal pharmaceuticals production and allocating determined resources between the chemical pesticide production and bio-pesticide production for each company. Except for the company 1, 7, and 17, all other companies have the different values for the 'whole' AE, the first-stage AE, and the second-stage AE. This further suggests that decomposing the 'whole' allocative efficiency for multi-product system is necessary.

There are two interesting propositions of decomposed AE components and 'whole' AE as follows:

Proposition 1, if 'whole' AE is 1 (full efficient), then all the decomposed AE components must be 1.

Proposition 2, 'whole' AE must be lower or equal to each decomposed AE component.

The proof of above propositions is simple. According to the function (9), 'whole' AE is the product of decomposed AE components. Since the AE components are all < or = to 1, the product of these AE components must be lower or equal to any of these AE components which are the multiplicand or multiplier. Therefore, if the 'whole' AE is 1, then its AE components must be 1, such as company 1, 7, and 17. If the 'whole' AE and its components are all lower than 1, then the 'whole' AE must be lower than its multiplicand and multiplier since they are all lower than 1. If the 'whole' AE as well as one of its components are lower than 1 while the other AE component is equal to 1, the 'whole' AE must be equal to the non-one component, such as company 3, 6, 11, and 16.

Conclusions

This study mainly concerns the method to decomposing allocative efficiency using the influence of data aggregation on DEA measurement. Although most of other papers before this study focused on the estimation bias in technical efficiency, we analyze the relationship between data aggregation and allocative efficiency and finally give a method for decomposing allocative efficiency for multi-output production system. An empirical example to show how to use the method in decomposing allocative efficiency for multi-product production system is also presented in the paper. In addition, some empirical situations (in both the agricultural sector and the industrial sector) are provided to tell us when we should use this method. Although this paper only provides a two-stage allocating process, the method developed here can be easily extended for the three-stage allocating process or even more complicated production system. The method in this paper and its application can provide useful information for researchers who concern the decision-making process for multi-product production systems.

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Tao Zhang
National University of Ireland, Galway
Department of Economics
(School of Public Administration
Macao Polytechnic Institute
Rua Luis Gonzaga Gomes
Macao)
Taozhang7608@hotmail.com

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ABSTRACT

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Data envelopment analysis (DEA), the non-parametric approach to measuring efficiency, was widely used in the literature as a linear programming model. Since the DEA is non-parametric linear program model, the estimated efficiency might be biased if there is data aggregation in DEA. It is proposed that both the inter-input aggregation and inter-output aggregation will make the estimated technical efficiency biased downwardly. Following some discussions on the technical efficiency bias caused by data aggregation in data envelopment analysis, this study presents the up-ward bias in the allocative efficiency caused by inter-output aggregation. However, hitherto, we have not observed any studies on considering how to utilize this character of DEA in methodology extensions for multi-output production system. Therefore, this paper originally proposes that the 'traditional' allocative efficiency can be decomposed for multi-product system. Then, the method to obtain decomposed allocative efficiency components is provided. In fact, the meaning of results from this paper is outside the DEA method, and the generality of the findings in this paper can provide useful information for researchers who concern the decision-making process in allocating resources for multi-product production system. Finally, an empirical example to show how to use the method in decomposing allocative efficiency for multi-product production system is also presented in the paper. In addition, some empirical situations (in both the agricultural sector and the industrial sector) are provided to tell us when we should use this method. Although this paper only provides a two-stage allocating process, the method developed here can be easily extended for the three-stage allocating process or even more complicated production system.

Key Words: *multi-product production systems, aggregate, DEA, technical efficiency, decomposing, allocative efficiency, economic efficiency, component.*

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