Conversion of biquadratic rational Bézier surfaces into patches of particular Dupin cyclides: the torus and the double sphere

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ABSTRACT

Toruses and double spheres are particular cases of Dupin cyclides. In this paper, we study the conversion of rational biquadratic Bézier surfaces into Dupin cyclide patches. We give the conditions that the Bézier surface should satisfy to be convertible, and present a new conversion algorithm to construct the torus or double sphere patch corresponding to a given Bézier surface, some conversion examples are illustrated and commented.

Keywords: *Torus and Dupin cyclides surfaces, rational biquadratic Bézier surfaces.*

1 Introduction

Rational biquadratic Bézier surfaces are tensor product parametric surfaces widely used in the first generation of computer graphics applications and geometric modelling systems. Good introductions to these surfaces may be found in [PT89, For68, DP98, HL93].

Dupin cyclide surfaces represent a family of ringed surfaces, i.e., surfaces generated by a circle of variable radius sweeping through space [Pra90, Deg94]. It is possible to formulate them either as algebraic or parametric surfaces. In recent decades, the interest of several authors in these surfaces relates to their potential value in the development of CAGD tools [Pra95, DMP93, Gar07]. Also, cyclide intersections and the use of cyclides as blending surfaces have been investigated [BP98, She98].

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The primary aim of this paper is to present an algorithm to convert a rational biquadratic Bézier surface into a particular Dupin cyclide patch. This conversion allows the obtention of parameters of the implicit equation of Dupin cyclide corresponding to the converted surface. Section 2 recalls the definition of Bézier curves and surfaces, and Dupin cyclides. Section 3 shows the construction of the control points and the computation of the weights of a Bézier surface which can be represented by a Dupin cyclide patch. Section 4 details the conversion algorithm. Section 5 presents our conclusions and suggests directions for future work.

2 Background

2.1 Rational Bézier curves and surfaces

Rational quadratic Bézier curves are second degree parametric curves defined by:

$$\overrightarrow{OM(t)} = \frac{\sum_{i=0}^{2} w_i B_i(t) \overrightarrow{OP_i}}{\sum_{i=0}^{2} w_i B_i(t)},\ t \in [0,1] \qquad (1)$$

where $B_i(t)$ are quadratic Bernstein polynomials defined as:

$$B_0(t) = (1-t)^2\,, B_1(t) = 2t\,(1-t) \text{ and } B_2(t) = t^2$$

and for $i \in \{0,1,2\}$, w_i are weights associated with the control points P_i. For a standard rational quadratic Bézier curve, w_0 and w_2 are equal to 1, while w_1 can be used to control the type of conic defined by the curve [Far93, Far99, Gar07].

Rational biquadratic Bézier surfaces are defined by control points $(P_{ij})_{0\leq i,j\leq 2}$ and weights $(w_{ij})_{0\leq i,j\leq 2}$ as:

$$\overrightarrow{OM(u,v)} = \frac{\sum_{i=0}^{2}\sum_{j=0}^{2} w_{ij} B_i(u) B_j(v) \overrightarrow{OP_{ij}}}{\sum_{i=0}^{2}\sum_{j=0}^{2} w_{ij} B_i(u) B_j(v)} \quad (2)$$

More details on Bézier surfaces can be found in [Far93, Far99, Gar07]. In the remaining of this paper, we only consider rational Bézier curves and surfaces of degree two to which we refer, for short, by Bézier curves and Bézier surfaces.

2.2 Dupin cyclides

Non-degenerate Dupin cyclides, figure 1(a), have been definied by P. Dupin [Dup22]. A. R. Forsyth [For12] and G. Darboux [For12, Dar17] have given two equivalent implicit equations:

$$\left(x^2+y^2+z^2-\mu^2+b^2\right)^2 = 4\left(ax-c\mu\right)^2+4b^2y^2 \quad (3)$$

$$\left(x^2+y^2+z^2-\mu^2-b^2\right)^2 = 4\left(cx-a\mu\right)^2-4b^2z^2 \quad (4)$$

in an orthonormal basis $(O, \overrightarrow{\imath_0}, \overrightarrow{\jmath_0}, \overrightarrow{k_0})$ where O is called Dupin cyclide center. Parameters a, b and c are related by $c^2 = a^2 - b^2$. The parameter a is always greater than or equal to c. Parameters a, c and μ determine the type of the cyclide. When $c < \mu \leq a$ it is a ring cyclide, when $0 < \mu \leq c$ it is a horned cyclide, and when $\mu > a$ it is a spindle cyclide.

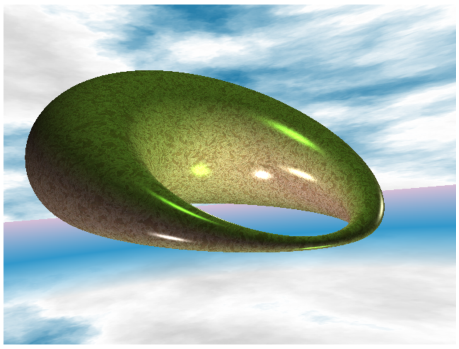
A Dupin cyclide admits two planes of symmetry $\mathcal{P}_y : y = 0$ and $\mathcal{P}_z : z = 0$ which define two couples of circles, called principal circles, figures 2(a) and 2(b). From the knowledge of a couple of principal circles and the Dupin cyclide type, it is easy to calculate Dupin cyclide parameters [Gar07].

If $c = 0$ and $a \neq 0$ then the Dupin cyclide is a torus, figure 1(b), and then :

- principal circles of the Dupin cyclide in $\mathcal{P}_y$ have the same radius, they represent a torus meridian;
- principal circles of the Dupin cyclide in $\mathcal{P}_z$ become concentric circles.

If $a = c = b$ then a Dupin cyclide is a double sphere and principal circles in both planes are identical.

The planes containing circles of curvature of a Dupin cyclide form two pencils of planes, figure 3, and define two



(a)

(b)

Figure 1: A ring Dupin cyclide (a) and a ring torus (b).

straight lines Δ_θ as the intersection of the planes of the first pencil and Δ_ψ as the intersection of the planes of the other pencil. If the Dupin cyclide is a torus, the line Δ_ψ belongs to the infinity plane (the planes containing circles of curvature are parallel).

Figure 4 shows lines Δ_θ and Δ_ψ with the principal circles of the ring Dupin cyclide in plane $\mathcal{P}_y$ ($\mathtt{C}_1^\theta$, $\mathtt{C}_2^\theta$) and in plane $\mathcal{P}_z$ ($\mathtt{C}_1^\psi$, $\mathtt{C}_2^\psi$). Δ_0 is the common perpendicular to Δ_θ and Δ_ψ. More details about properties of Dupin cyclide can be found in [Pra90, She98, AD96, DMP93].

Several authors have proposed algorithms to convert a Dupin cyclide patch into a Bézier surface [Pra90, Ued95, AD96, FGP05, Gar07] and vice-versa [Gar07, GFN06]. Table 1 gives the four most important properties of control points of a Bézier surface obtained by the conversion of a Dupin cyclide patch.

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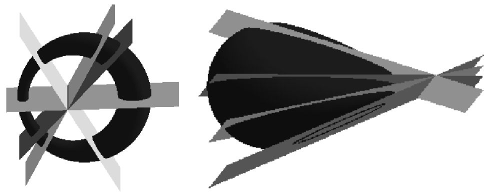


(a) (b)

Figure 3: Two pencils of planes generated by Dupin cyclide curvature circles defining Δ_θ (a) and Δ_ψ (b).

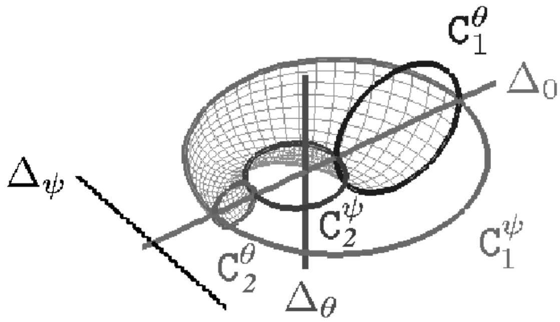
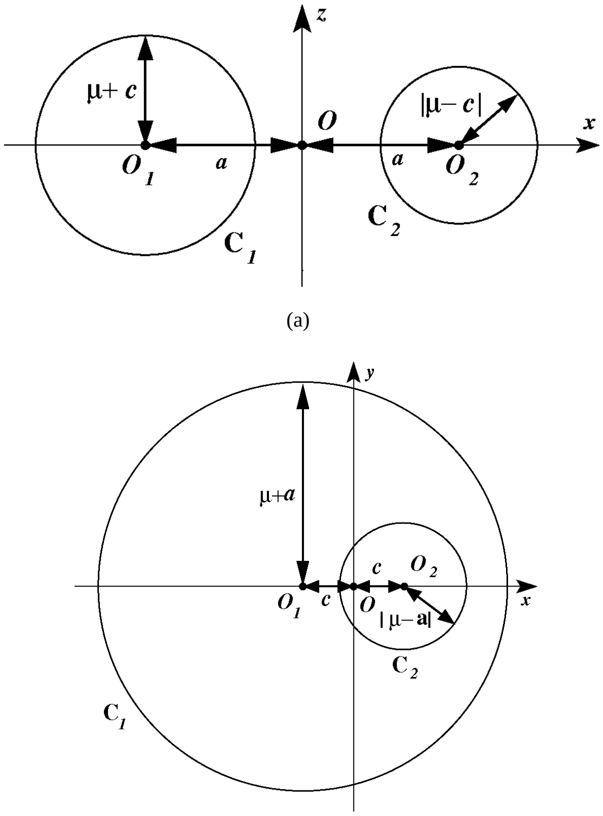


Figure 4: Straight lines Δ_θ and Δ_ψ obtained as intersections of two pencil planes.



(a)

(b)

Figure 2: Principal circles of ring Dupin cyclides. (a) : in plane $\mathcal{P}_y : y = 0$. (b) : in plane $\mathcal{P}_z : z = 0$.

(PG1)	P_{00}, P_{02} , P_{22} et P_{20} are cocyclical
(PG2)	$P_{00}P_{01} = P_{01}P_{02}$ $\quad P_{02}P_{12} = P_{12}P_{22}$ $P_{22}P_{21} = P_{21}P_{20}$ $\quad P_{00}P_{10} = P_{10}P_{20}$
(PG3)	$\overrightarrow{P_{00}P_{10}} \perp \overrightarrow{P_{00}P_{01}}$ $\quad \overrightarrow{P_{02}P_{01}} \perp \overrightarrow{P_{02}P_{12}}$ $\overrightarrow{P_{22}P_{12}} \perp \overrightarrow{P_{22}P_{21}}$ $\quad \overrightarrow{P_{20}P_{21}} \perp \overrightarrow{P_{20}P_{10}}$
(PG4)	$\overrightarrow{P_{00}P_{11}} \bullet \left(\overrightarrow{P_{00}P_{10}} \times \overrightarrow{P_{00}P_{01}}\right) = 0$ $\overrightarrow{P_{02}P_{11}} \bullet \left(\overrightarrow{P_{02}P_{01}} \times \overrightarrow{P_{02}P_{12}}\right) = 0$ $\overrightarrow{P_{22}P_{11}} \bullet \left(\overrightarrow{P_{22}P_{12}} \times \overrightarrow{P_{22}P_{21}}\right) = 0$ $\overrightarrow{P_{20}P_{11}} \bullet \left(\overrightarrow{P_{20}P_{21}} \times \overrightarrow{P_{20}P_{10}}\right) = 0$

Table 1: Geometrical properties of a Bézier surface obtained by conversion of a Dupin cyclide.

In table 1, property (PG4) can be presented as:

$$\begin{aligned} P_{11} \quad \in \quad & Aff\{P_{00};P_{01};P_{10}\} \cap Aff\{P_{02};P_{12};P_{01}\} \\ \cap \quad & Aff\{P_{20};P_{21};P_{10}\} \cap Aff\{P_{22};P_{21};P_{12}\} \end{aligned} \tag{5}$$

where $Aff\{A;B;C\}$ designate the affine space generated by points A, B and C.

3 Construction of the Bézier surface

In this section, we construct a Bézier surface convertible to a Dupin cyclide patch and so properties of table 1 must be satisfied [Gar07, GFN06]. The construction of a Bézier surface convertible to a torus patch or a double sphere patch is also considered.

3.1 Dupin cyclide case

As Dupin cyclide curvature lines are circles, the border lines of the Bézier surface to be convertible must be circular arcs. To ensure the convertibility of the Bézier surface, the following three conditions, on weight computation, has to be satisfied:
(i) we have $w_{00} = w_{02} = w_{20} = 1$ and value of w_{22} is calculated using Ueda's method [Ued95];
(ii) as border lines of a Bézier surface are Bézier curves reprenting circular arcs, it is easy to determine the weights w_{10}, w_{01},w_{21} and w_{12} [Gar07];
(iii) the computation of weight w_{11} is more complex and can be done using theorem 1 where $\mathsf{b}ar\left\{(A_i, \alpha_i)_{i \in I}\right\}$ indicates the barycentre of collection $(A_i, \alpha_i)_{i \in I}$ of level-headed points:

Theorem 1 *Barycentric middle curve*
Let us consider a Bézier surface defined by control points $(P_{ij})_{0 \leq i,j \leq 2}$ and weights $(w_{ij})_{0 \leq i,j \leq 2}$.
Let $G_i^u = \mathsf{b}ar\left\{(P_{i0}; w_{i0}), (P_{i1}; 2w_{i1}), (P_{i2}; w_{i2})\right\}$ and $\alpha_i^u = w_{i0} + 2w_{i1} + w_{i2}$ where $i \in [\![0; 2]\!]$.
Let $G_i^v = \mathsf{b}ar\left\{(P_{0i}; w_{0i}), (P_{1i}; 2w_{1i}), (P_{2i}; w_{2i})\right\}$ and $\alpha_i^v = w_{0i} + 2w_{1i} + w_{2i}$ where $i \in [\![0; 2]\!]$.
If $\sum_{i=0}^{2} \alpha_i^u \neq 0$, the barycentric middle curve $u \mapsto M\left(u, \frac{1}{2}\right)$
is a Bézier curve with control points $(G_i^u; \alpha_i^u)_{i \in [\![0;2]\!]}$.
If $\sum_{i=0}^{2} \alpha_i^v \neq 0$, the barycentric middle curve $v \mapsto M\left(\frac{1}{2}, v\right)$
is a Bézier curve with control points $(G_i^v; \alpha_i^v)_{i \in [\![0;2]\!]}$.

Proof:

$$\overrightarrow{OM\left(u, \tfrac{1}{2}\right)} = \frac{\sum_{i=0}^{2}\sum_{j=0}^{2} w_{ij} B_i(u) B_j\left(\frac{1}{2}\right)\overrightarrow{OP_{ij}}}{\sum_{i=0}^{2}\sum_{j=0}^{2} w_{ij} B_i(u) B_j\left(\frac{1}{2}\right)}$$

$$= \frac{\sum_{i=0}^{2} B_i(u)\left(w_{i0}B_0\left(\frac{1}{2}\right)\overrightarrow{OP_{i0}} + w_{i1}B_1\left(\frac{1}{2}\right)\overrightarrow{OP_{i1}} + w_{i2}B_2\left(\frac{1}{2}\right)\overrightarrow{OP_{i2}}\right)}{\sum_{i=0}^{2} B_i(u)\left(w_{i0}B_0\left(\frac{1}{2}\right) + w_{i1}B_1\left(\frac{1}{2}\right) + w_{i2}B_2\left(\frac{1}{2}\right)\right)}$$

$$= \frac{\sum_{i=0}^{2} B_i(u)\left(w_{i0}\overrightarrow{OP_{i0}} + 2w_{i1}\overrightarrow{OP_{i1}} + w_{i2}\overrightarrow{OP_{i2}}\right)}{\sum_{i=0}^{2} B_i(u)(w_{i0} + 2w_{i1} + w_{i2})}$$

$$= \frac{1}{\sum_{i=0}^{2} B_i(u)\alpha_i}\sum_{i=0}^{2} B_i(u)\,\alpha_i\overrightarrow{OG_i^u}$$

where $G_i^u = \mathsf{b}ar\left\{(P_{i0}; w_{i0}), (P_{i1}; 2w_{i1}), (P_{i2}; w_{i2})\right\}$
with $\alpha_i = w_{i0} + 2w_{i1} + w_{i2}$.
The second proof is similar.
■

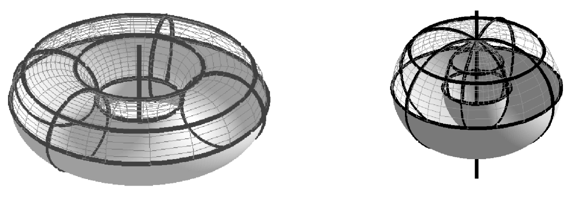
To determine the weigth w_{11}, we impose that the point G_1^u (resp. G_1^v) belongs to the perpendicular bissector plane of $[G_0^u G_2^u]$ (resp. $[G_0^v G_2^v]$).

The following section considers the conversion of a Bézier surface into a patch of a torus or a patch of a double sphere which are particular cases of Dupin cyclides. The general cases (conversion of a Bézier surface into a regular Dupin cyclide) has been consider earlier [Gar07, GFN06]. To summerize, a convertible Bézier surface must satisfy control points properties of table 1 and the three weight conditions given above.

3.2 Torus and double sphere case

We consider that two opposite edges (circular arcs) of the Bézier surface to be converted are in two parallel planes. In this case, the line Δ_ψ belongs to the infinity space and the conversion result will be a patch of a torus ($c = 0$, $a \neq 0$) or a patch of a double sphere ($a = c = 0$). To distinguish between these two cases, we have to consider the remaining edges of the Bézier surface: if the circles containing the edges have the same diameter line, it is a double sphere patch, otherwise, it is a torus patch. The type of torus can be determined throught the following three tests:

- if the circles are disjoint, the result is a ring torus patch;
- if the circles are secant in two points, the result is a spindle torus patch and the two points of intersection define Δ_θ which will be used as a frame axis in the conversion algorithm, figure 5(b);
- if the circles are tangent, the result is a horn torus patch.



(a) (b)

Figure 5: Torus as Dupin cyclide. (a) : ring torus. (b) : spindle torus.

We note here that the frame $(O, \overrightarrow{\imath_0}, \overrightarrow{\jmath_0}, \overrightarrow{k_0})$ in which the resulting patch is defined is not the same as the one of the initial Bézier surface. Vectors $\overrightarrow{\imath_0}$ and $\overrightarrow{\jmath_0}$ are perpendicular and belong to the vector plane attached to the affine planes containing the parrallel circles. The third vector of the frame is $\overrightarrow{k_0} = \overrightarrow{\imath_0} \times \overrightarrow{\jmath_0}$.

4 The conversion algorithm

Let γ be a standard Bézier curve definied by level-headed control points $(P_0; 1)$, $(P_1; w_1)$ and $(P_2; 1)$ such that:

- $w_1 = f_w\left(P_0; P_1; P_2\right)$ where $f_w : \mathcal{E}_3{}^3 \to \mathbb{R}$ [Gar07];

- γ is a circular arc.

Algorithm 1 details the steps required to convert a Bézier surface into a torus patch or a double sphere patch. Figure 6(a) shows the original Bézier surface, its control polyhedron, its two barycentric middle curves (theorem 1) as well as Bézier curves γ_3^+, γ_4^+, γ_5^+ and γ_6^+ which determine the edge of the Bézier surface.

First step of algorithm 1 is the determination of circles delimiting the Bézier surface. Each circle is reprented by an union of two Bézier curves having extremal weights equal to 1, figure 6(b). e.g., circle C_3 is the union of Bézier curves having control points P_{00}, P_{01} and P_{02}, and opposite median weights $f_w(P_{00}; P_{01}; P_{02})$ and $-f_w(P_{00}; P_{01}; P_{02})$ [Gar07].

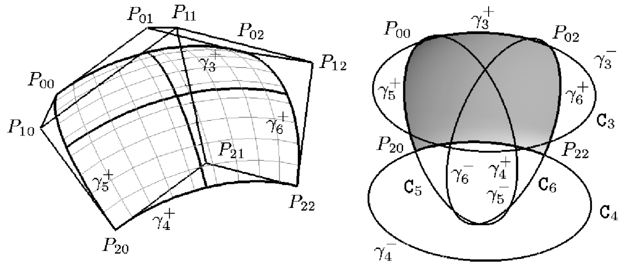


Figure 6: The conversion algorithm.(a) : the Bézier surface. (b) : circular Bézier surface edges.

Figure 7(a) shows the third step of the algorithm. Straight line Δ_θ is perpendicular to planes generated by parallel circles. It passes through the center of one of these circles (C_3 in the figure). Figure 7(b) shows the fourth step of the algorithm. Plane $\mathcal{P}_z$ is the plane passing through Ω with $\overrightarrow{k_0}$ as an orthogonal vector, where Ω is the perpendicular projection of center Ω_5 of circle C_5 onto the straight line $(\Omega_3, \overrightarrow{k_0})$.

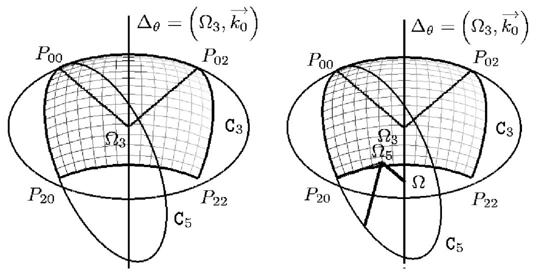


Figure 7: The conversion algorithm. (a) : determination of Δ_θ. (b) : determination of new frame origin Ω and plane $\mathcal{P}_z$.

Figure 8(a) permits, using Dupin cyclide plane of symmetry $\mathcal{P}_z$, the construction of points A et B belonging to

Algorithme 1 : Conversion of a Bézier surface into a torus patch or a double sphere.

Input data:

Let S be a Bézier surface defined by level-headed control points $(P_{ij}; w_{ij})_{0 \le i,j \le 2}$ such as:

$$Aff(P_{00}; P_{01}; P_{02}) \,//_{\neq} Aff(P_{20}; P_{21}; P_{22}) \quad (6)$$

Begin

1. Bézier surface edges are represented by standard Bézier curves:

Name	Control points	Intermediate weight
γ_3^+	$(P_{00}, P_{01}; P_{02})$	$f_w(P_{00}; P_{01}; P_{02})$
γ_3^-	$(P_{00}, P_{01}; P_{02})$	$-f_w(P_{00}; P_{01}; P_{02})$
γ_4^+	$(P_{20}; P_{21}; P_{22})$	$f_w(P_{20}; P_{21}; P_{22})$
γ_4^-	$(P_{20}; P_{21}; P_{22})$	$-f_w(P_{20}; P_{21}; P_{22})$
γ_5^+	$(P_{00}; P_{10}; P_{20})$	$f_w(P_{00}; P_{10}; P_{20})$
γ_5^-	$(P_{00}; P_{10}; P_{20})$	$-f_w(P_{00}; P_{10}; P_{20})$
γ_6^+	$(P_{02}; P_{12}; P_{22})$	$f_w(P_{02}; P_{12}; P_{22})$
γ_6^-	$(P_{02}; P_{12}; P_{22})$	$-f_w(P_{02}; P_{12}; P_{22})$

   Given circles $\mathsf{C}_3 = \gamma_3^+ \cup \gamma_3^-$, $\mathsf{C}_4 = \gamma_4^+ \cup \gamma_4^-$, $\mathsf{C}_5 = \gamma_5^+ \cup \gamma_5^-$ and $\mathsf{C}_6 = \gamma_6^+ \cup \gamma_6^-$.

   Condition (6) implies $\mathsf{C}_3 \,//\, \mathsf{C}_4$, figure 6(b).

2. New reference frame determination: $\overrightarrow{\imath_0}$ and $\overrightarrow{\jmath_0}$ are two unit orthogonal vectors generating vector plan $Vect(P_{00}; P_{01}; P_{02})$ and $\overrightarrow{k_0}$ is determined by :

$$\overrightarrow{k_0} = \overrightarrow{\imath_0} \times \overrightarrow{\jmath_0}$$

3. Altitude axis is $\Delta_\theta = (\Omega_3, \overrightarrow{k_0})$ where Ω_3 is the center of circle C_3, figure 7(a).

4. Let Ω_5 be the center of circle C_5. The origin of new reference frame is Ω, the orthogonal projection of Ω_5 onto $(\Omega_3, \overrightarrow{k_0})$. $\mathcal{P}_z$ is the plane passing through Ω with $\overrightarrow{k_0}$ as the orthogonal vector, figure 7(b).

5. Determination of points A and B such that: $\{A; B\} = \mathsf{C}_5 \cap \mathcal{P}_z$ and $\Omega B \le \Omega A$, figure 8(a).

6. In $\mathcal{P}_z$, principal circles C_1 and C_2 are determined by center Ω and radius $\rho_1 = \Omega A$ and $\rho_2 = \Omega B$ respectively, figure 8(b).

7. Dupin cyclide parameters computation.

   If $\mathsf{C}_1 = \mathsf{C}_2$, we obtain a double sphere with $a = c = 0$ and $\mu = \rho_1$.

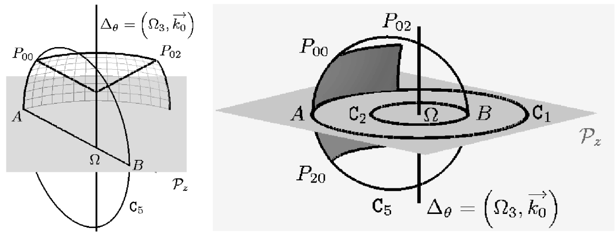
   If $\breve{\sharp}(\mathsf{C}_6 \cap \mathsf{C}_5) = 2$, we obtain a spindle torus with $c = 0$, $a = \frac{\rho_1 - \rho_2}{2}$ and $\mu = \frac{\rho_1 + \rho_2}{2}$, else we obtain a ring torus or a horned torus with $c = 0$, $a = \frac{\rho_1 + \rho_2}{2}$ and $\mu = \frac{\rho_1 - \rho_2}{2}$, figure 9(a).

8. Determination of values θ_0, θ_1, ψ_0 and ψ_1 delimiting the obtained patch [Gar07], figure 9(b).

End

Output: A torus patch or a double sphere patch representing the input Bézier surface.

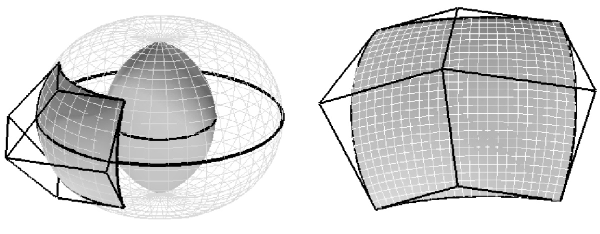
principal circles. Condition $\Omega B < \Omega A$ allows to identify immediately the great and the small torus principal circles. Condition $\Omega B = \Omega A$ implies that Dupin cyclide is a double sphere. Obviously, the center of these circles is the point Ω. Figure 8(b) shows two principal circles in $\mathcal{P}_z$.



(a) (b)

Figure 8: The conversion algorithm. (a) : plane $\mathcal{P}_z$. (b) : torus principal circles belonging to $\mathcal{P}_z$.

Figure 9(a) shows the torus, two principal circles, the Bézier surface with its control polyhedron. Obtained parameter values are $c = 0$, $a \simeq 1,63$ and $\mu \simeq 4,32$. Figure 9(b) shows the Bézier surface, its control polyhedron and the resulting torus patch which is delimited by curvature lines situated at: $\theta_0 \simeq 2,526112925$, $\theta_1 \simeq 3,757072362$, $\psi_0 \simeq 2,427868285$ and $\psi_1 \simeq 3,85531702$.

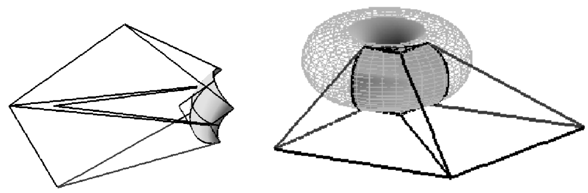


(a) (b)

Figure 9: The conversion algorithm. (a) : the Bézier surface with the control polyhedron and the spindle torus. (b) : the spindle torus patch with the control polyhedron of the initial Bézier surface.

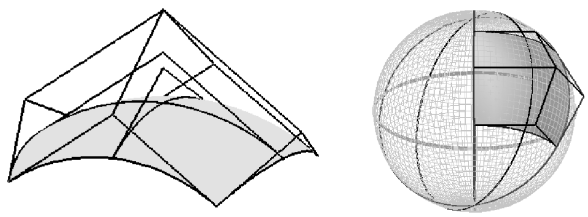
Figure 10(a) shows the Bézier surface with its control polyhedron, barycentric middle curves with their control polygons. Figure 10(b) shows the Bézier surface with its control polyhedron and the ring torus.

Figure 11 shows the conversion of a Bézier surface into a double sphere patch. Figure 11(a) shows the Bézier surface with its control polyhedron and the barycentric middle curves with their control polygons. Figure 11(b) shows the resulting double sphere patch with the control polyhedron of the initial Bézier surface.



(a) (b)

Figure 10: Conversion of a Bézier surface into a patch of a ring torus. (a) : the Bézier surface with the control polyhedron. (b) : the resulting patch of ring torus togother with the control polyhedron of the initial Bézier surface.



(a) (b)

Figure 11: Conversion of a Bézier surface into a double sphere patch. (a) : the Bézier surface with the control polyhedron. (b) : the resulting patch of the double sphere together with the control polyhedron of the initial Bézier surface.

5 Conclusion

In this paper, we have presented an algorithm which permits the conversion of a rational biquadratic Bézier surface into a torus patch or a double sphere patch which are particular Dupin cyclide patches. So, the rational biquadratic Bézier surface is fully represented by an implicit equation of degree 4. Moreover, if Dupin cyclide is a double sphere, it is possible to use an equation of degree 2 (the equation of the sphere).

An interesting extension of this work is to find the sufficiant conditions to construct rational biquadratic Bézier surfaces fully convertible into Dupin cyclide patches. The study of conversion of rational biquadratic Bézier surfaces into supercyclide patches will also be considered.

References

[AD96] G. Albrecht and W. Degen. Construction of Bézier rectangles and triangles on the symetric Dupin horn cyclide by means of inversion. *Computer Aided Geometric Design*, 14(4):349–375, 1996.

[BP98] W. Boehm and M. Paluszny. General cyclides as joining pipes. *Computer Aided Geometric Design*, 15:699–710, 1998.

[Dar17] G. Darboux. *Principes de géométrie analytique*. Gauthier-Villars, 1917.

[Deg94] W. L. F. Degen. Generalized Cyclides for Use in CAGD. In A. Bowyer, editor, *The Mathematics of Surfaces IV*, pages 349–363, Oxford, 1994. Clarendon Press.

[DMP93] D. Dutta, R. R. Martin, and M. J. Pratt. Cyclides in surface and solid modeling. *IEEE Computer Graphics and Applications*, 13(1):53–59, January 1993.

[DP98] G. Demengel and J. P. Pouget. *Mathématiques des Courbes et des Surfaces. Modèles de Bézier, des B-Splines et des NURBS*. Ellipse, 1998.

[Dup22] C. P. Dupin. *Application de Géométrie et de Méchanique à la Marine, aux Ponts et Chaussées, etc.* Bachelier, Paris, 1822.

[Far93] G. Farin. *Curves And Surfaces*. Academic Press, 3rd edition, 1993.

[Far99] G. Farin. *NURBS from Projective Geometry to Pratical Use*. A K Peters, Ltd, 2 edition, 1999. ISBN 1-56881-084-9.

[FGP05] S. Foufou, L. Garnier, and M. Pratt. Conversion of Dupin Cyclide Patches into Rational Biquadratic Bézier Form. In R. Martin, H. Bez, and M. Sabin, editors, *Proceedings of the 11th Conference on the Mathematics of Surfaces*, pages 201–218. Springer-Verlag Berlin Heidelberg, September 2005. ISBN: 3-540-28225.

[For12] A. R. Forsyth. *Lecture on Differential Geometry of Curves and Surfaces*. Cambridge University Press, 1912.

[For68] A. Forest. *Curves and Surfaces for Computer-Aided Design*. PhD thesis, University of Cambridge, 1968.

[Gar07] L. Garnier. *Mathématiques pour la modélisation géométrique, la représentation 3D et la synthèse d'images*. Ellipses, 2007.

[GFN06] L. Garnier, S. Foufou, and M. Neveu. Conversion d'un carreau de Bézier rationnel biquadratique en un carreau de cyclide de Dupin quartique. *TSI*, 25(6):709–734, 2006. numéro spécial AFIG'04, Hermès.

[HL93] J. Hoschek and D. Lasser. *Fundamentals of Computer Aided Geometric Design*. A.K.Peters, Wellesley, MA., 1993.

[Pra90] M. J. Pratt. Cyclides in computer aided geometric design. *Computer Aided Geometric Design*, 7(1-4):221–242, 1990.

[Pra95] M. J. Pratt. Cyclides in computer aided geometric design II. *Computer Aided Geometric Design*, 12(2):131–152, 1995.

[PT89] L. Piegl and W. Tilles. A managerie of rational b-spline circles. *IEEE Computer Graphics and Applications*, 9(5):46–56, 1989.

[She98] C. K. Shene. Blending two cones with Dupin cyclides. *Computer Aided Geometric Design*, 15(7):643–673, 1998.

[Ued95] K. Ueda. Normalized Cyclide Bézier Patches. In *Mathematical Methods for Curves and Surfaces*, pages 507–516, Nashville, USA, 1995. Vanderbilt University Press.