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Sensitivity analysis for the optimal perforation problem in acoustic transmission

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Abstract

The paper deals with the acoustic transmission through perforated interface and its sensitivity w.r.t. the perforation design. The homogenized transmission conditions are imposed on an interface plane separating two halfspaces occupied by the acoustic medium. The conditions were obtained recently as the two-scale homogenization limit of the standard acoustic problem imposed in the layer perforated by a sieve-like obstacle with periodic structure. The limit model involves some homogenized impedance coefficients depending on the so-called microscopic problems; these are imposed in the reference computational cell, Y embedding obstacle S the shape of which can be designed. This homogenization approach allows for an efficient treatment of complicated perforation designs of perforations. Acoustic response to the global acoustic problem involving the transmission conditions is subject to the sensitivity analysis. Namely the total variation of an objective function depending on the acoustic pressure w.r.t. shape of S at the "microlevel" is derived.

Keywords: linear acoustics, homogenization, transmission condition, sensitivity analysis

1. Introduction

Minimization of noise produced by flowing acoustic medium (inviscid compressible fluid) belongs to important challenges of the aerospace and automotive engineering. For example, in the exhaust silencers of the combustion engines the gas flows through ducts equipped with various sieve-like structures which in part may influence the transmission losses associated with acoustic waves propagating in the exhaust gas. Apart from optimization of the exhaust silencers, obviously there are other devices involving sieve-like structures for which the acoustic transmission is an important figure to look at.

In the paper we deal with the *optimal* acoustic transmission through perforated interface. In [10], using the asymptotic analysis we developed the homogenized transmission conditions to be imposed on an interface plane representing the periodic perforation which in reality is designed by obstacles having possibly complicated shapes.

We consider the acoustic medium occupying domain Ω which is subdivided by perforated plane Γ_0 in two disjoint subdomains Ω^+ and Ω^-, so that $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma_0$, see fig. 1. In a case of no convection flow (the linear acoustics), the acoustic waves in Ω are described by the

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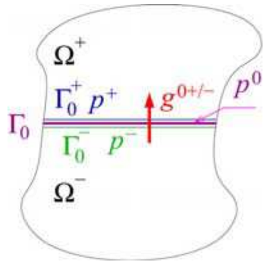

Fig. 1. Illustration of the transmission coupling — the acoustic pressure jump is proportional to the transverse acoustic velocity g^0

following equations

$$\begin{aligned} & c^2\nabla^2 p + \omega^2 p = 0 \quad \text{in } \Omega^+ \cup \Omega^-\,, \\ \text{transmission conditions} \quad & \left\{ \begin{aligned} c^2 \frac{\partial p}{\partial n^+} &= \mathrm{i}\omega g_0 \\ c^2 \frac{\partial p}{\partial n^-} &= -\mathrm{i}\omega g_0 \end{aligned} \right. \qquad \text{on } \Gamma_0\,, \\ \text{boundary conditions} \quad & \text{on } \partial\Omega\,. \end{aligned} \tag{1}$$

where $\frac{\partial p}{\partial n^\pm} = n^\pm \cdot \nabla p$ are the normal derivatives on Γ_0 w.r.t. normals outward to Ω^+ and Ω^-, respectively. The transmission conditions on interface Γ_0 involve the transversal acoustic velocity g_0 (up to the factor of the wave number squared); this variable satisfies additional integral identities the were developed in [10] using the asymptotic analysis performed in the δ-interface layer, see Fig. 2.

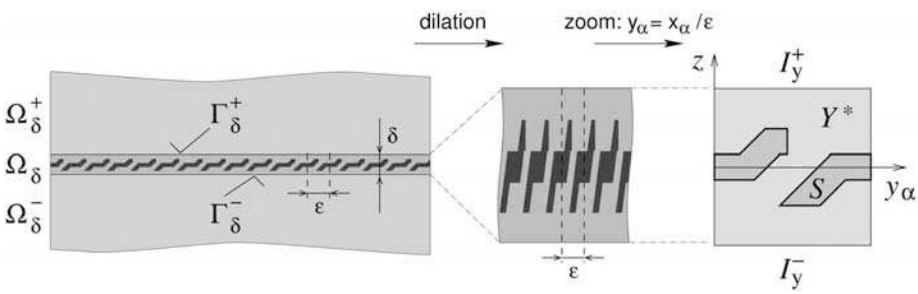

Fig. 2. The perforated interface layer, Ω_δ embedded in Ω; illustration of the thickness *dilatation* $z = x_3/\delta$ and the *periodic unfolding* related to rescaling $y_\alpha = x_\alpha/\varepsilon$, $\alpha = 1, 2$ for $\delta = \varkappa\varepsilon$. In the reference periodic cell, the perforation geometry is represented by $Y^* \subset Y$

2. Homogenized interface conditions on perforated layer

In this section we record the homogenized model of the perforated transmission layer, see [10]. This serves the transmission conditions closing the boundary value problem (1).

The homogenized coefficients governing the acoustic transmission are introduced below using so called corrector functions defined in the reference periodic cell $Y =]0, 1[^2 \times]-1/2, +1/2[$,

$Y \subset \mathbb{R}^3$. The acoustic medium occupies domain $Y^* = Y \setminus S$, where $S \subset Y$ is the solid (rigid) obstacle. For clarity we use notation $I_y =]0,1[^2$. The upper and lower boundaries are translations of $(I_y, 0)$; we define $I_y^+ = \{y \in \partial Y : \ z = 1/2\}$ and $I_y^- = \{y \in \partial Y : \ z = -1/2\}$. By $H^1_{\#(1,2)}(Y)$ we denote the space of $H^1(Y)$ functions which are "1-periodic" in coordinates y_α, $\alpha = 1,2$; such functions will be called "transversely Y-periodic".

2.1. *Limit macroscopic equations of the transmission layer*

The homogenized transmission conditions is expressed in terms of *interface mean acoustic pressure* $p^0 \in H^1(\Gamma_0)$, and *fictitious acoustic transverse velocity* $g^0 \in L^2(\Gamma_0)$; these quantities satisfy the interface problem constituted by two integral identities

$$
\begin{aligned}
\int_{\Gamma_0} A_{\alpha\beta} \partial^x_\beta p^0 \partial^x_\alpha q - \frac{|Y^*|}{|Y|}\omega^2 \int_{\Gamma_0} p^0 q + \mathrm{i}\omega \int_{\Gamma_0} B_\alpha \partial^x_\alpha q \, g^0 = 0 \,, & \\
-\mathrm{i}\omega \int_{\Gamma_0} D_\beta \partial^x_\beta p^0 \psi + \omega^2 \int_{\Gamma_0} F g^0 \psi = -\mathrm{i}\omega \frac{1}{\varepsilon_0} \int_{\Gamma_0} (p^+ - p^-)\psi \,, &
\end{aligned}
\tag{2}
$$

to hold for all $q \in H^1(\Gamma_0)$ and $\psi \in L^2(\Gamma_0)$. These equations involve the homogenized coefficients $A_{\alpha\beta}, B_\alpha, D_\alpha$ and F expressed in terms of the local corrector functions π^β and ξ defined in Y^*, the solutions of the microscopic auxiliary problems introduced below.

We remark that p^0 presents an internal variable describing the acoustic wave distributed in the interface layer, being driven by g^0; this phenomenon is featured by $\partial_\alpha p^0 \neq 0$ and it appears only if the coupling coefficients do not vanish, i.e. $B_\beta, D_\beta \neq 0$. For the discretized form (using FEM) one can introduce an effective *non-local acoustic impedance* which relates the pressure jump $p^+ - p^-$ on Γ_0 to the transverse velocity represented by g^0, see also Fig. 3.

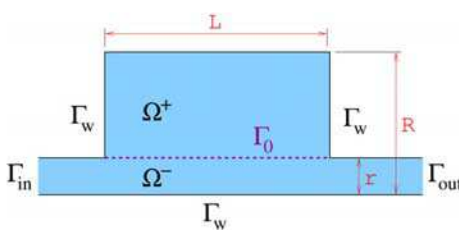

Fig. 3. The domain and boundary decomposition of the global acoustic problem considered. This layout is inspired by [4]

2.2. *Microscopic auxiliary problems and homogenized coefficients*

In order to compute the homogenized coefficients involved in (2) the following local problems must be solved ($\varkappa$ is the scaling parameter determining the ratio "thickness/period length"):

- (Corrector of the *tangent interface velocity* $v_t \approx \partial_\alpha p^0$) Find $\pi^\beta \in H^1_{\#(1,2)}(Y)$, $\beta = 1,2$, such that

$$
\int_{Y^*} \left[\partial^y_\alpha \pi^\beta \, \partial^y_\alpha q + \frac{1}{\varkappa^2} \partial_z \pi^\beta \partial_z q \right] = - \int_{Y^*} \partial^y_\beta q \qquad \forall q \in H^1_{\#(1,2)}(Y)
$$

- (Corrector of the *normal interface velocity* $v_n \approx g^{0\pm}$) Find $\xi^\pm \in H^1_{\#(1,2)}(Y)/\mathbb{R}$, such that

$$\int_{Y^*} \left[\partial^y_\alpha \xi^\pm \, \partial^y_\alpha q + \frac{1}{\varkappa^2} \partial_z \xi^\pm \partial_z q\right] = -\frac{|Y|}{c^2 \varkappa} \left(\int_{I_y^+} q \, dS_y - \int_{I_y^-} q \, dS_y \right) ,$$

for all $q \in H^1_{\#(1,2)}(Y)/\mathbb{R}$

The homogenized coefficients are now defined as follows:

- Tangent acoustic diffusion coefficients

$$A_{\alpha\beta} = \frac{c^2}{|Y|} \int_{Y^*} \partial^y_\gamma (y^\beta + \pi^\beta) \, \partial^y_\gamma (y^\alpha + \pi^\alpha) + \frac{c^2}{\varkappa^2 |Y|} \int_{Y^*} \partial_z \pi^\beta \partial_z \pi^\alpha .$$

- Coefficients of transversal-to-tangent coupling of velocity

$$\begin{aligned} B_\alpha &= \frac{c^2}{|Y|} \int_{Y^*} \partial^y_\alpha \xi^\pm , \\ \varkappa B_\alpha = D_\alpha &= \frac{1}{|I_y|} \left(\int_{I_y^+} \pi^\alpha \, dS_y - \int_{I_y^-} \pi^\alpha \, dS_y \right) , \end{aligned}$$

- Local transversal impedance

$$F = \frac{1}{|I_y|} \left(\int_{I_y^+} \xi^\pm \, dS_y - \int_{I_y^-} \xi^\pm \, dS_y \right) .$$

2.3. Acoustic problem in duct with transmission condition

As explained above, in domains with a perforated obstacle Γ_0 the acoustic pressure is discontinuous along Γ_0, which in general can be a fissure embedded in a connected domain Ω. For this we need $H^1_{-1}(\Omega, \Gamma_0)$, the space of discontinuous solutions defined at once in the whole of Ω: $H^1_{-1}(\Omega, \Gamma_0) = \{q \in L^2(\Omega) \;:\; q|_{\Omega^r} \in H^1(\Omega^r),\; r = +, -\}$. By q^+ and q^- we denote traces on Γ_0 of $q \in H^1(\Omega^+)$ and $q \in H^1(\Omega^-)$, respectively. Thus, in what follows by p we denote the solution in $\Omega \subset \Gamma_0$, whereas on Γ_0 the pressure is introduced by traces p^+ and p^- of $p \in H^1(\Omega^+)$ and $p \in H^1(\Omega^-)$, respectively; these traces are involved in the interface problem (2).

We also need to specify boundary conditions on boundary $\partial\Omega = \Gamma_{\rm in} \cup \Gamma_{\rm out} \cup \Gamma_{\rm w}$ consisting of the planar surfaces $\Gamma_{\rm in}$, $\Gamma_{\rm out}$ and the channel walls $\Gamma_{\rm w}$, see Fig. 3. On $\Gamma_{\rm in}$ we assume an incident wave of the form $\tilde{p}(x,t) = \bar{p} e^{-{\rm i} k n_l \cdot x_l} e^{{\rm i}\omega t}$, where (n_l) is the outward normal vector of Ω, on $\Gamma_{\rm out}$ we impose the radiation condition in the form of the anechoic output, so that

$$\begin{aligned} {\rm i}\omega p + c\frac{\partial p}{\partial n} &= 2{\rm i}\omega \bar{p} \quad \text{on } \Gamma_{\rm in} , \\ {\rm i}\omega p + c\frac{\partial p}{\partial n} &= 0 \quad \text{on } \Gamma_{\rm out} , \\ \frac{\partial p}{\partial n} &= 0 \quad \text{on } \Gamma_{\rm w} . \end{aligned} \tag{3}$$

The boundary value problem (1) with $(1)_3$ specified by conditions (3) can be formulated weakly as follows. Given amplitude $\bar{p}$ of incident plane wave with frequency ω, the weak solution $p \in H^1_{-1}(\Omega, \Gamma_0)$ to our acoustic problem is obtained by

$$
\begin{aligned}
c^2 \int_\Omega \nabla p \cdot \nabla q - \omega^2 \int_\Omega pq + \mathrm{i}\omega c \int_{\Gamma_{\rm in} \cup \Gamma_{\rm out}} pq \, d\Gamma & \\
- \int_{\Gamma_0^+} g^0 q^+ \, d\Gamma + \int_{\Gamma_0^-} g^0 q^- \, d\Gamma = \mathrm{i}2\omega c \int_{\Gamma_{\rm in}} \bar{p} q \, d\Gamma \quad \forall q \in H^1_{-1}(\Omega, \Gamma_0) \,, &
\end{aligned}
\tag{4}
$$

where $q^{+/-}$ are the traces on Γ_0 and g^0 is the solution of interface problem (2).

3. Formulation of the optimal perforation design problem

In this section we shall formulate problem of optimal shape of the periodic perforations targeted to maximize the transmission loss measured in an acoustic device which is equipped with the perforated interface.

3.1. *State problem hierarchical formulation*

We shall first summarize the structure of the state problem describing acoustic waves in a duct $\Omega \subset \mathbb{R}^3$ wherein the perforation represented by interface Γ_0 is placed; we adhere the same decomposition as introduced earlier; namely we may consider the following placement of the flat (homogenized) perforation:

$$
\begin{aligned}
\Gamma_0 &= \{x \in \Omega | \; x_3 = 0\} \,, \\
\Omega^+ &= \{x \in \Omega | \; x_3 > 0\} \,, \\
\Omega^- &= \{x \in \Omega | \; x_3 < 0\} \,.
\end{aligned}
\tag{5}
$$

The state problem has a hierarchical structure incorporating *3 levels*, as will be recognized when developing the sensitivity analysis.

Let p be the acoustic pressure in $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma_0$ and p^+, p^- be traces of $p|_+$, $p|_-$ on Γ_0, respectively, where $p|_\pm$ are restrictions of p on $\Omega^\pm$. The *level 1 state problem* is to find $p \in H^1_{-1}(\Omega, \Gamma_0)$ such that (by virtue of (4) we employ a self-explaining notation)

$$
a_\Omega \left(p, \, q\right) - \omega^2 \left(p, \, q\right)_\Omega + \omega c \left\langle p, \, q\right\rangle_{\Gamma_{\rm in-out}} - \mathrm{i}\omega \left\langle g^0, \, q^+\right\rangle_{\Gamma_0} + \mathrm{i}\omega \left\langle g^0, \, q^-\right\rangle_{\Gamma_0} = 2\mathrm{i}\omega c \left\langle \bar{p}, \, q\right\rangle_{\Gamma_{\rm in}}
\tag{6}
$$

for all $q \in H^1_{-1}(\Omega, \Gamma_0)$, where $g^0 \in L^2(\Gamma_0)$ satisfies the interface conditions represented by the two homogenized equations (2); the *level 2 state problem* related to (2) is to find $g^0 \in L^2(\Gamma_0)$ and $p^0 \in H^1(\Gamma_0)$ (an internal variable) such that

$$
\begin{aligned}
\mathcal{A}(p^0, \phi) - \omega^2 \varsigma^* \left\langle p^0, \, \phi\right\rangle_{\Gamma_0} + \mathrm{i}\omega \mathcal{B}(g^0, \phi) = 0 \,, \quad \forall \phi \in H^1(\Gamma_0) \,, & \\
-\mathrm{i}\omega \varkappa_0 \mathcal{B}(\psi, p^0) + \omega^2 \mathcal{F}(g^0, \psi) = -\mathrm{i}\omega \frac{1}{\varepsilon^0} \left\langle p^+ - p^-, \, \psi\right\rangle_{\Gamma_0} \,, \quad \forall \psi \in L^2(\Gamma_0) \,, &
\end{aligned}
\tag{7}
$$

where $\varsigma^* = |Y^*|/|Y|$ and $\varkappa_0 = \varkappa/|I_y|$; the bilinear forms involved in (7) are defined in terms of the homogenized coefficients (recall $\alpha = 1, 2$ for 3D problems):

$$
\begin{aligned}
\mathcal{A}(p, q) &= \int_{\Gamma_0} A_{\alpha\beta} \partial_\beta p \partial_\alpha q \, d\Gamma, \\
\mathcal{B}(g, q) &= \int_{\Gamma_0} B_\alpha g \partial_\alpha q \, d\Gamma, \\
\mathcal{F}(g, h) &= \int_{\Gamma_0} F g h \, d\Gamma.
\end{aligned} \tag{8}
$$

The homogenized coefficients, A, B, F and, thereby, the bilinear forms $\mathcal{A}, \mathcal{B}, \mathcal{F}$ are determined by the solution of the *level 3 state problem* constituted by the local corrector problems. To simplify the notation, we introduce

$$
\begin{aligned}
\hat{\nabla} q &= (\partial_\alpha^y q, \varkappa^{-1} \partial_z q), \\
a_Y^* (\pi, \xi) &= \int_{Y^*} \hat{\nabla} \pi \cdot \hat{\nabla} \xi = \int_{Y^*} \left(\partial_\alpha^y \pi \partial_\alpha^y \xi + \frac{1}{\varkappa^2} \partial_z \pi \partial_z \xi \right) , \\
\gamma^\pm(\xi) &= \int_{I_y^+} \xi - \int_{I_y^-} \xi
\end{aligned} \tag{9}
$$

and rewrite the local corrector problems as follows: Find $\pi^\beta, \xi \in H^1_{\#(1,2)}(Y)/\mathbb{R}$ such that

$$
\begin{aligned}
a_Y^* \left(\pi^\beta + y_\beta, \, \phi \right) &= 0 \, , \quad \forall \phi \in H^1_{\#(1,2)}(Y), \quad \beta = 1, 2 \, , \\
a_Y^* \left(\xi, \, \phi \right) &= - \frac{|Y|}{\varkappa c^2} \gamma^\pm(\phi) \, , \quad \forall \phi \in H^1_{\#(1,2)}(Y).
\end{aligned} \tag{10}
$$

Using the notation just introduced, the homogenized coefficients can be expressed, as follows:

$$
\begin{aligned}
A_{\alpha\beta} &= \frac{c^2}{|Y|} a_Y^* \left(\pi^\beta + y^\beta, \, \pi^\alpha + y^\alpha \right) \, , \\
B_\alpha &= \frac{c^2}{|Y|} a_Y^* \left(\xi, \, y_\alpha \right) \, , \\
F &= \frac{1}{|I_y|} \gamma^\pm(\xi) \, .
\end{aligned} \tag{11}
$$

3.2. Optimal perforation problem

We now consider an objective function $\Phi(p)$, e.g. expressing the transmission loss evaluated using two pressures p^a, p^b,

$$
\Phi(p) = \hat{\Phi}(p^a, p^b) = 20 \log \left(\frac{|p^a|}{|p^b|} \right) , \quad p^a = p(x = x^a), \quad p^b = p(x = x^b), \quad x^a, x^b \in \Omega \, , \tag{12}
$$

where p satisfies the *state problem*, as represented by (6). In the 3D case, the shape parameterization of ∂S (the boundary of the obstacle placed in Y) is introduced through a one-to-one mapping (diffeomorphism) $\Sigma(\alpha, \cdot) : \mathbb{R}^2 \to \mathbb{R}^3$, which for a given design variable $\alpha \in \mathbb{R}^{N_{\mathrm{dv}}}$ associates a reference placement $t \in T \subset \mathbb{R}^2$ with a corresponding spatial position y on the manifold ∂S, i.e. $\partial S \ni y = \Sigma(\alpha, t)$, $t \in T$.

In fact $\Sigma(\alpha)$ shapes domain Y^* where the microscopic auxiliary problems (10) are posed to computed corrector functions π^β, ξ; these determine homogenized coefficients A, B, F involved in transmission conditions (7), being coupled with (6) in terms of transversal "velocity flux" g^0 and the pressure discontinuity $p^+ - p^-$.

We can now define the *optimal perforation design problem*:

$$
\begin{aligned}
&\min_{\alpha \in D_{adm}} \Phi(p) \\
\text{subject to:}\quad & p \text{ solves (6)–(7),} \\
& \quad \text{where } A, B, F \text{ are given by (10)–(11).}
\end{aligned}
\tag{13}
$$

D_{adm} is the set of admissible designs; besides shape smoothness requirements it should reflect some constraints concerning the size of the obstacle (thickness) and porosity of the interface.

Due to the *hierarchical structure* of the state problem, the homogenized coefficients can be viewed as "intermediate" optimization parameters. Then the optimal perforation problem splits into the following two:

- *Shape Optimization*: using (micro) geometry of perforation represented by α, optimize homogenized transmission coefficients, A, B, F on Γ_0,
- *Material Optimization*: using A, B, F on Γ_0 optimize acoustic pressure in Ω.

4. Obstacle shape parameterization and shape derivative

By virtue of the hierarchical setting of the problem, we are interested in the shape sensitivity of the microscopic response described by the corrector functions, π^β, ξ on the *perforation design*, as represented by ∂S.

4.1. Design velocity field

In Section 4.2, in the standard way, we consider a "flux" of material points which is given in terms of a vectorial (design velocity) field $\vec{\mathcal{V}}(y)$, $y \in Y$ so that for $y \in \partial S$ it describes the "flux" of points on the design boundary. Such velocity field, in general, must be differentiable w.r.t. y and must vanish on that part of the boundary of the optimized structure which is not subject of the design modification (so-called "fixed boundary"); in our case all exterior boundary ∂Y is fixed. A possible construction of $\vec{\mathcal{V}} : \overline{Y} \longrightarrow \mathbb{R}^3$ (or $\mathbb{R}^2$ in the 2D situation) is performed by following steps:

- use a finite set of the *design variables* $\{\alpha^k\}$, $k = 1, \ldots, N_{\text{dv}}$ which shape the design boundary ∂S; in this way we introduce the mapping $\Sigma(\alpha, T) \rightarrow \partial S$;
- consider a *design perturbation* $\delta\alpha$ which modifies the design boundary
$$\{\delta\alpha^k\} \rightarrow \delta(\partial S) \equiv \{\vec{\mathcal{V}}(y)\},\ y \in \partial S\,; \tag{14}$$
- compute $\vec{\mathcal{V}} = (\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3)$ as a solution of the *auxiliary Dirichlet boundary value problems* for an elastic medium occupying domain Y^*: in our case
$$
\begin{aligned}
\partial_j^y \sigma_{ij}(\vec{\mathcal{V}}) &= 0 \quad \text{in } Y^*\,, \\
\vec{\mathcal{V}} &= 0 \quad \text{on } I_y^\pm \\
n_i \mathcal{V}_i &= 0 \quad \text{on } \partial Y^* \setminus (\partial S \cup I_y^\pm) \\
\vec{\mathcal{V}} &= \delta(\partial S) \quad \text{on } \partial S\,,
\end{aligned}
\tag{15}
$$
which gives $\vec{\mathcal{V}}$ in Y^*. We shall consider extension of $\vec{\mathcal{V}}$ by *zero* over $Y \setminus \overline{Y^*}$.

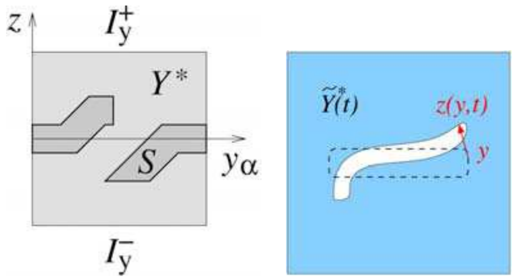

Fig. 4. Left: design velocity field is supported in Y^*, boundary ∂S is shaped by parameters $\{\alpha\}$, Right: Domain perturbation using (17); parameter t corresponds to τ used in the text

Above we consider $\sigma_{ij} = c_{ijkl}e_{ij}^y(\vec{\mathcal{V}})$ with arbitrary elasticity c_{ijkl}. It is worth noting, that by means of an elasticity defined inhomogeneously in Y^*, this machinery allows for controlling the flux of finite element mesh in the design (optimization) process.

The design variables $\alpha = \{\alpha^k\}$, $k = 1, \ldots, N_{\mathrm{dv}}$ influence the design boundary in terms of the shape functions which satisfy certain regularity conditions with respect to the curve parameterization $T \ni t \to \partial S$ (we adhere the 2D situation). As an example of the *design parameterization*, we may consider a given set of shape functions $\{w_i^k(t)\}_k$, $t \in T$, $i = 1, 2$, $k = 1, \ldots, N_{\mathrm{dv}}$; these guarantee regularity of $\partial S(\alpha)$ (but also restrict its variability) by virtue of the following definition

$$\begin{aligned} \partial S(\alpha) &= \{y(t,\alpha) \mid t \in T\,,\ \alpha \in D_{adm}\}\,, \\ \text{where } y_i(t,\alpha) &= \bar{y}_i(t) + \sum_{k=1}^{N_{\mathrm{dv}}} \alpha^k w_i^k(t)\,,\ i = 1,2 \\ \{\bar{y}(t)\}_{t\in T} &= \partial S^0\,. \end{aligned} \tag{16}$$

Above D_{adm} is a given set of admissible design parameters and ∂S^0 is some reference (initial) design attained for $\alpha = 0$. Obviously, the shape functions must be chosen so that for any fixed $\alpha \in D_{adm}$ we have a one-to-one mapping $T \ni t \to y(t,\alpha) \in Y$.

4.2. Elements of material and shape derivatives

We are interested in variation of the shape of the obstacle S placed in the domain, Y, thereby in variation of $Y^* \subset Y$. On introducing the velocity field $\vec{\mathcal{V}}$ in Y, as suggested in the previous section, see (14)–(15), we parameterize the material points constituting the domain Y by

$$z_i(y,\tau) = y_i + \tau\mathcal{V}_i(y)\,, \quad y \in Y\,, \quad i = 1, 2\,, \tag{17}$$

where τ is the "time-like" variable, see Fig. 4; for all details on the concept of shape and material derivatives we refer to [6] and [5]. Throughout the text below we shall use the notion of the following derivatives:

$\delta(\cdot)$	...	total (*material*) derivative
$\delta_\tau(\cdot)$	...	partial (*local*) derivative *w.r.t.* τ .

The derivatives just introduced are computed as the directional derivatives in the direction of $\vec{\mathcal{V}}(y)$, $y \in Y$; for reader's convenience we recall the definitions of both the material and local derivatives, as considered e.g. in [5]. Let $f(y)$ be a smooth function, e.g. $f \in C^1(Z)$, where $Z \supset Y$ is such that for τ small enough $z_i(y,\tau) \in Z$ for any $y \in Y$. We assume that f depends on the actual shape of Y which is perturbed by the velocity field $\vec{\mathcal{V}}$, as introduced in (17). Therefore, by $\tilde{f}(z,\tau)$ we denote the function value evaluated at $z = z(\tau)$ and associated with the perturbed design $\tilde{Y}(\tau) = \{z | \; z(y,\tau) = y + \tau\mathcal{V}(y)\,,\; y \in Y\}$. Due to mapping (17) one can trace the "motion" of a selected material point. The *material derivative* reflects the change of the function value in the material point which is convected with velocity $\mathcal{V}$:

$$
\begin{aligned}
\delta f(y) \circ \mathcal{V} &\equiv \lim_{\tau \to 0_+} \frac{\tilde{f}(z(y,\tau),\tau) - f(y)}{\tau} \\
&= \lim_{\tau \to 0_+} \frac{\tilde{f}(z(y,\tau),\tau) - \tilde{f}(y,\tau)}{\tau} + \lim_{\tau \to 0_+} \frac{\tilde{f}(y,\tau) - f(y)}{\tau} \\
&= \delta_\tau f(y) \circ \mathcal{V} + \nabla f(y) \cdot \mathcal{V}(y)\,,
\end{aligned}
\tag{18}
$$

where the *partial derivative* is defined by

$$
\delta_\tau f(y) \circ \mathcal{V} = \lim_{\tau \to 0_+} \frac{\tilde{f}(y,\tau) - f(y)}{\tau}\,,
\tag{19}
$$

so that it corresponds to the local change in f evaluated at fixed position $y \in Y$. Whenever a particular function of interest $\tilde{h}(z,\tau)$ can be expressed *explicitly* in the form $\tilde{h}(z(y,\tau),\tau) = h(y,\tau)$, the *shape derivative* makes sense, so that $\delta_\tau h = \delta h$ holds. Therefore, any function depending "directly" on the design modification (17) is differentiated using (19), however, typically the solutions to the problems formulated on domains subject to design modifications are differentiated in the sense of (18); such treatment is naturally pursued when the finite element solution is considered (values defined at mesh nodes) and finite difference calculation is applied to approximate the sensitivity of the solution w.r.t. a particular design change.

In Section 5.2 below, we shall use extensively the following formulae, which are easy to verify (note $J(z(y,\tau)) = \det[\partial z_i(y,\tau)/\partial y_j]$)

$$
\begin{aligned}
\delta_\tau \left(\frac{\partial z_i}{\partial y_j}\right) \circ \mathcal{V} &= \frac{\mathrm{d}}{\mathrm{d}\,\tau} \left(\frac{\partial z_i(y,\tau)}{\partial y_j}\right)_{\tau=0} = \frac{\partial \mathcal{V}_i(y)}{\partial y_j}\,, \\
\delta_\tau \left(\frac{\partial y_k}{\partial z_j}\right) \circ \mathcal{V} &= \frac{\mathrm{d}}{\mathrm{d}\,\tau} \left(\frac{\partial y_k}{\partial z_j(y,\tau)}\right)_{\tau=0} = -\frac{\partial \mathcal{V}_k(y)}{\partial y_j}\,, \\
\delta_\tau \left(J(z)\right) \circ \mathcal{V} &= \frac{\mathrm{d}}{\mathrm{d}\,\tau} \left(J(z(y,\tau))\right)_{\tau=0} = \frac{\partial \mathcal{V}_i(y)}{\partial y_i} = \mathrm{div}\vec{\mathcal{V}}\,,
\end{aligned}
\tag{20}
$$

where by $\frac{\mathrm{d}}{\mathrm{d}\,\tau}$ we mean the partial derivative w.r.t. τ. For completeness,

$$
\delta_\tau |Y^*| \circ \mathcal{V} = \delta_\tau \left(\int_{Y^*} dy\right) \circ \mathcal{V} = \int_{Y^*} \mathrm{div}\vec{\mathcal{V}}\, dy = \int_{\partial Y^*} \vec{\mathcal{V}} \cdot \vec{n} d\Gamma_y\,,
\tag{21}
$$

where we recall $\vec{\mathcal{V}} \cdot \vec{n} = 0$ on $\partial Y^* \cap \partial Y$.

5. Sensitivity analysis for uniformly designed perforation

We shall now develop the sensitivity of functional Φ which depends on the perforation design through the hierarchy of the state sub-problems declared in definition (13). Assuming that the perforation is uniform on entire Γ_0, there is only one set of homogenized coefficients A, B, F (i.e. they are not functions of $x \in \Gamma_0$). In order to derive the sensitivity formulae, we proceed in the following steps:

1. we define the Lagrangian of problem (13) respecting constraints (6)–(7) only;
2. using the adjoint problem technique, we derive sensitivity of Φ w.r.t. homogenized coefficients A, B, F;
3. we derive the sensitivity of A, B, F w.r.t. the perforation design represented by mapping Σ dependent on design variables α.

In what follows, by expression $\delta_\phi f(\phi, \dots) \circ \delta\phi$ we mean the (partial) Gateaux differential of f w.r.t. ϕ, so that $\delta_\phi f(\phi, \dots)$ is the (partial Fréchet) derivative.

5.1. Sensitivity w.r.t. the homogenized transmission coefficients

Because of complicated structure of the state problem (6)–(7), it is useful to introduce its abstract form which will allow us to derive efficiently the sensitivity formulas. Let $\boldsymbol{u} = (p, p^0, g^0)$ be a solution to (6)–(7) and define $\boldsymbol{V} = H^1_{-1}(\Omega, \Gamma_0) \times H^1(\Gamma_0) \times L^2(\Gamma_0)$, the space of the state problem solutions. We shall use the following notation:

$$\begin{aligned} \Psi(\boldsymbol{u}, \boldsymbol{v}) \equiv\, & a_\Omega\,(p,\, q) - \omega^2\,(p,\, q)_\Omega + \omega c\,\langle p,\, q\rangle_{\Gamma_{\text{in}-\text{out}}} - \mathrm{i}\omega\,\left\langle g^0,\, q^+ - q^-\right\rangle_{\Gamma_0} \\ & + \mathcal{A}(p^0, \varphi) - \omega^2 \varsigma^*\,\left\langle p^0,\, \varphi\right\rangle_{\Gamma_0} + \mathrm{i}\omega\mathcal{B}(g^0, \varphi) \\ & - \mathrm{i}\omega\varkappa_0\mathcal{B}(\psi, p^0) + \omega^2\mathcal{F}(g^0, \psi) + \mathrm{i}\omega\frac{1}{\varepsilon^0}\left\langle p^+ - p^-,\, \psi\right\rangle_{\Gamma_0}\,, \\ f(\boldsymbol{v}) =\, & 2\mathrm{i}\omega c\,\langle \bar{p},\, q\rangle_{\Gamma_{\text{in}}}\,, \\ \text{where } & \boldsymbol{v} = (q, \varphi, \psi)\,. \end{aligned} \tag{22}$$

Now the state problem can be rewritten in the abstract form: find $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$\Psi(\boldsymbol{u}, \boldsymbol{v}) = f(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,. \tag{23}$$

Let us denote by $u^\star$ the complex conjugate of u. Obviously, if $\boldsymbol{u}$ solves (23), then also

$$\Psi^\star(\boldsymbol{u}^\star, \boldsymbol{v}) = f^\star(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,, \tag{24}$$

where $\Psi^\star(\cdot,\cdot)$ and $f^\star(\cdot)$ are complex conjugate to $\Psi(\cdot,\cdot)$ and $f(\cdot)$, respectively.

The Lagrangean associated to step 1 involves the triple of primary variables $\boldsymbol{u} = (p, p^0, g^0)$ and two Lagrange multipliers, $\boldsymbol{w}_k = (q_k, \varphi_k, \psi_k)$, $k = 1, 2$, associated to the state problem defined equivalently by (23) and (24):

$$\mathcal{L}(\boldsymbol{u}; \boldsymbol{w}_1, \boldsymbol{w}_2) = \Phi(\boldsymbol{u}) + \Psi(\boldsymbol{u}, \boldsymbol{w}_1) - f(\boldsymbol{w}_1) + \Psi^\star(\boldsymbol{u}^\star, \boldsymbol{w}_2) - f^\star(\boldsymbol{w}_2)\,, \tag{25}$$

where $\Phi(\boldsymbol{u}) \equiv \Phi(p)$. Let us consider $\Phi(\boldsymbol{u})$ evaluated just for *admissible states* $\boldsymbol{u}$, i.e. for any admissible design giving the homogenized coefficients we consider $\boldsymbol{u}$ satisfying (23). Then optimality condition

$$\delta_{\boldsymbol{u}}\mathcal{L} \circ \delta\boldsymbol{u} = \delta_p\mathcal{L} \circ \delta p + \delta_{p^0}\mathcal{L} \circ \delta p^0 + \delta_{g^0}\mathcal{L} \circ \delta g^0 = 0 \tag{26}$$

must hold for any variation $\delta \boldsymbol{u} \in \boldsymbol{V}$, where

$$
\begin{aligned}
\delta_p \mathcal{L} \circ \delta p &= \delta_p \Phi(p) \circ \delta p + a_\Omega(\delta p,\, q) - \omega^2 (\delta p,\, q)_\Omega + \omega c \langle \delta p,\, q\rangle_{\Gamma_{\rm in-out}} + \\
&\quad + \mathrm{i}\omega \frac{1}{\varepsilon^0} \left\langle \delta p^+ - \delta p^-,\, \psi \right\rangle_{\Gamma_0} \\
\delta_{p^0} \mathcal{L} \circ \delta p^0 &= \mathcal{A}(\delta p^0, \varphi) - \omega^2 \varsigma^* \left\langle \delta p^0,\, \varphi \right\rangle_{\Gamma_0} - \mathrm{i}\omega \varkappa_0 \mathcal{B}(\psi, \delta p^0) \\
\delta_{g^0} \mathcal{L} \circ \delta g^0 &= -\mathrm{i}\omega \left\langle \delta g^0,\, q^+ - q^- \right\rangle_{\Gamma_0} + \mathrm{i}\omega \mathcal{B}(\delta g^0, \varphi) + \omega^2 \mathcal{F}(\delta g^0, \psi)\,.
\end{aligned}
\tag{27}
$$

Since we deal with complex functions, it is worth to recall the sense of differentiation employed above; $p = \Re(p) + \mathrm{i}\Im(p)$, hence

$$
\delta_p \Phi(p) \circ \delta p = \delta_{\Re(p)} \Phi(p) \circ \Re(\delta p) + \mathrm{i}\delta_{\Im(p)} \Phi(p) \circ \Im(\delta p).
$$

If condition (26) is satisfied for various designs $\alpha \in D_{adm}$, we obtain a *path of admissible states* $\boldsymbol{u}(\alpha)$ which form a manifold in the *design–state* space. Thus, (13) may be considered as minimization of $\Phi(p)$ w.r.t. α on the manifold $p(\alpha)$. Then, by virtue of optimality (26), multipliers $\boldsymbol{w}_k$, $k = 1, 2$, called the adjoint variables, must satisfy

$$
\begin{aligned}
\left[\delta_{\Re(\boldsymbol{u})} \Psi(\boldsymbol{u}, \boldsymbol{w}_1) + \delta_{\Re(\boldsymbol{u})} \Psi^\star(\boldsymbol{u}^\star, \boldsymbol{w}_2) + \delta_{\Re(\boldsymbol{u})} \Phi(\boldsymbol{u})\right] \circ \delta \Re(\boldsymbol{u}) = 0\,, \\
\left[\delta_{\Im(\boldsymbol{u})} \Psi(\boldsymbol{u}, \boldsymbol{w}_1) + \delta_{\Im(\boldsymbol{u})} \Psi^\star(\boldsymbol{u}^\star, \boldsymbol{w}_2) + \delta_{\Im(\boldsymbol{u})} \Phi(\boldsymbol{u})\right] \circ \delta \Im(\boldsymbol{u}) = 0\,.
\end{aligned}
\tag{28}
$$

Since, due to linearity, $\delta_{\Re(\boldsymbol{u})} \Psi^\star(\boldsymbol{u}^\star, \boldsymbol{w}_2) = \delta_{\Re(\boldsymbol{u})} \Psi^\star(\boldsymbol{u}, \boldsymbol{w}_2) = \mathrm{i}\delta_{\Im(\boldsymbol{u})} \Psi^\star(\boldsymbol{u}^\star, \boldsymbol{w}_2)$ and $\mathrm{i}\delta_{\Im(\boldsymbol{u})} \Psi(\boldsymbol{u}, \boldsymbol{w}_1) = -\delta_{\Re(\boldsymbol{u})} \Psi(\boldsymbol{u}, \boldsymbol{w}_1)$, on multiplying $(28)_2$ subsequently by $-\mathrm{i}$ and i and on adding the result to $(28)_1$, the following equivalents of (28) can be obtained:

$$
\begin{aligned}
2\delta_{\Re(\boldsymbol{u})} \Psi(\boldsymbol{u}, \boldsymbol{w}_1) \circ \delta \Re(\boldsymbol{u}) &= -\left[\delta_{\Re(\boldsymbol{u})} \Phi(\boldsymbol{u}) - \mathrm{i}\delta_{\Im(\boldsymbol{u})} \Phi(\boldsymbol{u})\right] \circ \delta \Re(\boldsymbol{u})\,, \\
2\delta_{\Re(\boldsymbol{u})} \Psi^\star(\boldsymbol{u}, \boldsymbol{w}_2) \circ \delta \Re(\boldsymbol{u}) &= -\left[\delta_{\Re(\boldsymbol{u})} \Phi(\boldsymbol{u}) + \mathrm{i}\delta_{\Im(\boldsymbol{u})} \Phi(\boldsymbol{u})\right] \circ \delta \Re(\boldsymbol{u})\,,
\end{aligned}
\tag{29}
$$

where the r.h.s. of the two equations are mutually complex conjugate. Hence $\boldsymbol{w}_2^\star = \boldsymbol{w}_1 \equiv \boldsymbol{w}$ and just one *adjoint equation* must be solved for $\boldsymbol{w} \in \boldsymbol{V}$:

$$
2\Psi(\boldsymbol{v}, \boldsymbol{w}) = -\left[\delta_{\Re(\boldsymbol{u})} \Phi(\boldsymbol{u}) - \mathrm{i}\delta_{\Im(\boldsymbol{u})} \Phi(\boldsymbol{u})\right] \circ \boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,,
\tag{30}
$$

which reads as: compute $(q, \varphi, \psi) \in H^1_{-1}(\Omega, \Gamma_0) \times H^1(\Gamma_0) \times L^2(\Gamma_0)$ satisfying adjoint equations

$$
\begin{aligned}
\mathcal{A}(\tilde{\varphi}, \varphi) - \omega^2 \varsigma^* \langle \tilde{\varphi},\, \varphi \rangle_{\Gamma_0} - \mathrm{i}\omega \varkappa_0 \mathcal{B}(\psi, \tilde{\varphi}) = 0\,, \\
\mathrm{i}\omega \mathcal{B}(\tilde{\psi}, \varphi) + \omega^2 \mathcal{F}(\tilde{\psi}, \psi) - \mathrm{i}\omega \left\langle \tilde{\psi},\, q^+ - q^- \right\rangle_{\Gamma_0} = 0\,, \\
\mathrm{i}\omega \frac{1}{\varepsilon^0} \left\langle \tilde{q}^+ - \tilde{q}^-,\, \psi \right\rangle_{\Gamma_0} + a_\Omega(\tilde{q},\, q) - \omega^2 (\tilde{q},\, q)_\Omega + \omega c \langle \tilde{q},\, q \rangle_{\Gamma_{\rm in-out}} = \\
-\frac{1}{2}\left[\delta_{\Re(p)} \Phi(p) - \mathrm{i}\delta_{\Im(p)} \Phi(p)\right] \circ \tilde{q}
\end{aligned}
\tag{31}
$$

for all $(\tilde{q}, \tilde{\varphi}, \tilde{\psi}) \in H^1_{-1}(\Omega, \Gamma_0) \times H^1(\Gamma_0) \times L^2(\Gamma_0)$. Note the order of equations which was changed w.r.t. to (22) to make the symmetry of (31) more apparent. Now the Lagrangian (25) can be rewritten as

$$
\mathcal{L}(\boldsymbol{u}; \boldsymbol{w}) = \Phi(\boldsymbol{u}) + \Psi(\boldsymbol{u}, \boldsymbol{w}) - f(\boldsymbol{w}) + \Psi^\star(\boldsymbol{u}^\star, \boldsymbol{w}^\star) - f(\boldsymbol{w}^\star)\,.
\tag{32}
$$

Once the adjoint state $\boldsymbol{w} = (q, \varphi, \psi)$ has been computed, one can evaluate sensitivity of Φ w.r.t. the homogenized coefficients (which depend further on design α). For this we consider Lagrangian (25) as the function of homogenized coefficients A, B, F. Then the total variation of $\mathcal{L}$ involves partial derivatives w.r.t. A, B, F; it holds that

$$\delta\mathcal{L} \circ \delta(A, B, F) = \left[\delta_{(p,p^0,g^0)}\mathcal{L} \circ \delta(p, p^0, g^0) + \delta_{(A,B,F)}\mathcal{L}\right] \circ \delta(A, B, F)\,, \tag{33}$$

where δ is the total variation w.r.t. A, B, F. Since we consider only admissible states, i.e. (31) holds also for $(\tilde{q}, \tilde{\varphi}, \tilde{\psi}) = \delta(p, p^0, g^0)$, the first r.h.s. term in (33) vanishes. Moreover, $\delta\mathcal{L} = \delta\Phi$ w.r.t. to any variation on the path of admissible states, hence

$$\delta\Phi(p) \circ \delta(A, B, F) = \delta_{(A,B,F)}\mathcal{L}(p, p^0, g^0, q, \varphi, \psi) \circ \delta(A, B, F)\,, \tag{34}$$

where (p, p^0, g^0) is the state problem solution and (q, φ, ψ) satisfies (31), for a given (A, B, F). The sensitivity $\delta(A, B, F)$ w.r.t. the microstructure is derived below.

5.2. *Shape sensitivity of the homogenized transmission coefficients*

Through the following text, for simplicity of the notation, we shall write just $\delta_\tau(\cdot)$ and $\delta(\cdot)$ instead of $\delta_\tau(\cdot) \circ \mathcal{V}$ and $\delta(\cdot) \circ \mathcal{V}$, respectively, to refer to the directional derivatives (18)–(19).

In order to complete the sensitivity formula (34), we shall derive sensitivity formulae for computing the shape derivatives of the homogenized coefficients defined in (11). For this we need to differentiate the local equations (10); thus, we obtain

$$\begin{aligned} \delta_\tau a_Y^*\left(\pi^\alpha,\, \phi\right) + a_Y^*\left(\delta\pi^\alpha + \mathcal{V}_\alpha,\, \phi\right) = 0\,, \\ \delta_\tau a_Y^*\left(\xi,\, \phi\right) + a_Y^*\left(\delta\xi,\, \phi\right) = 0\,, \end{aligned} \tag{35}$$

for all $\phi \in H^1_{\#(1,2)}(Y)$, where using (20)

$$\begin{aligned} \delta_\tau a_Y^*\left(\phi,\, \psi\right) &= \int_{Y^*} \left[\mathrm{div}\mathcal{V}\hat{\nabla}\phi \cdot \hat{\nabla}\psi - (\hat{\nabla}\mathcal{V} \cdot \nabla\phi) \cdot \hat{\nabla}\psi - \hat{\nabla}\phi \cdot (\hat{\nabla}\mathcal{V} \cdot \nabla\psi)\right] \\ &= \int_{Y^*} \Big[\mathrm{div}\mathcal{V}\hat{\nabla}\phi \cdot \hat{\nabla}\psi - \partial_\alpha\mathcal{V}_k\partial_k\phi\partial_\alpha\psi - \partial_\alpha\phi\partial_\alpha\mathcal{V}_l\partial_l\psi \\ &\qquad -\frac{1}{\varkappa^2}\partial_z\mathcal{V}_k\partial_k\phi\partial_z\psi - \frac{1}{\varkappa^2}\partial_z\phi\partial_z\mathcal{V}_l\partial_l\psi\Big]\,. \end{aligned} \tag{36}$$

This expression is derived by virtue of the definition in (18), using (20),

$$\begin{aligned} \delta_\tau a_Y^*\left(\pi,\, \phi\right) &= \lim_{\tau\to 0} \tau^{-1}\left[a^*_{\tilde{Y}(\tau)}(\pi, \phi) - a_Y^*\left(\pi,\, \phi\right)\right] \\ \text{where}\quad a^*_{\tilde{Y}(\tau)}(\pi, \phi) &= \int_{Y^*}\left[\frac{\partial y_k}{\partial z_\alpha}\frac{\partial \pi}{\partial y_k}\frac{\partial y_l}{\partial z_\alpha}\frac{\partial \phi}{\partial y_l} + \frac{1}{\varkappa^2}\frac{\partial y_k}{\partial z_3}\frac{\partial \pi}{\partial y_k}\frac{\partial y_l}{\partial z_3}\frac{\partial \phi}{\partial y_l}\right] J(z)\,. \end{aligned}$$

On differentiating (11) we obtain the sensitivity of $A_{\alpha\beta}$:

$$\delta A_{\alpha\beta} = \frac{c^2}{|Y|}\left[\delta_\tau a_Y^*\left(\pi^\beta + y_\beta,\, \pi^\alpha + y_\alpha\right) \circ \mathcal{V} + a_Y^*\left(\mathcal{V}_\beta,\, \pi^\alpha + y_\alpha\right) + a_Y^*\left(\pi^\beta + y_\beta,\, \mathcal{V}_\alpha\right)\right] \tag{37}$$

where the following identity was employed $a_Y^*\left(\pi^\beta + y_\beta,\, \delta\pi^\alpha\right) = 0$. From this and using $(35)_2$ one obtains

$$a_Y^*\left(\delta\xi,\, y_\beta\right) = -a_Y^*\left(\delta\xi,\, \pi^\beta\right) = \delta_\tau a_Y^*\left(\xi,\, \pi^\beta\right)\,, \tag{38}$$

which is used to simplify the sensitivity of B_α:

$$\begin{aligned} \delta B_\alpha &= \frac{c^2}{|Y|} \left[a_Y^* \left(\delta\xi,\, y_\alpha\right) + a_Y^* \left(\xi,\, \mathcal{V}_\alpha\right) + \delta_\tau a_Y^* \left(\xi,\, y_\alpha\right) \right] \\ &= \frac{c^2}{|Y|} \left[\delta_\tau a_Y^* \left(\xi,\, \pi^\alpha + y_\alpha\right) + a_Y^* \left(\xi,\, \mathcal{V}_\alpha\right) \right] \,. \end{aligned} \tag{39}$$

In order to derive the sensitivity of F, we apply subsequently $(10)_2$ with $\phi = \delta\xi$ and $(35)_2$, thus

$$\delta F = \frac{1}{|I_y|} \gamma^\pm(\delta\xi) = -\frac{\varkappa c^2}{|I_y||Y|} a_Y^* \left(\xi,\, \delta\xi\right) = \frac{\varkappa c^2}{|I_y||Y|} \delta_\tau a_Y^* \left(\xi,\, \xi\right) \,. \tag{40}$$

We remark that, as usually in such a case, the sensitivities of the homogenized coefficients can be expressed by the partial derivatives only, without need of any adjoint variables.

5.3. Shape sensitivity of the objective function

We shall summarize the sensitivity procedure to evaluate the total variation of $\Phi(p)$ w.r.t. the shape variation. Assuming *given design*, $\{\alpha\}$, and a fixed non-resonant frequency ω, we proceed as follows:

- compute the state (p, p^0, g^0),
- evaluate $\Phi(p)$ and $\delta_p \Phi(p)$ in the sense of distributions ("two-point-pressure function"),
- using (31) compute the adjoint state (q, φ, ψ) for given state p and $\delta_p \Phi(p)$,
- using (37)–(40) compute the *shape sensitivity* of $\delta A, \delta B, \delta F$ (independently of the state level 1,2)
- evaluate the total variation (recalling (32) and (34)):

$$\begin{aligned} \delta\Phi(p) \circ \delta(A, B, F) &= \delta_{(A,B,F)} \mathcal{L}(p, p^0, g^0, q, \varphi, \psi) \circ \delta(A, B, F) \\ &= \delta_{(A,B,F)} \mathcal{L}(\boldsymbol{u}, \boldsymbol{w}) \circ \delta(A, B, F) \\ &= \left[\delta_{(A,B,F)} \Psi(\boldsymbol{u}, \boldsymbol{w}) + \delta_{(A,B,F)} \Psi^\star(\boldsymbol{u}^\star, \boldsymbol{w}^\star) \right] \circ \delta(A, B, F) \\ &= 2\delta_{(A,B,F)} \Re(\Psi(\boldsymbol{u}, \boldsymbol{w})) \circ \delta(A, B, F) \\ &= 2\Re \left\{ \int_{\Gamma_0} \delta A_{\alpha\beta} \partial_\beta p^0 \partial\varphi + \omega^2 \int_{\Gamma_0} \delta F g^0 \psi \right. \\ &\qquad \left. + \mathrm{i}\omega \left[\int_{\Gamma_0} \delta B_\alpha \partial_\alpha \varphi \, g^0 - \varkappa_0 \int_{\Gamma_0} \delta B_\alpha \partial_\alpha p^0 \, \psi \right] \right\} \,. \end{aligned} \tag{41}$$

6. Conclusion

We have developed sensitivity formulas which describe influence of the perforation design change on a real objective function based, in general, on the acoustic pressure field in an area surrounding the perforation. The model of the acoustic transmission condition imposed was developed in [10] using the asymptotic homogenization analysis; some numerical simulation aspects related to this model are reported in this issue, [7]. The further step in the research will be aimed at numerical implementation of the sensitivity analysis and at solving numerically an optimal perforation design problem to maximize the transmission loss. Such problem is an important issue in the automotive industry, namely in the exhaust silencer design, [3, 4]. Obviously, optimal designing the perforated obstacles, like sieves is just a part of tools employed in the structural optimization related to acoustics, cf. [2, 11]. In this context, it is worthy to note that the homogenized transmission conditions we are dealing with are non-local, involving spatial gradients of the acoustic pressure.

The perforated sieve-like structures were considered as rigid obstacles without mechanical interaction between the acoustic fluid (air) and the structure itself. However, for some applications (thin structures) it might be important to treat deflections of the structure due to the acoustic pressure field fluctuations in the fluid. Then the mechanical interaction can be influenced by mechanical properties of the perforated "smart" structure, which may contain some distributed elements to control the vibrations, see e.g. [1, 8, 9].

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