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Stiffness Reconstruction of 3D Frame by Reduced Redundant Measurements

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Abstract

The paper describes the way how the recently published method of redundant measurements is suitable for reducing the number of measured parameters. The developed method enables to determine the deformation and the stiffness of a truss that is equipped with redundant measurements during varying loading which is partially measured. The paper is focus on ability to help to provide better comfort during measurements.

Keywords: redundant measurements, frame, displacement of nodes, rotation of joint, static condensation

1. Introduction

A new promising method was published [1]. Presented method shows how to identify the structural element stiffness from the redundant measurements. The goal was to apply static non-destructive loads on framed structures [3] and measure displacements and rotations of the separate joints. From this measurement can be derived a complete set of the structural element stiffness parameters.

2. Stiffness Reconstruction of 3D Frame

The fundamental equation [2] is

$$^{g}Ku_j = f_j, \tag{1}$$

where f_j is a column vector of j-th applied load. K is a stiffness matrix of a solved structure in the global coordinate system and u_j is a column vector of the displacements and rotations of each joint for j-th applied load.

The particular members of a spatial structure can transfer axial forces and bending moment too. A network of sensors for measurement of displacements and rotations is considered. The stiffness matrix of a single member is expressed in the local coordinate system

$$^{l}K_n = k_{n1}A_n + k_{n2}B_n + k_{n3}C_n, \tag{2}$$

where k_{n1} is the unknown stiffness parameter corresponding to the tension or compression of the member, k_{n2} is the unknown stiffness parameter corresponding to the bending according to a bending moment about the neutral axis y of the local coordinate system and k_{n3} is the unknown stiffness parameter corresponding to the bending according to a bending moment

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about the neutral axis z of the local coordinate system. Schemes of the matrices A_n, B_n and C_n were previously discussed [2]. By the assumption of the linear strength theory it is possible to transform the stiffness matrix of a member in the local coordinate system into the global one [2] using the transformation matrix T

$$\begin{aligned} {}^{g}K_n &= T({}^{l}K_n)T^T, \\ {}^{g}K_n &= T(k_{n1}A_n + k_{n2}B_n + k_{n3}C_n)T^T. \end{aligned} \tag{3}$$

By adoption of this assignment, it is possible to express (assuming a one loading case) the matrix equation (1) in the form

$$\begin{aligned} &[k_{1,1}{}^{gx}K_{1,1} + k_{2,1}{}^{gx}K_{2,1} + \ldots + k_{Nprutu,1}{}^{gx}K_{Nprutu,1} + \\ &k_{1,2}{}^{gx}K_{1,2} + k_{2,2}{}^{gx}K_{2,2} + \ldots + k_{Nprutu,2}{}^{gx}K_{Nprutu,2} + \\ &k_{1,3}{}^{gx}K_{1,3} + k_{2,3}{}^{gx}K_{2,3} + \ldots + k_{Nprutu,3}{}^{gx}K_{Nprutu,3}]u = f. \end{aligned} \tag{4}$$

Where ${}^{gx}K_{n,i}$, $i = 1, 2, 3$ are modified stiffness matrices of a particular member expressed in the global coordinate system. The dimension of this matrix is $[5N_{uzlu}, 5N_{uzlu}]$. Index $i = 1$ corresponds to the matrices which represents tension or compression in the structure, $i = 2, 3$ corresponds to the matrices which represents bending moment about the neutral axis y and about the neutral axis z.

In the equation (1) are all displacements and rotations u known from the measurement. The unknown parameter is the column vector k of the axial and bending stiffness. It is necessary to reformulate the equation (4) in a different form

$$[{}^{gx}K_{1,1}u, \ldots, {}^{gx}K_{Nprutu,1}u, \ldots, {}^{gx}K_{1,3}u, \ldots, {}^{gx}K_{Nprutu,3}u] \begin{bmatrix} k_{1,1} \\ \ldots \\ k_{n,1} \\ k_{1,2} \\ \ldots \\ k_{n,2} \\ k_{1,3} \\ \ldots \\ k_{n,3} \end{bmatrix} = f. \tag{5}$$

Equation (5) can be expressed (for j load cases) to a matrix form

$$\begin{bmatrix} A_1 \\ \ldots \\ A_j \end{bmatrix} k = \begin{bmatrix} f_1 \\ \ldots \\ f_j \end{bmatrix}, \tag{6}$$

The dimensions of the separate matrices are

$$\lfloor m_{z-stavu} \cdot (5N_{uzlu} - N_{reakci}), 3N_{prutu} \rfloor \cdot \lfloor 3N_{rutu}, 1 \rfloor = \lfloor m_{z-stavu} \cdot (5N_{uzlu} - N_{reakci}), 1 \rfloor, \tag{7}$$

where $m_{z-stavu}$ is number of load cases, N_{reakci} is the number of the boundary conditions.

For determination of the unknown stiffness parameters of a particular member the number of equations, which are represented by the matrix A, must be greater or equal to number of unknown parameters, which represent a column vector of unknown stiffness k.

$$(5N_{uzlu} - N_{reakci})m_{z-stavu} \geq 3N_{prutu}. \tag{8}$$

For the frame structure (Fig. 1) is

$$\begin{aligned} (5 \cdot 50 - 3 \cdot 4)m_{z-stavu} &\geq 3 \cdot 176, \\ m_{z-stavu} &\geq 2.22. \end{aligned} \tag{9}$$

It is necessary to apply at least 3 different load cases and measure 714 parameters (displacements or rotations) of the construction.

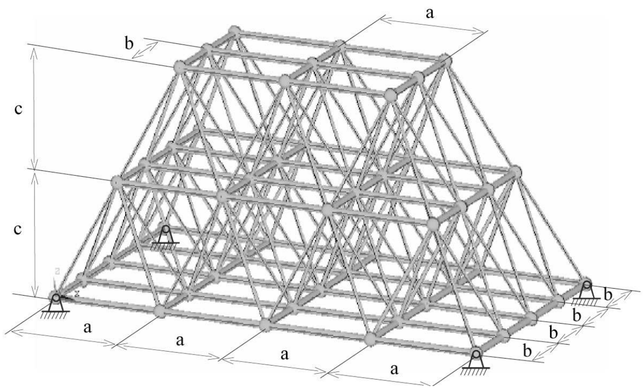

Fig. 1. Solved frame structure

3. Numerical method

System of the linear equations (6) is over-constrained. It can be solved by Singular Value Decomposition method or Least Square Method.

3.1. Singular Value Decomposition (SVD)

Singular Value Decomposition method decomposes the system matrix A into sub-matrices

$$A = USV^T, \tag{10}$$

such that

$$U^T U = E, \quad V^T V = E, \tag{11}$$

where E is an identity matrix. Using the substitution

$$k = Vy \tag{12}$$

the equation (6) can be modified by (7) and (12) to

$$\left(USV^T\right)(Vy) = f, \tag{13}$$

Multiplying equations (13) from the left hand side by U^T it is achieved

$$\left(U^T U\right) S \left(V^T V\right) y = U^T f, \tag{14}$$

Equation (14) can be formally rewritten

$$Sy = c, \tag{15}$$

where S is a diagonal matrix of singular values. Provided that SVD not necessarily leads us to all unknown stiffness parameters (e.g. singular cases), it is useful rewrite (15) into

$$\begin{bmatrix} \mathrm{diag}\,(s_i > \varepsilon) & 0 \\ 0 & \mathrm{diag}\,(s_i \leq \varepsilon) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_D \\ y_I \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \tag{16}$$

where ε is a small parameter. The computed parameters are

$$k = V_d[\mathrm{diag}\,(s_i > \varepsilon)]^{-1}c_1. \tag{17}$$

The main reason for SVD failure is a bad position of the load cases. Load case can generate a too small or even almost none forces or bending moments in some members of the structure. Thus a too small or even almost none displacement or rotation can be measured.

The main advantage of SVD method is its speed of the computation (according to least square method).

3.2. Least Square Method (LSQ)

Let us consider a bunch of inaccurate data (a_i, f_i). To perform a linear regression to this data (find the representing straight line) means to find parameters to fit equation

$$f_i = k_1 a_i + k_2. \tag{18}$$

The conditions (18) can be reformulated to a matrix form

$$Ak = f, \tag{19}$$

where

$$A = \begin{bmatrix} a_1 & 1 \\ a_2 & 1 \\ \dots & \dots \\ a_q & 1 \end{bmatrix}, \quad k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \\ \dots \\ f_q \end{bmatrix}. \tag{20}$$

The over-constrained system (19) can be solved by the Least Square Method (LSQ). The residuum of the equation (19) must be minimized for finding of the parameters k_1, k_2

$$\sum_{i=1}^{q} (f_i - k_1 a_i - k_2)^2. \tag{21}$$

Due to the physical reason must be the equation (22) valid

$$k \geq 0. \tag{22}$$

The function *lsqnonneg* from the MATLAB Optimization Toolbox was used for solution equation (21) and (22)

$$\text{find } \min t\|Ak - f\|, \quad k \geq 0. \tag{23}$$

The LSQ method is more time consuming than SVD. But in most cases it gives more precise results.

4. Reconstruction of the 3D frame

Reconstruction of unknown stiffness parameters was performed on the 3D spatial structure (See Fig. 1). The SVD method was used. It was possible to determine all stiffness parameters at 781 load case combinations composed by 3 forces (in direction of acceleration due to gravity) applied to different joints from 15180 possible combinations (Fig. 2).

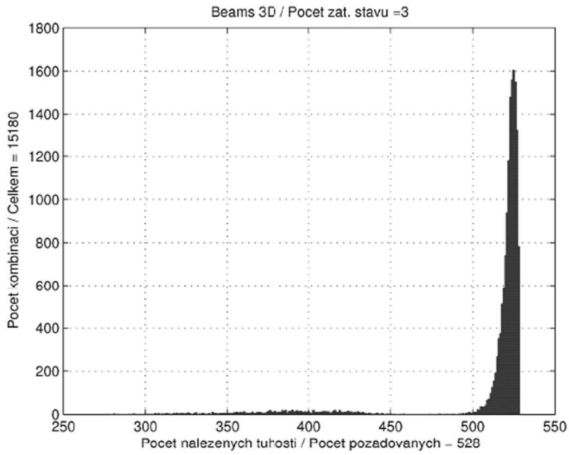

Fig. 2. Number of determined stiffness parameters

5. Reduced Redundant Measurements

Disadvantage of the presented approach is the necessity to obtain the vector of deformation in direction of particular degrees of freedom (DOFs). Thus a question rises whether the number of measuring points on the structure can be reduced. The method of static condensation has been investigated in order to reduce the number of measuring DOFs of the structure. A column vector of deformations has been divided into so called master (m) DOFs that should be preserved in the problem being solved and slave (s) DOFs that should be eliminated from the equation.

$$\begin{gathered}\left[{k_{1,1}}^{gx}\begin{bmatrix}K_{mm} & K_{ms}\\ K_{sm} & K_{ss}\end{bmatrix}_{1,1}+\ldots+k_{Nprutu,1}{}^{gx}\begin{bmatrix}K_{mm} & K_{ms}\\ K_{sm} & K_{ss}\end{bmatrix}_{Nprutu,1}\right]\begin{bmatrix}u_m\\ u_s\end{bmatrix}+\\ \left[{k_{1,2}}^{gx}\begin{bmatrix}K_{mm} & K_{ms}\\ K_{sm} & K_{ss}\end{bmatrix}_{1,2}+\ldots+k_{Nprutu,2}{}^{gx}\begin{bmatrix}K_{mm} & K_{ms}\\ K_{sm} & K_{ss}\end{bmatrix}_{Nprutu,2}\right]\begin{bmatrix}u_m\\ u_s\end{bmatrix}+\\ \left[{k_{1,3}}^{gx}\begin{bmatrix}K_{mm} & K_{ms}\\ K_{sm} & K_{ss}\end{bmatrix}_{1,3}+\ldots+k_{Nprutu,3}{}^{gx}\begin{bmatrix}K_{mm} & K_{ms}\\ K_{sm} & K_{ss}\end{bmatrix}_{Nprutu,3}\right]\begin{bmatrix}u_m\\ u_s\end{bmatrix}=\begin{bmatrix}f_m\\ 0\end{bmatrix}.\end{gathered}\tag{24}$$

From the equation (24) can be derived the system of nonlinear equations

$$f(k_{n1}, k_{n2}, k_{n3}) = 0. \tag{25}$$

From the system of nonlinear equations (25) it is possible to compute the required stiffness characteristics of the system being solved. The solution of these equations is a very time consuming due to the number of equations and number of the unknown parameters.

Therefore it was considered direct iterative evaluation of stiffness parameters. According to preparations shown at [2] it can be obtained system of linear equations

$$\begin{bmatrix} A_{1,1}u_{m1}, \ldots, A_{Nprvku,1}u_{m1}, \ldots, A_{1,1}u_{m1}, \ldots, A_{Nprvku,3}u_{m1} \\ \ldots \\ A_{1,1}u_{mn}, \ldots, A_{Nprvku,1}u_{mn}, \ldots, A_{1,1}u_{mn}, \ldots, A_{Nprvku,3}u_{mn} \end{bmatrix} \begin{bmatrix} k_{1,1} \\ \ldots \\ k_{Nprvku,1} \\ k_{1,2} \\ \ldots \\ k_{Nprvku,2} \\ k_{1,3} \\ \ldots \\ k_{Nprvku,3} \end{bmatrix} = \begin{bmatrix} f_{m1} \\ \ldots \\ f_{mm} \end{bmatrix}, \quad (26)$$

where is assumed $1, \ldots, n$ load cases. The term A can be written

$$A_{i,j} = \left({}^{gx}K_{mm}\right)_{i,j} + \left({}^{gx}K_{ms}\right)_{i,j} Z_{sm}. \quad (27)$$

Matrix Z_{sm} deals with supposed relations between the stiffness parameters

$$u_s = Z_{sm}u_m. \quad (28)$$

This method was used in next simulations.

5.1. Elimination of inner nodes

The possibility of elimination of inner nodes has been investigated. It can be supposed the structure has e.g. some sort of cover and thus we can obtain its geometry from its documentation.

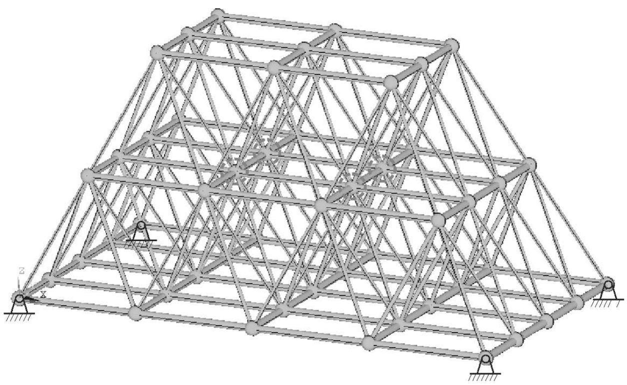

Fig. 3. Inner nodes

But we cannot perform even nondestructive static load to these inner nodes and of course measurements of displacements or rotation. Inner nodes are shown at Fig. 3.

It was used SVD method to obtain results for all possible load cases. This method was chosen mainly for its speed and due to a large number of combinations that was tested. Only one inner node was eliminated for each inner node was found only one load case combination which leads to solution with error rate under 1 %[1]. Results are shown in Tab. 1. Next the finding

[1] As we use the known construction with known parameters the results can be easily verified (using the strength theory). And there is no need for a measurement at this time.

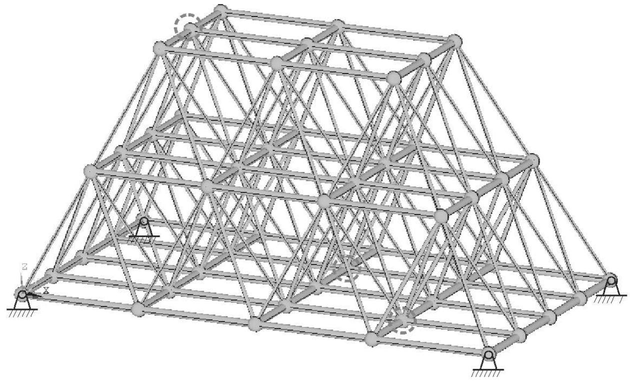

Fig. 4. First advantageous combination — load case A

of the most advantageous load cases (it means three different loads in direction of acceleration due to gravity (Fig. 4) were tested. In tests where 2 or more inner nodes were eliminated (maximum 4) SVD method for two most advantageous load cases combination failed.

On the other hand, with usage of LSQ we were able to eliminate all inner nodes using previously gained load cases combinations. If we eliminate two neighbor nodes it is not possible (using previously described reduction method) to compute unknown stiffness parameters of member between these nodes. All other unknown stiffness parameters were successfully solved.

Table 1. Report of elimination of inned nodes

Number of eliminated inner nodes	**Method**	**Load cases**	**Max error rate for unknown parameters**
1	SVD, LSQ	load case A	0.,002 %[1]
1	SVD, LSQ	load case B	0.002 %[1]
4	SVD	load case A load case B	solution not found
4	LSQ	load case A load case B	0.002 %[1] for solved parameters (not all parameters solved)

5.2. Saving costs for redundant measurements

Another approach is elimination as much DOFs as possible to decrease amount of necessary measurements. At the same time it is necessary to solve all unknown stiffness parameters. For system (6) and for bigger spatial structure it is necessary to satisfy (9). If we assume 3 load cases it is necessary to perform 714 measurements of displacements or rotations. But there is only 528 unknown parameters. If all equations were linearly independent, it could be possible to measure only 528 DOFs.

It has been shown than it is possible to reduce number of meassurement from 714 to 582 and still solve all unknown stiffness parameters. The goal was to eliminate as much rotations measurements as possible. We eliminate in chosen joints both rotation DOFs (Fig. 5).

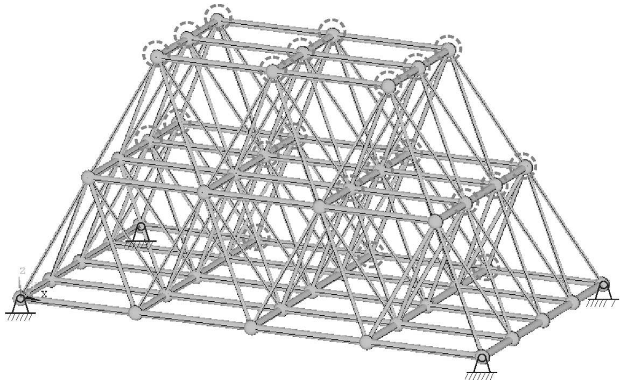

Fig. 5. Eliminate as much rotations measurements as possible

6. Conclusion

The previously presented method [2] was validated towards new construction. There was shown Least Square Method — a way to solve final equations. This method gives in most cases better results than previously used Singular Value Decomposition. Presented way to eliminate inner nodes should be further inspect to be able to keep count of all unknown stiffness parameters.

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