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Differential transform method to study free transverse vibration of monoclinic rectangular plates resting on Winkler foundation

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Abstract

This paper analyses free transverse vibration of a monoclinic rectangular plate of uniform thickness resting on Winkler foundation using differential transform method (DTM). Two parallel edges of the plate are taken according to Levy approach i.e., simply supported and other two edges may be either clamped-clamped or clamped-simply supported. This semi-numerical-analytical technique converts the governing differential equation and boundary conditions into a set of algebraic equations. Characteristic equations have been obtained for above two combinations of boundary conditions in the form of infinite series and solved numerically by truncating these equations to finite number of terms. Robustness and convergence of the method is confirmed through numerical results. Two dimensional and three dimensional mode shapes have been plotted for both the cases.

Keywords: differential transform method, monoclinic, rectangular, Winkler foundation

1. Introduction

Recently, solutions of engineering problems have appeared in the literature using differential transform method (DTM). In 1986, Zhou [14] developed it to solve initial value problems occurring in electrical circuits. It works as an alternative approach for getting Taylor series solution. Using this method, the solution of the problem is obtained in the form of a polynomial. So far, the eigenvalue problems have been solved using Frobenius method [3], finite difference method [9], finite element method [11], differential quadrature method [12], Chebyshev collocation technique [7], discrete singular convolution method [10] and Rayleigh-Ritz method [6], etc. DTM seems quite easily applicable for getting the solution of eigenvalue problems. Very few vibration problems have been solved using DTM [1, 2, 5, 8, 13]. In this paper, DTM has been applied to fourth order boundary value problem that represents free transverse vibration of monoclinic thin rectangular plate of uniform thickness resting on Winkler foundation. The first three natural frequencies have been presented for two boundary configurations. This paper has been organized as follows: In section 2, mathematical model of the problem under study is presented. Section 3 presents the respective boundary conditions. Section 4 is concerned with the solution and results of the problem. Conclusions are presented in section 5.

2. Mathematical model of the problem

A monoclinic rectangular plate of uniform thickness h with the domain $0 \leq x \leq a$, $0 \leq y \leq b$, where a and b are the length and the breadth of the plate, respectively, is considered. The z-axis

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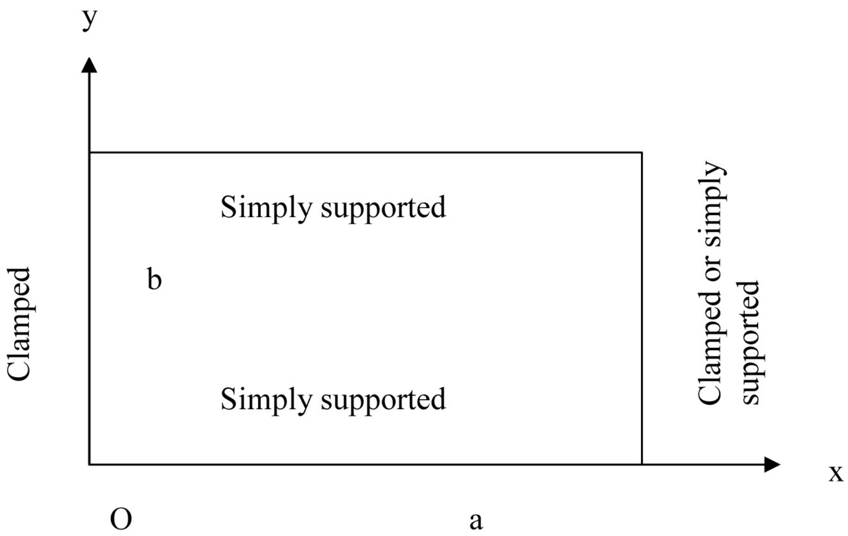
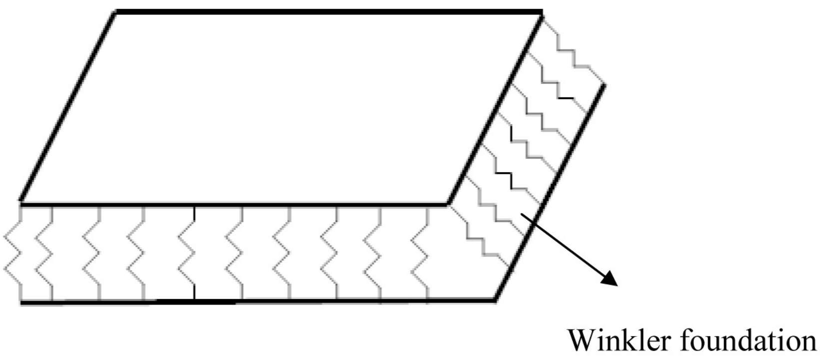
is taken in the perpendicular direction of xy-plane. Middle surface of the plate is denoted by $z = 0$. One of the corners of the plate is designated as the origin of the plate. The plate is resting on Winkler foundation having the foundation modulus K_f. Two opposite edges $y = 0$ and $y = b$ are taken to be simply supported (see Fig. 1).



(i)



(ii)

Fig. 1. (i) Geometry of the plate with boundary conditions and (ii) plate resting on Winkler foundation

Following Kumar and Tomar [7], the differential equation describing the motion of a monoclinic rectangular plate of uniform thickness resting on Winkler foundation is given as follows:

$$a_0 \frac{\mathrm{d}^4\bar{w}}{\mathrm{d}X^4} + a_1 \frac{\mathrm{d}^3\bar{w}}{\mathrm{d}X^3} + a_2 \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}X^2} + a_3 \frac{\mathrm{d}\bar{w}}{\mathrm{d}X} + a_4\bar{w} = 0, \tag{1}$$

where

$$\begin{aligned} &a_0 = 1, \qquad a_1 = 0, \qquad a_2 = -2\lambda^2(c_{12} + c_{21} + 2c_{66})/c_{11}, \\ &a_3 = 0, \qquad a_4 = \lambda^4\,(c_{22}/c_{11}) + (12K/h^3) - \Omega^2, \qquad X = \frac{x}{a}, \\ &K = \frac{aK_f}{c_{11}}, \qquad \lambda^2 = \frac{m^2\pi^2a^2}{b^2}, \qquad \Omega^2 = \frac{12\rho a^4\omega^2}{c_{11}h^2}. \end{aligned}$$

Here c_{11}, c_{12}, c_{21}, c_{22}, c_{66} are elastic coefficients, ρ is density of the plate material, ω is the circular frequency, K is the foundation parameter and Ω is the frequency parameter.

3. Boundary conditions

Two boundary conditions, namely, C-C and C-S have been considered where first letter represents the boundary condition at the edge $X = 0$ and second one at the edge $X = 1$. Here, C is used for clamped edge and S for simply supported edge. The edge $X = 0$ is clamped and the edge $X = 1$ is either clamped or simply supported. The conditions that should be satisfied by clamped and simply supported edges are

$$\bar{w} = \frac{\mathrm{d}\bar{w}}{\mathrm{d}X} = 0 \text{ for clamped edge} \tag{2}$$

and

$$\bar{w} = \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}X^2} - \frac{(c_{12} + c_{21})}{c_{11}}\lambda^2\bar{w} = 0 \text{ for simply supported edge.} \tag{3}$$

4. Solution and results of the problem

The Taylor's series expansion of a function $\bar{w}(X)$ may be written as

$$\bar{w}(X) = \sum_{i=0}^{\infty}(X - X_0)^i\bar{W}_i, \tag{4}$$

where $\bar{W}_i = \frac{1}{i!}\left[\frac{\mathrm{d}^i\bar{w}}{\mathrm{d}X^i}\right]_{X=X_0}$ is called i-th order differential transform of $\bar{w}(X)$ about a point $X = X_0$. The series is truncated to finite number of terms i.e., N while solving practical problems.

Taking the differential transform of equation (1) at $X_0 = 0$, we get

$$a_0\frac{(i+4)!}{i!}\bar{W}_{i+4} + a_2\frac{(i+2)!}{i!}\bar{W}_{i+2} + a_4\bar{W}_i = 0, \tag{5}$$

as differential transform of $\frac{\mathrm{d}^k\bar{w}}{\mathrm{d}X^k}$ is given by $\frac{(i+k)!}{i!}\bar{W}_{i+k}$.

After taking the differential transform, the boundary conditions (2) and (3) may be written as follows:

$$\sum_{i=0}^{N}(X - X_0)^i\bar{W}_i = 0,$$
$$\sum_{i=0}^{N}i(X - X_0)^{i-1}\bar{W}_i = 0 \tag{6}$$

and

$$\sum_{i=0}^{N}(X - X_0)^i\bar{W}_i = 0,$$
$$\sum_{i=0}^{N}i(i-1)(X - X_0)^{i-2}\bar{W}_i - \frac{c_{12} + c_{21}}{c_{11}}\lambda^2\sum_{i=0}^{N}(X - X_0)^i\bar{W}_i = 0. \tag{7}$$

The equation (5) can be re-written in the following manner

$$\bar{W}_{i+4} = \frac{R}{(i+4)(i+3)}\bar{W}_{i+2} + \frac{S}{(i+4)(i+3)(i+2)(i+1)}\bar{W}_i, \quad i = 0, 1, 2, 3, \ldots, N, \tag{8}$$

where

$$R = \frac{2(c_{12} + c_{21} + 2c_{66})\lambda^2}{c_{11}}, \qquad S = \Omega^2 - (c_{22}/c_{11})\lambda^4 - (12K/h^3).$$

Now, characteristic equations for both the cases can be obtained by adopting the following mathematical procedure:

Case 1: Clamped at $X = 0$ and clamped at $X = 1$

Let $X_0 = 0$. At $X = 0$, equations (6) become

$$\begin{aligned} \bar{W}_0 + 0\bar{W}_1 + 0\bar{W}_2 + 0\bar{W}_3 + 0\bar{W}_4 + 0\bar{W}_5 + \ldots &= 0, \\ 0\bar{W}_0 + \bar{W}_1 + 0\bar{W}_2 + 0\bar{W}_3 + 0\bar{W}_4 + 0\bar{W}_5 + \ldots &= 0, \end{aligned} \tag{9}$$

i.e.,

$$\bar{W}_0 = \bar{W}_1 = 0.$$

Using equation (8), first few terms can be written as follows:

$$\begin{aligned} \bar{W}_4 &= \frac{R}{12}\bar{W}_2, & \bar{W}_6 &= \frac{R}{30}\bar{W}_4 + \frac{S}{360}\bar{W}_2, & \bar{W}_8 &= \frac{R}{56}\bar{W}_6 + \frac{S}{1\,680}\bar{W}_4, \ldots \\ \bar{W}_5 &= \frac{R}{20}\bar{W}_3, & \bar{W}_7 &= \frac{R}{42}\bar{W}_5 + \frac{S}{840}\bar{W}_3, & \bar{W}_9 &= \frac{R}{72}\bar{W}_7 + \frac{S}{3\,024}\bar{W}_5, \ldots \end{aligned} \tag{10}$$

It is evident from equations (10) that $\bar{W}_{2i}$ and $\bar{W}_{2i+1}$ can be represented in terms of $\bar{W}_2$ and $\bar{W}_3$, respectively.

At $X = 1$, equations (6) become

$$\begin{aligned} \bar{W}_0 + \bar{W}_1 + \bar{W}_2 + \bar{W}_3 + \bar{W}_4 + \bar{W}_5 + \ldots &= 0, \\ 0\bar{W}_0 + \bar{W}_1 + 2\bar{W}_2 + 3\bar{W}_3 + 4\bar{W}_4 + 5\bar{W}_5 + \ldots &= 0. \end{aligned} \tag{11}$$

Using (10), equations (11) can be written as follows:

$$\begin{aligned} p_{11}(\Omega)\bar{W}_2 + p_{12}(\Omega)\bar{W}_3 &= 0, \\ p_{21}(\Omega)\bar{W}_2 + p_{22}(\Omega)\bar{W}_3 &= 0. \end{aligned} \tag{12}$$

The characteristic equation is obtained from the non-trivial condition of (12) i.e.,

$$\begin{vmatrix} p_{11}(\Omega) & p_{12}(\Omega) \\ p_{21}(\Omega) & p_{22}(\Omega) \end{vmatrix} = 0, \tag{13}$$

where p_{ij}, $i, j = 1, 2$ are polynomials in Ω^2.

In particular,

$$\begin{aligned} p_{11}(\Omega) &= 1 + \frac{R}{12} + \frac{R^2 + S}{360} + \frac{R^3 + 2RS}{20\,160} + \ldots, \\ p_{12}(\Omega) &= 1 + \frac{R}{20} + \frac{R^2 + S}{840} + \frac{R^3 + 2RS}{60\,480} + \ldots, \\ p_{21}(\Omega) &= 2 + \frac{4R}{12} + \frac{6(R^2 + S)}{360} + \frac{8(R^3 + 2RS)}{20\,160} + \ldots, \\ p_{22}(\Omega) &= 3 + \frac{5R}{20} + \frac{7(R^2 + S)}{840} + \frac{9(R^3 + 2RS)}{60\,480} + \ldots \end{aligned}$$

Case 2: Clamped at $X = 0$ and simply supported at $X = 1$
Differential transform of equations (7) at $X = 1$ leads to

$$\begin{aligned} \bar{W}_2 + \bar{W}_3 + \bar{W}_4 + \bar{W}_5 + \ldots &= 0, \\ 2\bar{W}_2 + 6\bar{W}_3 + 12\bar{W}_4 + 20\bar{W}_5 + \ldots &= 0. \end{aligned} \tag{14}$$

Hence, the non-trivial condition of (14) after incorporating (10) provides the following characteristic equation

$$\begin{vmatrix} p_{11}(\Omega) & p_{12}(\Omega) \\ p_{21}(\Omega) & p_{22}(\Omega) \end{vmatrix} = 0, \tag{15}$$

where

$$\begin{aligned} p_{11}(\Omega) &= 1 + \frac{R}{12} + \frac{R^2+S}{360} + \frac{R^3+2RS}{20\,160} + \ldots, \\ p_{12}(\Omega) &= 1 + \frac{R}{20} + \frac{R^2+S}{840} + \frac{R^3+2RS}{60\,480} + \ldots, \\ p_{21}(\Omega) &= 2 + \frac{12R}{12} + \frac{30(R^2+S)}{360} + \frac{56(R^3+2RS)}{20\,160} + \ldots, \\ p_{22}(\Omega) &= 6 + \frac{20R}{20} + \frac{42(R^2+S)}{840} + \frac{72(R^3+2RS)}{60\,480} + \ldots \end{aligned}$$

Displacements of the plates are obtained using the following function:

$$\bar{W}(X) = \sum_{i=0}^{N} (X - X_0)^i \bar{W}_i, \tag{16}$$

$$\begin{aligned} \bar{W}(X) &= X^2\bar{W}_2 + X^4\bar{W}_4 + X^6\bar{W}_6 + X^8\bar{W}_8 + X^{10}\bar{W}_{10} + \ldots + \\ &\quad X^3\bar{W}_3 + X^5\bar{W}_5 + X^7\bar{W}_7 + X^9\bar{W}_9 + X^{11}\bar{W}_{11} + \ldots \\ &= \left[X^2 + \frac{R}{12}X^4 + \frac{(R^2+S)}{360}X^6 + \frac{(R^3+2RS)}{20\,160}X^8 + \ldots\right]\bar{W}_2 + \\ &\quad \left[X^3 + \frac{R}{20}X^5 + \frac{(R^2+S)}{840}X^7 + \frac{(R^3+2RS)}{60\,480}X^9 + \ldots\right]\bar{W}_3 \\ &= \left[X^2 + \frac{R}{12}X^4 + \frac{(R^2+S)}{360}X^6 + \frac{(R^3+2RS)}{20\,160}X^8 + \ldots\right]\bar{W}_2 - \\ &\quad \frac{p_{11}(\Omega)}{p_{12}(\Omega)}\left[X^3 + \frac{R}{20}X^5 + \frac{(R^2+S)}{840}X^7 + \frac{(R^3+2RS)}{60\,480}X^9 + \ldots\right]\bar{W}_2. \end{aligned} \tag{17}$$

For numerical simulation, rock gypsum has been taken as an example of monoclinic material and the values of elastic constants for the same have been taken from Haussuhl [4] as

$$\begin{aligned} c_{11} &= 7.859 \times 10^6 \text{ erg/cm}^3, & c_{12} = c_{21} &= 4.1 \times 10^6 \text{ erg/cm}^3, \\ c_{22} &= 6.287 \times 10^6 \text{ erg/cm}^3, & c_{66} &= 1.044 \times 10^6 \text{ erg/cm}^3. \end{aligned}$$

Apart from it, the values of other parameters considered are

$$K = 0.01, 0.02, 0.03, 0.04, 0.05, \qquad a/b = 0.5,\ 1.0.$$

To obtain the values of frequency parameter Ω, the characteristic equations (13) and (15) have been solved using bisection method with the help of a computer program developed in C++ for different values of aspect ratio $\lambda(= a/b)$ and foundation parameter K for both the boundary configurations. This program was run for different values of N until we get first three values of frequency parameter Ω correct to four decimal places and the value of N has been taken as 36. The value of m has been fixed as 1. The convergence of first three values of frequency parameter Ω for monoclinic square C-C and C-S plates with increasing number of terms N is shown in Table 1. The desired accuracy of first mode can be achieved by using 26 terms in both the cases and higher modes can be obtained by increasing the number of terms. First three values of frequency parameter Ω for different combinations of aspect ratio and foundation parameter are presented in Tables 2 and 3 for monoclinic and isotropic plates, respectively. It is concluded that the value of frequency parameter Ω for C-C plate is greater than that for C-S plate.

Table 1. Convergence of first three values of frequency parameter Ω for monoclinic square plates for $K = 0.05$

	C-C			C-S		
	mode					
N	I	II	III	I	II	III
5	21.7242	–	–	19.1580	–	–
10	29.2641	–	–	26.6908	–	–
15	37.7442	290.2750	–	33.9770	293.4680	–
20	38.6039	360.0430	–	34.7363	360.5910	–
25	38.6257	75.3754	112.7500	34.7544	65.4812	102.8810
30	38.6257	75.2865	130.1810	34.7544	65.4272	116.4180
35	38.6257	75.2672	133.5870	34.7544	65.4267	118.2090
36	38.6257	75.2672	133.5870	34.7544	65.4267	118.2090

Table 2. First three values of frequency parameter Ω for monoclinic plates

		C-C		C-S	
		a/b			
K	mode	0.5	1.0	0.5	1.0
0.00	I	24.1786	29.8655	17.7925	24.6549
	II	64.0740	71.1699	52.7101	60.6684
	III	123.5360	131.3220	107.1440	115.6430
0.01	I	26.5444	31.8111	20.8943	26.9790
	II	65.0037	72.0080	53.8364	61.6494
	III	124.0210	131.7780	107.7030	116.1610
0.02	I	28.7159	33.6444	23.5918	29.1182
	II	65.9203	72.8365	54.9395	62.6151
	III	124.5040	132.2330	108.2580	116.6760
0.03	I	30.7344	35.3829	26.0110	31.1106
	II	66.8243	73.6556	56.0210	63.5661
	III	124.9850	132.6860	108.8110	117.1890
0.04	I	32.6283	37.0398	28.2236	32.9828
	II	67.7162	74.4658	57.0820	64.5031
	III	125.4640	133.1370	109.3610	117.7000
0.05	I	34.4181	38.6257	30.2750	34.7544
	II	68.5965	75.2672	58.1236	65.4267
	III	125.9410	133.5870	109.9080	118.2090

Table 3. First three values of frequency parameter Ω for isotropic $\left(\frac{c_{12}+c_{21}+2c_{66}}{c_{11}} = \upsilon,\ \frac{c_{22}}{c_{11}} = 1,\ c_{11} = \frac{E}{1-\upsilon^2}\right)$ plates

		C-C		C-S	
		a/b			
K	mode	0.5	1.0	0.5	1.0
0.00	I	23.8156	28.9509	17.3318	23.6463
	II	63.5345	69.3270	52.0979	58.6464
	III	122.9290	129.0900	106.4790	113.2260
0.01	I	26.2142	30.9540	20.5034	26.0605
	II	64.4720	70.1872	53.2372	59.6607
	III	123.4170	129.5540	107.0410	113.7550
0.02	I	28.4110	32.8352	23.2463	28.2692
	II	65.3960	71.0369	54.3525	60.6581
	III	123.9020	130.0170	107.6000	114.2810
0.03	I	30.4497	34.6143	25.6981	30.3175
	II	66.3071	71.8765	55.4454	61.6393
	III	124.3850	130.4770	108.1560	114.8050
0.04	I	32.3602	36.3064	27.9355	32.2358
	II	67.2059	72.7065	56.5172	62.6051
	III	124.8660	130.9360	108.7090	115.3270
0.05	I	34.1641	37.9230	30.0065	34.0463
	II	68.0928	73.5271	57.5691	63.5562
	III	125.3460	131.3940	109.2600	115.8460

Also, frequencies of monoclinic plates are greater than those for isotropic plates for same values of parameters. Further, it increases with the increasing values of foundation parameter K and aspect ratio a/b. More importantly, the difference in the values of frequency parameter Ω for first mode of vibration for monoclinic C-C ($a/b = 0.5$, $K = 0$, 0.01, 0.02, 0.03, 0.04, 0.05) and C-S ($a/b = 1.0$, $K = 0$, 0.01, 0.02, 0.03, 0.04, 0.05) plates is not considerable. Same conclusion is true for isotropic plate. The percentage variations in the value of frequency parameter are more for C-S plates than those for C-C plates and these percentage variations decrease with the increasing value of K when material changes from isotropic to monoclinic.

The percentage variations in the value of frequency parameter are 1.5, 1.3, 1.1, 0.9, 0.8, 0.7 when K changes from 0.0 to 0.05 for first mode of vibration ($a/b = 0.5$ and C-C plate). These percentage variations are 0.8 and 0.5 for second and third modes, respectively, for all K. These variations increase with increasing value of a/b. Displacements have been calculated using equation (17).

Two dimensional and three dimensional mode shapes of C-C and C-S plates for $K = 0.01$, $a/b = 1$ have been depicted in Figs. 2–4. Three dimensional mode shapes have been plotted using MATLAB software.

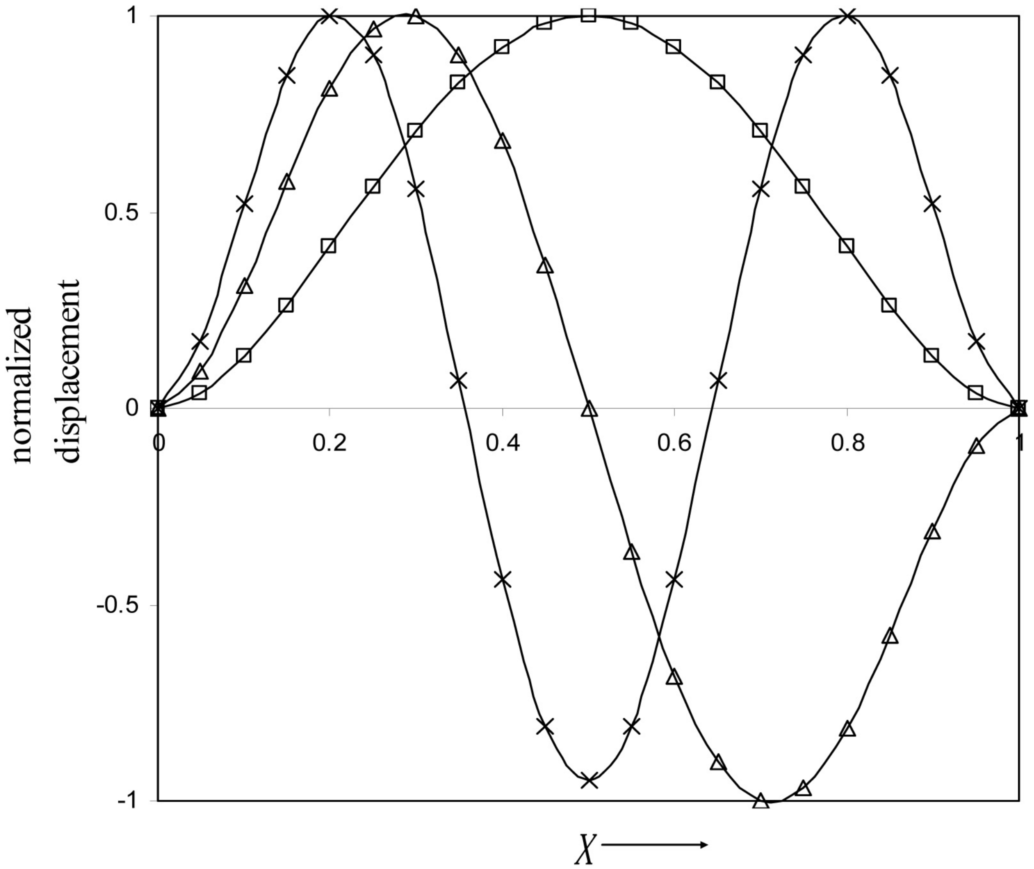


Fig. 2. Normalized displacements of C-C monoclinic plate for $K = 0.01$, $a/b = 1$. First mode (□), second mode (△) and third mode (×)

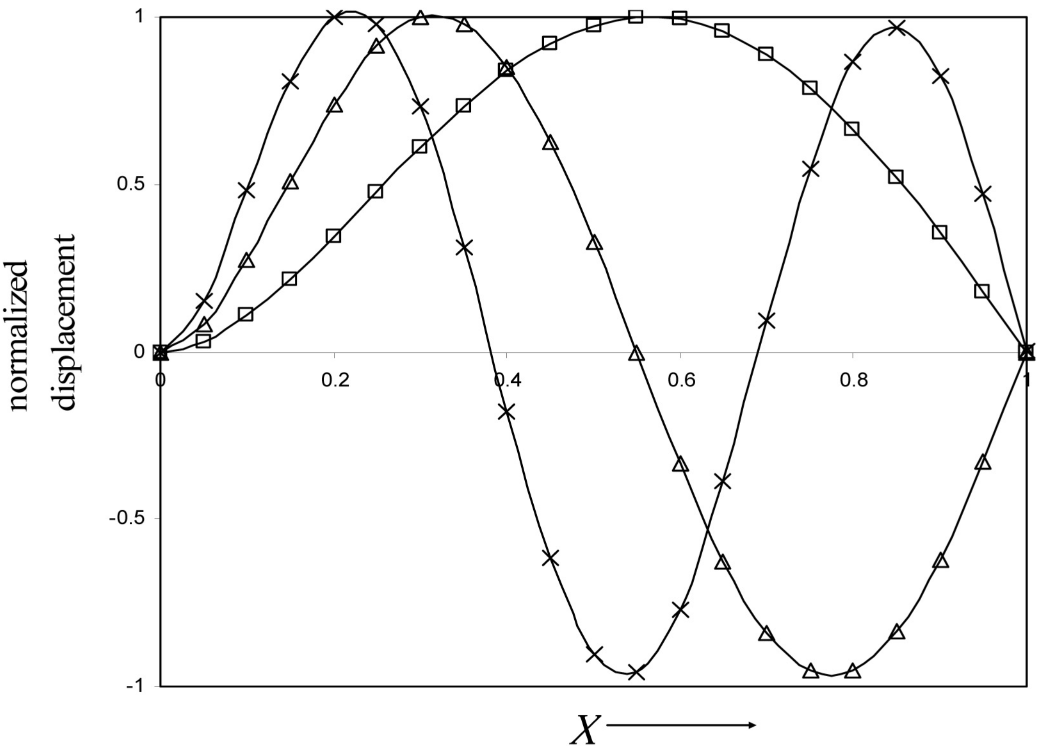


Fig. 3. Normalized displacements of C-S monoclinic plate for $K = 0.01$, $a/b = 1$. First mode (□), second mode (△) and third mode (×)

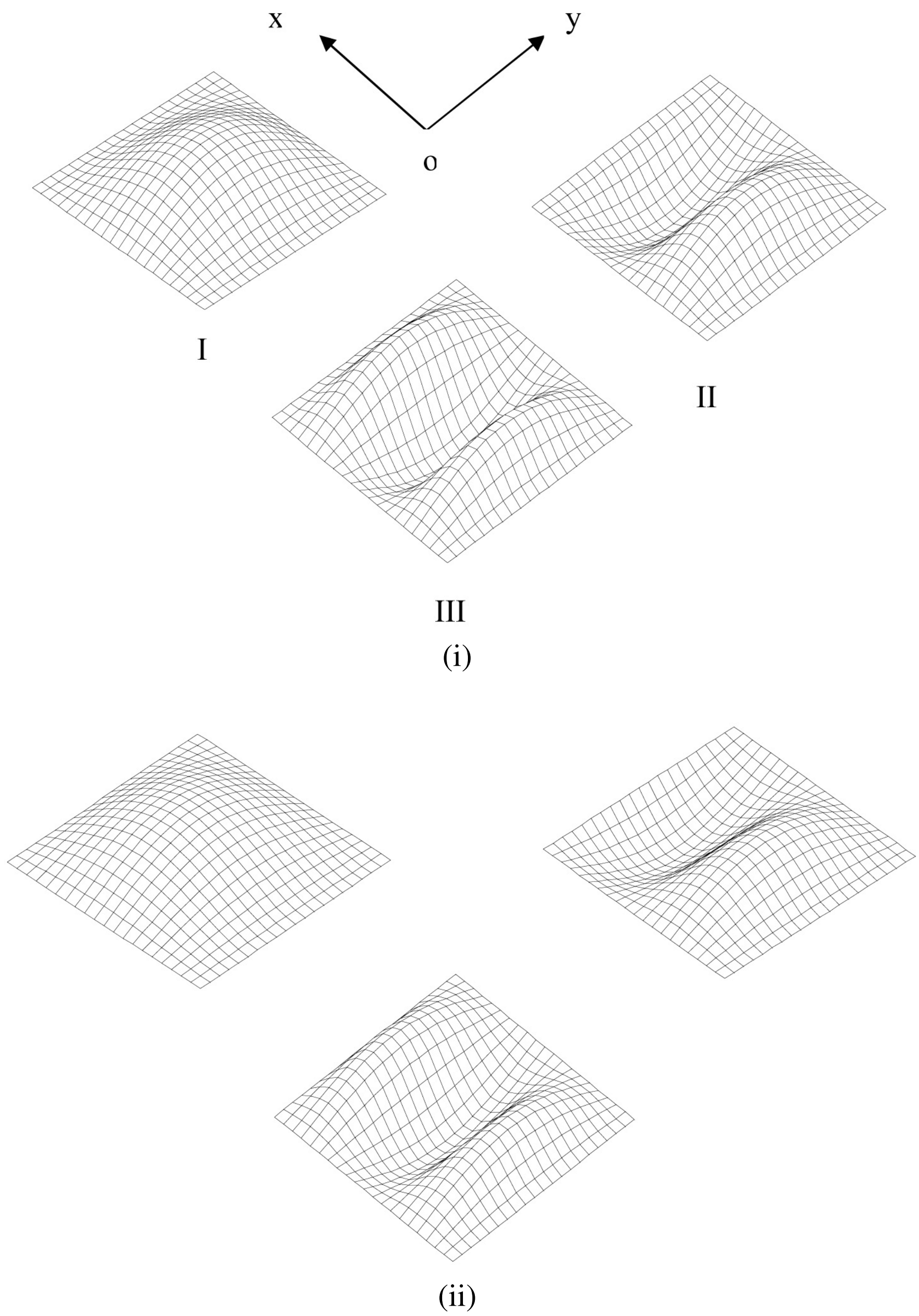


Fig. 4. First three mode shapes of (i) C-C and (ii) C-S monoclinic plates for $K = 0.01$, $a/b = 1$ using $w(x, y) = \bar{w}(x/a)\sin(m\pi y/b)$

5. Conclusions

Differential transform method is successfully applied to analyze free transverse vibration of monoclinic rectangular plates of uniform thickness resting on Winkler foundation. The two opposite edges of the plate are assumed to be simply supported. Two boundary conditions namely, clamped and simply supported have been taken on one of the other two parallel edges, keeping the other edge clamped. Characteristic equations have been obtained in the form of infinite series. The series have been truncated to finite number of terms and solved numerically to obtain first three natural frequencies using a computer program developed by the author in

C++ language. Displacements have been calculated and demonstrated in two dimensions as well as three dimensions. Analysis shows that present method performed really well for monoclinic plates in terms of simplicity and efficiency.

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