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Computational analysis of acoustic transmission through periodically perforated interfaces

V. Lukeš[a,*], E. Rohan[a]

[a] Faculty of Applied Sciences, University of West Bohemia, Univerzitní 22, 306 14 Plzeň, Czech Republic

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Abstract

The objective of the paper is to demonstrate the homogenization approach applied to modelling the acoustic transmission on perforated interfaces embedded in the acoustic fluid. We assume a layer, with periodically perforated obstacles, separating two half-spaces filled with the fluid. The homogenization method provides limit transmission conditions which can be prescribed at the homogenized surface representing the "limit" interface. The conditions describe relationship between jump of the acoustic pressures and the transversal acoustic velocity, on introducing the "in-layer pressure" which describes wave propagation in the tangent directions with respect to the interface.

This approach may serve as a relevant tool for optimal design of devices aimed at attenuation of the acoustic waves, such as the engine exhaust mufflers or other structures fitted with sieves and grillages. We present numerical examples of wave propagation in a muffler-like structure illustrating viability of the approach when complex 3D geometries of the interface perforation are considered.

Keywords: acoustic transmission, homogenization, perforated layer, numerical modelling

1. Introduction

The purpose of the paper is to demonstrate the homogenization approach applied to computational modelling of the acoustic transmission through perforated planar structure. We consider the acoustic medium occupying domain Ω which is subdivided by perforated plane Γ_0 in two disjoint subdomains Ω^+ and Ω^- so that $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma_0$, see Fig. 1.

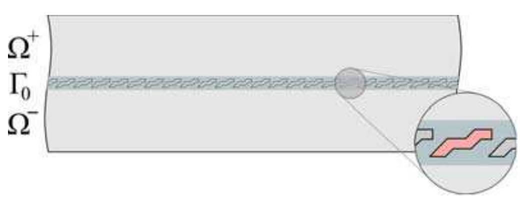


Fig. 1. Acoustic domain Ω and perforated plane Γ_0

In the differential form the problem for unknown acoustic pressures p^+, p^- reads as follows:

$$\begin{aligned} c^2\nabla^2 p^+ + \omega^2 p^+ &= 0 \quad \text{in } \Omega^+, \\ c^2\nabla^2 p^- + \omega^2 p^- &= 0 \quad \text{in } \Omega^-, \\ + \text{ boundary conditions} & \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

*Corresponding author. Tel.: +420 377 632 365, e-mail: lukes@kme.zcu.cz.

In a case of no convection flow the usual transmission conditions are given by

$$\begin{aligned} \frac{\partial p^+}{\partial n^+} &= -\mathrm{i}\frac{\omega\rho}{Z}(p^+ - p^-), \\ \frac{\partial p^-}{\partial n^-} &= -\mathrm{i}\frac{\omega\rho}{Z}(p^- - p^+), \end{aligned} \tag{2}$$

where n^+ and n^- are the outward unit normals to Ω^+ and Ω^-, respectively, ω is the frequency, ϱ is the density and Z is the *transmission impedance*; this complex number is characterized by features of the actual perforation considered and is determined using experiments in the acoustic laboratories, see e.g. [6].

The aim of our approach is to replace the transmission condition (2) by the two-scale homogenization limit of the standard acoustic problem and obtain some *homogenized coefficients* characterizing the perforated structure. The problem of acoustic transmission in muffler structures treated by means of the asymptotic method was studied in [1, 2], but the results are limited only for simple shapes of perforation.

2. Problem formulation

By indices $^\varepsilon$ we denote the dependence of variables on the scale parameter $\varepsilon > 0$; similar convention is adhered in the explicit reference to the layer thickness $\delta > 0$. By the Greek indices we refer to the coordinate index 1 or 2, so that $(x_\alpha, x_3) \in I\!R^3$.

Let $\Omega_\delta \subset I\!R^3$ be an open domain shaped as a layer bounded by $\partial\Omega_\delta$ which is split as follows

$$\partial\Omega_\delta = \Gamma_\delta^+ \cup \Gamma_\delta^- \cup \partial\Omega_\delta^\infty, \tag{3}$$

where $\delta > 0$ is the layer thickness, see Fig. 2. The acoustic medium occupies domain $\Omega_\delta \setminus S_\delta^\varepsilon$, where S_δ^ε is the solid obstacle which in a simple layout has a form of the periodically perforated sheet.

For homogenization technique, it is important to have a fixed domain, therefore the *dilatation* is considered, cf. [4, 6]; let Γ_0 be the plane spanned by coordinates $1, 2$ and containing the origin. Further let Γ_δ^+ and Γ_δ^- be equidistant to Γ_0 with the distance $\delta/2$. Therefore, $x_3 \in]-\delta/2, \delta/2[$ and we introduce the rescaling $x_3 = z\delta$.

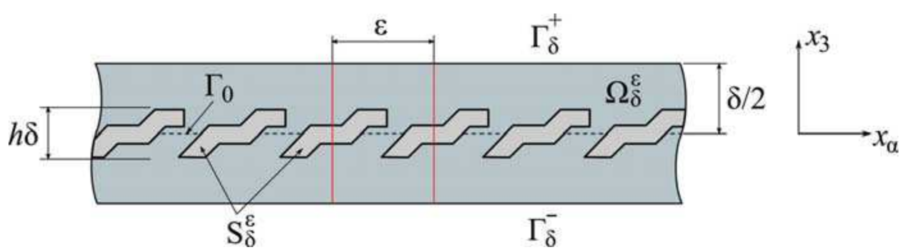


Fig. 2. The perforated interface layer with periodic solid perforations; Ω – acoustic medium, S – solid perforation (obstacle)

The problem of acoustics is defined in $\Omega^\varepsilon_\delta$. We assume a monochrome stationary incident wave with frequency ω and no convection velocity of the medium, so that

$$\begin{aligned} c^2\nabla^2 p^{\varepsilon\,\delta} + \omega^2 p^{\varepsilon\,\delta} &= 0 \quad \text{in } \Omega^\varepsilon_\delta, \\ c^2 \frac{\partial p^{\varepsilon\,\delta}}{\partial n^\delta} &= -\mathrm{i}\omega g^{\varepsilon\,\delta\pm} \quad \text{on } \Gamma^\pm_\delta, \\ \frac{\partial p^{\varepsilon\,\delta}}{\partial n^\delta} &= 0 \quad \text{on } \partial\Omega^\infty_\delta \cup \partial S^\varepsilon_\delta, \end{aligned} \tag{4}$$

where $c = \omega/k$ is the speed of sound propagation, $g^{\varepsilon\,\delta\pm}k^2$ is the interface normal acoustic momentum; by n^δ we denote the normal vector outward to Ω_δ.

3. Homogenization

The homogenized model can be derived using the Tartar method of oscillating functions, see [5, 8, 7] or, alternatively, the *periodic unfolding method*, see [4]. The homogenization technique itself is outside the scope of this paper, but we briefly describe the first approach.

The homogenization procedure is based on the following steps: 1) a priori estimation of the pressure gradient, which gives information for 2) the formal asymptotic expansion that allows to decompose the problem into local and global subproblems; 3) the homogenized coefficients are identified using the Tartar variational method; 4) correction to a finite scale of the obstacle thickness.

The homogenization process results in the limit macroscopic problem in the transmission layer and the local microscopic subproblem that is formulated using so called *corrector functions*. The local subproblem is solved within the reference periodic cell and gives some *homogenized acoustic coefficients* that characterize the specific shape of the perforation at the microscopic level. These acoustic coefficients allows to constitute the new *homogenized transmission condition* imposed on the perforated surface at the global (macroscopic) scale.

3.1. Local microscopic problems

The homogenized coefficients are introduced using *corrector functions* π^β, $\xi^\pm$ ($\beta = 1, \ldots, (N-1)$ where N is the problem dimension) computed for the reference periodic cell Y which is perforated by the solid (rigid) obstacle S, so that the acoustic medium occupies domain $Y^* = Y \setminus S$. We refer to the upper and lower boundaries of Y by I^+_Y and I^-_Y (see Fig. 3).

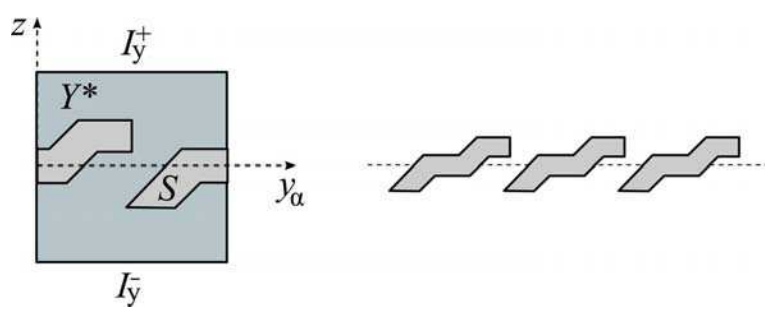


Fig. 3. Reference periodic cell Y; Y^* – acoustic medium, S – solid obstacle, y_α – "periodicity" direction, z – normal direction, I^+_Y and I^-_Y – upper and lower boundaries

The local microscopic problems can be formulated (see [7, 8]) in the discrete forms (in the sense of finite element approximation) as: Find $\boldsymbol{\pi}^\beta$ and $\boldsymbol{\xi}^\pm$ such that (notation $\mathbf{p}^H$ means the Hermitian transpose to $\mathbf{p}$)

$$\begin{aligned} \left(\mathbf{K} + \frac{1}{\kappa^2}\mathbf{K}_z\right)\boldsymbol{\xi}^\pm &= \frac{|Y|}{c^2\boldsymbol{\kappa}}\,\mathbf{f}, \\ \left(\mathbf{K} + \frac{1}{\kappa^2}\mathbf{K}_z\right)\boldsymbol{\pi}^\beta &= \mathbf{K}\mathbf{y}^\beta, \end{aligned} \tag{5}$$

where $\mathbf{K}$, $\mathbf{K}_z$, $\mathbf{f}$ are finite element approximations of integrals

$$\begin{aligned} \int_{Y^*} \frac{\partial q}{\partial y_\alpha}\frac{\partial p}{\partial y_\alpha} &\approx \mathbf{q}^H\mathbf{K}\mathbf{p}, \\ \int_{Y^*} \frac{\partial q}{\partial z}\frac{\partial p}{\partial z} &\approx \mathbf{q}^H\mathbf{K}_z\mathbf{p}, \\ \left(\int_{I_y^+} q - \int_{I_y^-} q\right) &\approx \mathbf{q}^H\mathbf{f} \end{aligned} \tag{6}$$

and $\boldsymbol{y}$ is the coordinate vector. Parameter κ relates the thickness and the period length, so that $\delta = \kappa\varepsilon$.

3.2. *Macroscopic problem in transmission layer*

Homogenized acoustic behaviour in the transmission layer is expressed in terms of *interface mean acoustic pressure* p^0 and *acoustic transverse momentum* g^0 which satisfy the interface problem

$$\begin{bmatrix} (\mathbf{A} - \phi^*\omega^2\mathbf{M}) & \mathrm{i}\omega\mathbf{B}^T \\ -\mathrm{i}\omega\mathbf{D} & \omega^2\mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{p}^0 \\ \mathbf{g^0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\mathrm{i}\omega\mathbf{M}\left(\bar{\mathbf{p}}^+ - \bar{\mathbf{p}}^-\right) \end{bmatrix}, \tag{7}$$

where $\phi^* = \frac{|Y^*|}{|Y|}$ and matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{D}$, $\mathbf{F}$, $\mathbf{M}$ are finite element approximations of the following integrals

$$\begin{aligned} \int_{\Gamma_0} A_{\alpha\beta}\frac{\partial q}{\partial x_\alpha}\frac{\partial p}{\partial x_\beta} &\approx \mathbf{q}^H\mathbf{A}\mathbf{p}, \\ \int_{\Gamma_0} B_\alpha\frac{\partial q}{\partial x_\alpha}\,p &\approx \mathbf{q}^H\mathbf{B}\mathbf{p}, \\ \int_{\Gamma_0} D_\alpha\, q\,\frac{\partial p}{\partial x_\alpha} &\approx \mathbf{q}^H\mathbf{D}\mathbf{p}, \\ \int_{\Gamma_0} F^\pm\, q\, p &\approx \mathbf{q}^H\mathbf{F}\mathbf{p}, \\ \int_{\Gamma_0} q\, p &\approx \mathbf{q}^H\mathbf{M}\mathbf{p}. \end{aligned} \tag{8}$$

These equations involve the *homogenized coefficients* which are expressed in terms of the corrector functions $\boldsymbol{\pi}^\beta$ and $\boldsymbol{\xi}^\pm$:

$$
\begin{aligned}
A_{\alpha\beta} &= \frac{c^2}{|Y|} \left(\mathbf{y}^\beta + \boldsymbol{\pi}^\beta\right)^T \mathbf{K} \left(\mathbf{y}^\alpha + \boldsymbol{\pi}^\alpha\right) + \frac{c^2}{|Y|\kappa^2} \left(\boldsymbol{\pi}^\beta\right)^T \mathbf{K}_z \boldsymbol{\pi}^\alpha, \\
D_\alpha &= \frac{1}{|I_Y|} \mathbf{f}^T \boldsymbol{\pi}^\alpha = \frac{\kappa}{|I_Y|} B_\alpha, \\
F^\pm &= \frac{1}{|I_Y|} \mathbf{f}^T \boldsymbol{\xi}^\pm.
\end{aligned} \tag{9}
$$

It is possible to compute the Schur complement (for ω out-of-resonance) of the discretized interface problem (7), so that

$$
\begin{aligned}
\mathbf{p}^0 &= -\mathrm{i}\omega \left(\mathbf{A} - \phi^* \omega^2 \mathbf{M}\right)^{-1} \mathbf{B}^T \mathbf{g}^0, \\
\omega^2 \left[\mathbf{F} - \mathbf{D} \left(\mathbf{A} - \phi^* \omega^2 \mathbf{M}\right)^{-1} \mathbf{B}^T\right] \mathbf{g}^0 &= -\mathrm{i}\omega \mathbf{M} \left(\bar{\mathbf{p}}^+ - \bar{\mathbf{p}}^-\right).
\end{aligned} \tag{10}
$$

Thus, it is possible to introduce the *coupled impedance*

$$
\mathbf{X}(\omega^2) = \omega^2 \left[\mathbf{F} - \mathbf{D} \left(\mathbf{A} - \phi^* \omega^2 \mathbf{M}\right)^{-1} \mathbf{B}^T\right], \tag{11}
$$

hence the discretized interface transmission condition reduces to

$$
\mathbf{X}(\omega^2)\, \mathbf{g}^0 = -i\, \omega\, \mathbf{M} \left(\bar{\mathbf{p}}^+ - \bar{\mathbf{p}}^-\right), \tag{12}
$$

which resembles the structure of the standard conditions (2), since $\mathbf{g}^0$ approximates the transversal velocities ($\mathbf{g}^0 \approx \partial p^+ / \partial n^+$).

4. Global acoustic problem

Macroscopic acoustic behaviour in Ω is described by acoustic pressures p^+, p^- which satisfy equations (1) and by the homogenized interface problem (7), so we can consider (see [4, 8])

$$
\begin{aligned}
c^2 \frac{\partial p^+}{\partial n^+} &= \mathrm{i}\omega g^0, \\
c^2 \frac{\partial p^-}{\partial n^-} &= -\mathrm{i}\omega g^0 \quad \text{on } \Gamma_0,
\end{aligned} \tag{13}
$$

instead of (2). The relationship between p^0, g^0 and pressure jump $p^+ - p^-$ is given by (7).

We need to specify boundary conditions on boundary $\partial\Omega = \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_w$ consisting of the planar surfaces Γ_{in}, Γ_{out} and the walls Γ_w, see Fig. 4. On Γ_{in} we assume an incident wave with given amplitude $\tilde{p}$ and on Γ_{out} we impose the radiation condition in the form of the anechoid output, so that

$$
\begin{aligned}
\mathrm{i}\omega p + c \frac{\partial p}{\partial n} &= 2\mathrm{i}\omega \tilde{p} \quad \text{on } \Gamma_{in}, \\
\mathrm{i}\omega p + c \frac{\partial p}{\partial n} &= 0 \quad \text{on } \Gamma_{out}, \\
\frac{\partial p}{\partial n} &= 0 \quad \text{on } \Gamma_w.
\end{aligned} \tag{14}
$$

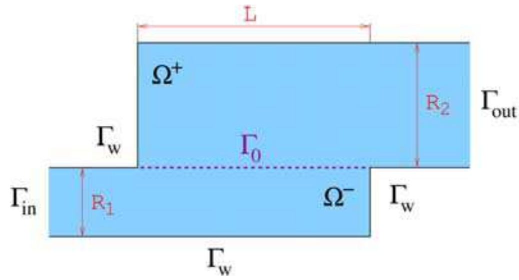


Fig. 4. Macroscopic domain Ω; $L = 1\,\mathrm{m}$, $R_1 = 0.3\,\mathrm{m}$, $R_2 = 0.6\,\mathrm{m}$

The global problem in the FEM discretized form can be expressed by the following linear system

$$\begin{bmatrix} \mathbf{C}(\omega) & (\mathbf{Q}^+)^H(\omega) & (\mathbf{Q}^-)^H(\omega) & 0 \\ \mathbf{Q}^+(\omega) & \bar{\mathbf{C}}^+(\omega) & \mathbf{0} & -\mathrm{i}\omega\mathbf{M} \\ \mathbf{Q}^-(\omega) & \mathbf{0} & \bar{\mathbf{C}}^-(\omega) & +\mathrm{i}\omega\mathbf{M} \\ \mathbf{0} & +\mathrm{i}\omega\mathbf{M} & -\mathrm{i}\omega\mathbf{M} & \mathbf{X}(\omega^2) \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \bar{\mathbf{p}}^+ \\ \bar{\mathbf{p}}^- \\ \mathbf{g}^0 \end{bmatrix} = \mathrm{i}\omega \begin{bmatrix} \bar{\mathbf{h}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{15}$$

where

$$\begin{aligned} c^2 \int_\Omega \left(\nabla p \,\nabla q - \omega^2 pq\right) &\approx \mathbf{q}^H \mathbf{C}(\omega)\mathbf{p}, \\ c^2 \int_{\Omega^{+/-}} \left(\nabla p \,\nabla q - \omega^2 pq\right) &\approx \mathbf{q}^H \bar{\mathbf{C}}^{+/-}(\omega)\mathbf{p}, \end{aligned} \tag{16}$$

matrices $\mathbf{Q}^+(\omega)$, $\mathbf{Q}^-(\omega)$ are associated with boundary conditions in $\bar{\Omega} \setminus \Gamma_0^{+/-}$, $\mathbf{p}$ is pressure in $\Omega^+ \cup \Omega^- \cup \partial\Omega$, $\bar{\mathbf{p}}^+$, $\bar{\mathbf{p}}^-$ are pressures on Γ_0^+, Γ_0^- and $\bar{\mathbf{h}}$ involves the right hand sides of boundary conditions (14).

We recall that coupled impedance $\mathbf{X}(\omega^2)$ is linear function of scale parameter ε, which reflects a given finite scale of the perforation.

5. Numerical simulations

This section presents some illustrative numerical examples of acoustic transmission showing influence of the perforation design. Examples were computed using our code based on Python language ("Sfepy" project, [3]) and Matlab system. We use Q1 finite element approximation for acoustic pressure in Ω and characteristic functions in Y and P1 line elements on Γ_0 to approximate p^0 and g^0.

5.1. Microstructure — various perforations

In Figs. 5 and 6, we compare the corrector functions $\xi^\pm$, π and homogenized parameters of three different perforations in 2D and three perforations in 3D. Due to the geometrical arrangement of the solid obstacles the coupling coefficients B, D vanish for perforation types 2D/#1, 3D/#1 and 3D/#2. For types 2D/#2, 2D/#3 and 3D/#3 these coefficients are nonzero, i.e. the transversal and tangential velocities in the interface layer are coupled.

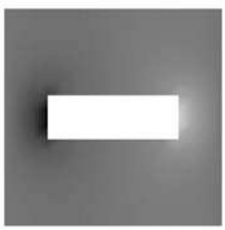
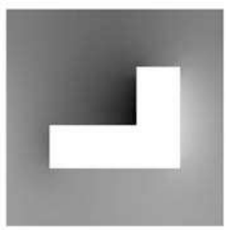
	Mic. 2D/#1	Mic. 2D/#2	Mic. 2D/#3
$\xi^{\pm}$			
π^1			
A [(m/s)^2]	99271.58	77973.96	51535.28
B [m]	0	-0.112957	-0.452692
F [s^2]	$1.405\,082 \times 10^{-5}$	$1.530\,726 \times 10^{-5}$	$3.344\,039 \times 10^{-5}$

Fig. 5. Distribution of the characteristic functions $\xi^{\pm}$, π^1 in the microscopic domain Y^* and homogenized acoustic coefficients for three shapes of 2D perforations

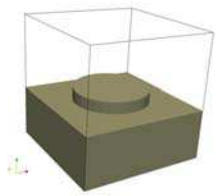
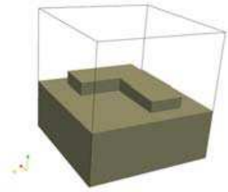
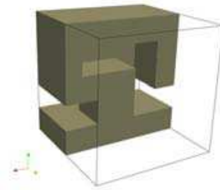
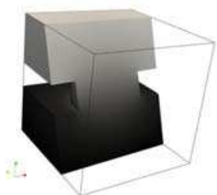
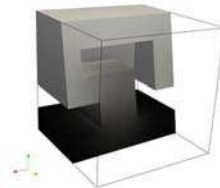
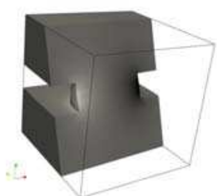
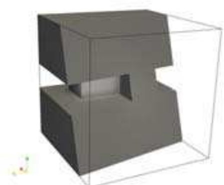
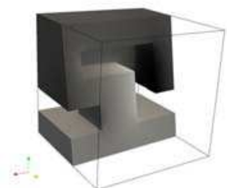
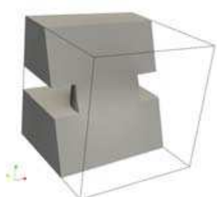
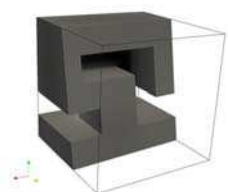
	Mic. 3D/#1	Mic. 3D/#2	Mic. 3D/#3
Geometry			
$\xi^{\pm}$			
π^1			
π^2			
A [(m/s)^2]	$\begin{bmatrix} 98\,415.75 & 0.0 \\ 0.0 & 98\,415.75 \end{bmatrix}$	$\begin{bmatrix} 98\,654.50 & 207.83 \\ 207.83 & 98\,155.32 \end{bmatrix}$	$\begin{bmatrix} 75\,295.34 & 0.0 \\ 0.0 & 81\,814.05 \end{bmatrix}$
B [m]	$\begin{bmatrix} 0.0 & 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.0 & 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.142\,330 & 0.142\,330 \end{bmatrix}$
F [s^2]	$1.754\,429 \times 10^{-5}$	$1.647\,584 \times 10^{-5}$	$2.838\,839 \times 10^{-5}$

Fig. 6. Distribution of the characteristic functions $\xi^{\pm}$, π^1 and π^2 in the microscopic domain Y^* and homogenized acoustic coefficients for three shapes of 3D perforations

5.2. Global problem — modelling acoustic waveguide

In Fig. 7 we show the global response of a waveguide with the homogenized transmission layer. The modulus of the acoustic pressure p is displayed and we illustrate how this response is sensitive to the type of perforation (2D/#1, 2D/#2 and 2D/#3). The results were obtained for the following parameters: density $\rho = 1.55$ kg/m^3, acoustic speed $c = 343$ m/s, $\omega = 5 \cdot c/L$ and the amplitude of the incident wave (see (14)) is $\tilde{p} = v_n/(\rho c)$, where $v_n = 1$ m/s. The geometry of the acoustic (macroscopic) waveguide is depicted in Fig. 4.

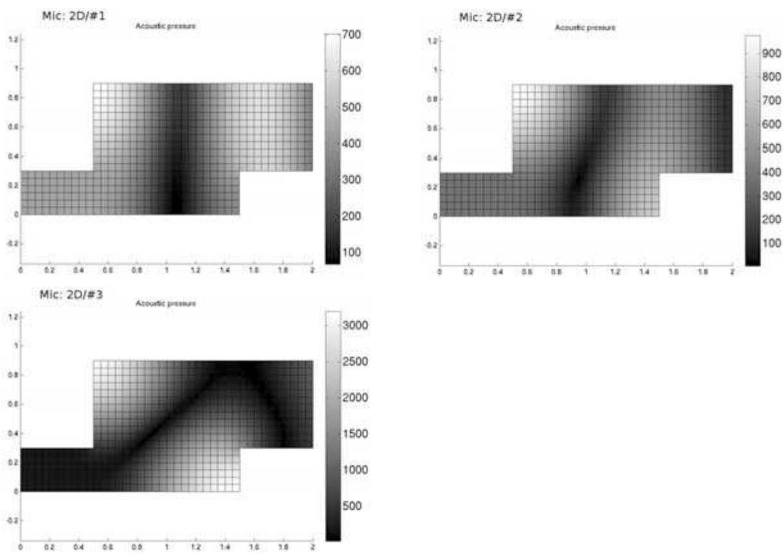


Fig. 7. Modulus of the acoustic pressure p [Pa] in the macroscopic waveguide for perforation types 2D/#1, 2D/#2 and 2D/#3 (see Fig. 5)

6. Conclusion

We demonstrated the homogenization approach applied to modelling the acoustic transmission on perforated interfaces. Our model involves the new transmission conditions, see [7, 8], with homogenized parameters which reflect specific geometry of the periodic perforation. In numerical examples we showed the sensitivity of the acoustic transmission coefficients on the shape of perforations. The presented approach can be applied to various engineering problems, such as modelling of muffler structures (see [1]), etc.

Acknowledgements

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